Bull. Math. Soc. Sci. Math. Roumanie
Tome 59(107) No. 2, 2016, 187–195

Depth in a pathological case

by
Dorin Popescu*

Abstract

Let I be a squarefree monomial ideal of a polynomial algebra over a field minimally generated by $f_1, \ldots, f_r$ of degree $d \geq 1$, and a set E of monomials of degree $\geq d+1$. Let $J \subsetneq I$ be a squarefree monomial ideal generated in degree $\geq d+1$. Suppose that all squarefree monomials of $I \setminus (J \cup E)$ of degree $d+1$ are some least common multiples of f_i. If J contains all least common multiples of two of (f_i) of degree $d+2$ then $\operatorname{depth}_S I/J \leq d+1$ and Stanley's Conjecture holds for I/J.

Key Words: Monomial Ideals, Depth, Stanley depth.
2010 Mathematics Subject Classification: Primary 13C15; Secondary 13F20, 13F55, 13P10.

Introduction

Let K be a field and $S = K[x_1, \ldots, x_n]$ be the polynomial K-algebra in n variables. Let $I \supsetneq J$ be two monomial ideals of S and suppose that I is generated by some monomials of degrees $\geq d$ for some positive integer d. After a multigraded isomorphism we may assume either that $J = 0$, or J is generated in degrees $\geq d+1$.

Suppose that $I \subset S$ is minimally generated by some monomials $f_1, \ldots, f_r$ of degrees d, and a set E of monomials of degree $\geq d+1$. Let B (resp. C) be the set of squarefree monomials of degrees $d+1$ (resp. $d+2$) of $I \setminus J$. Let w_{ij} be the least common multiple of f_i and f_j, $i < j$ and set W to be the set of all w_{ij}. By [4, Proposition 3.1] (see [7, Lemma 1.1]) we have $\operatorname{depth}_S I/J \geq d$. It is easy to see that if $d = 1$, $E = \emptyset$ and $B \subset W$ then $\operatorname{depth}_S I/J = d$ (see for instance [7, Lemma 1.8] and [6, Lemma 3]). Attempts to extend this result were made in [10, Proposition 1.3], [6, Lemma 4]. However [6, Example 1] (see here Example 1) shows that for $d = 2$, $E = \emptyset$ and $B \subset W$ it holds $\operatorname{depth}_S I/J = d+1 = 3$.

*The support from grant ID-PCE-2011-1023 of Romanian Ministry of Education, Research and Innovation is gratefully acknowledged.

If $B \cap (f_1, \dots, f_r) \subset W$ we call I/J a *pathological case*. It is the purpose of this paper to show the following theorem.

Theorem 1. *If* $B \cap (f_1, \dots, f_r) \subset W$ *and* $C \cap W = \emptyset$ *then* $\operatorname{depth}_S I/J \leq d + 1$.

In particular, if $C \cap W = \emptyset$ then the so called Stanley's Conjecture holds in the pathological case (see Corollary 1). But why is important this pathological case? The methods used in [11], [6], [9] to show a weak form of Stanley's Conjecture when $r \leq 4$ (see [9, Conjecture 0.1]) could be applied only when $B \cap (f_1, \dots, f_r) \not\subset W$, that is when I/J is not pathological. Thus the above theorem solves partially one of the obstructions to prove this weak form. We believe that the condition $C \cap W = \emptyset$ could be removed from the above theorem. The proof of Theorem 1 relies on Lemmas 2, 3 and Examples 2, 3, 5 found after many computations with the Computer Algebra System SINGULAR [3].

The above theorem hints a possible positive answer to the following question.

Question 1. *Let* $i \in [r-1]$. *Suppose that* $E = \emptyset$ *and every squarefree monomial from* $I \setminus J$ *of degree* $d+i$ *is a least common multiple of* $i+1$ *monomials* f_j. *Then is it* $\operatorname{depth}_S I/J \leq d + i$*?*

We owe thanks to A. Zarojanu and a Referee, who noticed some mistakes and a gap in Section 2 of some previous versions of this paper.

1 Depth and Stanley depth

Suppose that I is minimally generated by some squarefree monomials $f_1, \dots, f_r$ of degree d for some $d \in \mathbb{N}$ and a set E of some squarefree monomials of degree $\geq d+1$. Let C_3 be the set of all $c \in C \cap (f_1, \dots, f_r)$ having all degree $(d+1)$ divisors from $B \setminus E$ in W. In particular each monomial of C_3 is the least common multiple of at least three of the f_i.

Next lemma is closed to [10, Lemma 1.1].

Lemma 1. *Suppose that* $E = \emptyset$ *and* $\operatorname{depth}_S I/(J, b) = d$ *for some* $b \in B$. *Then* $\operatorname{depth}_S I/J \leq d + 1$.

Proof: If there exists no $c \in C$ such that $b | c$ then we have $\operatorname{depth}_S I/J \leq d+1$ by [10, Lemma 1.5]. Otherwise, in the exact sequence

$$0 \to (b)/J \cap (b) \to I/J \to I/(J, b) \to 0$$

the first term has depth $\geq d+2$ because for a multiple $c \in C$ of b all the variables of c form a regular system. By hypothesis the last term has depth $\geq d$ and so the middle one has depth d too using the Depth Lemma. □

We recall the following example from [6].

Example 1. Let $n = 5$, $r = 5$, $d = 2$, $f_1 = x_1x_2$, $f_2 = x_1x_3$, $f_3 = x_1x_4$, $f_4 = x_2x_3$, $f_5 = x_3x_5$ and $I = (f_1, \ldots, f_5)$, $J = (x_1x_2x_5, x_1x_4x_5, x_2x_3x_4, x_3x_4x_5)$. It follows that $B = \{x_1x_2x_3, x_1x_2x_4, x_1x_3x_4, x_1x_3x_5, x_2x_3x_5\}$ and so $s = |B| = r = 5$. Note that $B \subset W$. A computation with SINGULAR when char $K = 0$ gives $\operatorname{depth}_S I/J = \operatorname{depth}_S S/J = 3$ and $\operatorname{depth}_S S/I = 2$. Since depth depends on the characteristic of the field it follows in general only that $\operatorname{depth}_S S/J \leq 3$, $\operatorname{depth}_S S/I \leq 2$ using [1, Lemma 2.4]. In fact $\operatorname{depth}_S I/J \leq d + 1 = 3$ using [12, Proposition 2.4] because $q = |C| = 2 < r = 5$. Note that choosing any $b \in B$ we have $\operatorname{depth}_S I/(J, b) = 2$ because the corresponding $s' < r$ and we may apply [7, Theorem 2.2]. But then $\operatorname{depth}_S I/J \leq 3$ by Lemma 1.

Example 2. In the above example set $I' = (f_1, \ldots, f_4)$, $J' = J \cap I' = (x_1x_2x_5, x_1x_4x_5, x_2x_3x_4)$. Note that we have an injection $I'/J' \to I/J$ and so $\operatorname{depth}_S I'/J' > 2$ because otherwise we get $\operatorname{depth}_S I/J = 2$ which is impossible. Given B', W' for I'/J' we see that $B' \not\subset W'$ since $x_2x_3x_5 \in B' \setminus W'$, that is I'/J' is not anymore in the pathological case even this was the case of I/J. We have $I/(J, f_5) \cong I'/(J', x_1x_3x_5, x_2x_3x_5)$. Using SINGULAR when char $K = 0$ we see that $\operatorname{depth}_S I'/(J', x_1x_3x_5) = \operatorname{depth}_S I'/(J', x_1x_3x_5, x_2x_3x_5) = 2$, $\operatorname{depth}_S I'/(J', x_2x_3x_5) = 3$. It follows that always

$$\operatorname{depth}_S I'/(J', x_1x_3x_5), \operatorname{depth}_S I'/(J', x_1x_3x_5, x_2x_3x_5) \leq 2$$

using [1, Lemma 2.4]. These inequalities are in fact equalities because I' is generated in degree 2. Thus we cannot apply Lemma 1 for I', J', $b = x_2x_3x_5$ but we may apply this lemma for I', J', $b' = x_1x_3x_5$ to get $\operatorname{depth}_S I'/J' \leq 3$.

Lemma 2. *Suppose that $r > 1$ and $\operatorname{depth}_S I/(J, f_r) = d$. Then $\operatorname{depth}_S I/J \leq d + 1$.*

Proof: Let $B \cap (f_r) = \{b_1, \ldots, b_p\}$. As $I/(J, f_r)$ has a squarefree, multigraded free resolution we see that only the components of squarefree degrees of

$$\operatorname{Tor}_S^{n-d}(K, I/(J, f_r)) \cong H_{n-d}(x; I/(J, f_r))$$

are nonzero, the last module being the Koszul homology of $I/(J, f_r)$. Thus we may find

$$z = \sum_{i=1}^{r-1} y_i f_i e_{[n]\setminus \operatorname{supp} f_i} \in K_{n-d}(x; I/(J, f_r)),$$

$y_i \in K$ inducing a nonzero element in $H_{n-d}(x; I/(J, f_r))$. Here we set $\operatorname{supp} f_i = \{t \in [n] : x_t | f_i\}$ and $e_A = \wedge_{j \in A} e_j$ for a subset $A \subset [n]$. We have

$$\partial z = \sum_{b \in B} P_b(y) b e_{[n]\setminus \operatorname{supp} b} = \sum_{i=1}^{p} P_{b_i}(y) b_i e_{[n]\setminus \operatorname{supp} b_i},$$

where P_b are linear homogeneous polynomials in y. Note that $P_b(y) = 0$ for all $b \notin \{b_1, \ldots, b_p\}$. Choose $j \in \operatorname{supp} f_r$ and consider

$$z_j = \sum_{i=1, f_i \notin (x_j)}^{r-1} y_i f_i e_{[n]\setminus(\{x_j\}\cup \operatorname{supp} f_i)} \in K_{n-d-1}(x; I/J).$$

In ∂z_j appear only terms of type ue_A, $j \notin A$ with $|A| = n - d - 1$ and $u = \Pi_{i\in[n]\setminus(A\cup\{j\})} x_i$. Thus terms of type $b_i e_{[n]\setminus \operatorname{supp} b_i}$, $i \in [p]$ are not present in ∂z_j because $b_i \in (x_j)$. It follows that $\partial z_j = 0$ and so z_j is a cycle. Note that a cycle of $K_{n-d-1}(x; I/J)$ could contain also terms of type $ve_{A'}$ with $|A'| = n - d - 1$ and $v = \Pi_{i\in[n]\setminus A'} x_i \in B$, but z_j is just a particular cycle.

Remains to show that we may find j such that z_j is a nonzero cycle. Suppose that $y_m \neq 0$ for $m \in [r-1]$ and choose $j \in \operatorname{supp} f_r \setminus \operatorname{supp} f_m$. It follows that z_j is a nonzero cycle because $y_m f_m e_{[n]\setminus(\{x_j\}\cup \operatorname{supp} f_m)}$ is present in z_j. Thus $\operatorname{depth}_S I/J \leq d + 1$ by [2, Theorem 1.6.17]. ∎

Remark 1. Applying the above lemma to Example 2 we see that $\operatorname{depth}_S I/J \leq 3$ because $\operatorname{depth}_S I/(J, f_5) = 2$.

Let $P_{I\setminus J}$ be the poset of all squarefree monomials of $I \setminus J$ with the order given by the divisibility. Let P be a partition of $P_{I\setminus J}$ in intervals $[u, v] = \{w \in P_{I\setminus J} : u|w, w|v\}$, let us say $P_{I\setminus J} = \cup_i [u_i, v_i]$, the union being disjoint. Define $\operatorname{sdepth} P = \min_i \deg v_i$ and the *Stanley depth* of I/J given by $\operatorname{sdepth}_S I/J = \max_P \operatorname{sdepth} P$, where P runs in the set of all partitions of $P_{I\setminus J}$ (see [4], [13]). Stanley's Conjecture says that $\operatorname{sdepth}_S I/J \geq \operatorname{depth}_S I/J$.

In Example 1 we have $s = 5 < q + 3 = 7$ and so it follows that $\operatorname{depth}_S I/J \leq d + 1$ by [8, Theorem 1.3] (see also [6, Theorem 2]). Next example follows [9, Example 1.6] and has $s = q + r$, $\operatorname{sdepth}_S I/J = d + 2$ but $\operatorname{depth}_S I/J = d$.

Example 3. Let $n = 12$, $r = 11$, $f_1 = x_{12}x_1$, $f_2 = x_{12}x_2$, $f_3 = x_{12}x_3$, $f_4 = x_{12}x_4$, $f_5 = x_{12}x_5$, $f_6 = x_{12}x_6$, $f_7 = x_6x_7$, $f_8 = x_6x_8$, $f_9 = x_6x_9$, $f_{10} = x_6x_{10}$, $f_{11} = x_6x_{11}$, $J = (x_7, \ldots, x_{11})(f_1, \ldots, f_5) + (x_1, \ldots, x_5)(f_7, \ldots, f_{11}) + f_6(x_9, \ldots, x_{11})$, $I = (f_1, \ldots, f_{11})$. We have $B = \{w_{ij} : 1 \leq i < j \leq 5\} \cup \{w_{kt} : 6 < k < t \leq 11\} \cup \{w_{i6} : i \in [8], i \neq 6\}$, that is $s = |B| = 27$. Let $c_1 = x_6w_{12}$, $c_2 = x_6w_{23}$, $c_3 = x_6w_{34}$, $c_4 = x_6w_{45}$, $c_5 = x_6w_{15}$, $c_6 = x_8w_{67}$, $c_7 = x_9w_{78}$, $c_8 = x_{10}w_{89}$, $c_9 = x_{11}w_{9,10}$, $c_{10} = x_7w_{10,11}$, $c_{11} = x_7w_{8,11}$, $c'_{13} = x_4w_{13}$, $c'_{14} = x_5w_{14}$, $c'_{24} = x_6w_{24}$, $c'_{25} = x_3w_{25}$, $c'_{35} = x_6w_{35}$. These are all monomials of C, that is $q = |C| = 16$ and so $s = q + r$. The intervals $[f_i, c_i]$, $i \in [11]$ and $[w_{13}, c'_{13}]$, $[w_{14}, c'_{14}]$, $[w_{24}, c'_{24}]$, $[w_{25}, c'_{25}]$, $[w_{35}, c'_{35}]$ induce a partition P on I/J with sdepth 4.

We claim that $\operatorname{depth}_S S/J = 2$. Indeed, let $J' = (x_7, \ldots, x_{11})(f_1, \ldots, f_5) + (x_1, \ldots, x_5)(f_7, \ldots, f_{11}) = (x_{12}, x_6)(x_7, \ldots, x_{11})(x_1, \ldots, x_5)$. By [5, Theorem 1.4] we get $\operatorname{depth}_S S/J' = 2 = d$. Set $J_1 = J' + (x_{12}x_6x_9)$, $J_2 = J_1 + (x_{12}x_6x_{10})$. We have $J = J_2 + (x_{12}x_6x_{11})$. In the exact sequences

$$0 \to (x_{12}x_6x_9)/(x_{12}x_6x_9) \cap J' \to S/J' \to S/J_1 \to 0,$$

$$0 \to (x_{12}x_6x_{10})/(x_{12}x_6x_{10}) \cap J_1 \to S/J_1 \to S/J_2 \to 0,$$
$$0 \to (x_{12}x_6x_{11})/(x_{12}x_6x_{11}) \cap J_2 \to S/J_2 \to S/J \to 0$$

the first terms have depth ≥ 5. Applying the Depth Lemma by recurrence we get our claim.

Now we see that $\operatorname{depth}_S S/I = 6$. Set $I_j = (f_1, \ldots, f_j)$ for $6 \leq j \leq 11$. We have $I = I_{11}$, $I_6 = x_{12}(x_1, \ldots, x_6)$ and $\operatorname{depth}_S S/I_6 = 6$. In the exact sequences

$$0 \to (f_{j+1})/(f_{j+1}) \cap I_j \to S/J_j \to S/I_{j+1} \to 0,$$

$6 \leq j < 11$ we have $(f_{j+1}) \cap I_j = f_{j+1}(x_{12}, x_7, \ldots, x_j)$ and so $\operatorname{depth}_S(f_{j+1})/(f_{j+1}) \cap I_j = 12 - (j-5) \geq 7$ for $6 \leq j < 11$. Applying the Depth Lemma by recurrence we get $\operatorname{depth}_S S/I_{j+1} = 6$ for $6 \leq j < 11$ which is enough.

Finally using the Depth Lemma in the exact sequence

$$0 \to I/J \to S/J \to S/I \to 0$$

it follows $\operatorname{depth}_S I/J = 2 = d$.

The following lemma is the key in the proof of Theorem 1 and its proof is given in the next section.

Lemma 3. *Suppose that $E = \emptyset$, $C \subset C_3$, $C \cap W = \emptyset$ and Theorem 1 holds for $r' < r$. Then* $\operatorname{depth}_S I/J \leq d + 1$.

Proposition 1. *Suppose that $C \cap (f_1, \ldots, f_r) \subset C_3$, $C \cap W = \emptyset$ and Theorem 1 holds for $r' < r$. Then* $\operatorname{depth}_S I/J \leq d + 1$.

Proof: Suppose that $E \neq \emptyset$, otherwise apply Lemma 3. Set $I' = (f_1, \ldots, f_r)$, $J' = J \cap I'$. In the exact sequence

$$0 \to I'/J' \to I/J \to I/(I', J) \to 0$$

the last term is isomorphic to something generated by E and so its depth is $\geq d+1$. The first term satisfies the conditions of Lemma 3 which gives $\operatorname{depth}_S I'/J' \leq d + 1$. By the Depth Lemma we get $\operatorname{depth}_S I/J \leq d + 1$ too. □

Proof of Theorem 1

Apply induction on r. If $r < 5$ then $B \cap (f_1, \ldots, f_r) \subset W$ implies $|B \cap (f_1, \ldots, f_r)| < 2r$ and so $\operatorname{sdepth}_S I/J \leq d + 1$ and even $\operatorname{depth}_S I/J \leq d + 1$ by [12, Proposition 2.4] (we may also apply [9, Theorem 0.3]). Suppose that $r \geq 5$. Since all divisors of a monomial $c \in C \cap (f_1, \ldots, f_r)$ of degrees $d + 1$ are in B, they are also in W by our hypothesis. Thus $C \cap (f_1, \ldots, f_r) \subset C_3$ and we may apply Proposition 1 under induction hypothesis. □

Corollary 1. *Suppose that $B \cap (f_1, \ldots, f_r) \subset W$ and $C \cap W = \emptyset$. Then* $\operatorname{depth}_S I/J \leq \operatorname{sdepth}_S I/J$, *that is the Stanley Conjecture holds for I/J.*

Proof: If $\operatorname{sdepth}_S I/J = d$ then apply [7, Theorem 4.3], otherwise apply Theorem 1. □

2 Proof of Lemma 3.

We may suppose that $B \subset W$ because each monomial of B must divide a monomial of C, otherwise we get $\operatorname{depth}_S I/J \leq d+1$ by [10, Lemma 1.5]. Then we may suppose that $B \subset \cup_i \operatorname{supp} f_i$ and we may reduce to the case when $[n] = \cup_i \operatorname{supp} f_i$ because then $\operatorname{depth}_S I/J = \operatorname{depth}_{\tilde{S}}(I \cap \tilde{S})/(J \cap \tilde{S})$ for $\tilde{S} = K[\{x_t : t \in \cup_i \operatorname{supp} f_i\}]$.

On the other hand, we may suppose that for each $i \in [r]$ there exists $c \in C$ such that $f_i | c$, otherwise we may apply again [10, Lemma 1.5]. Since $c \in C_3$, let us say c is the least common multiple of f_1, f_2, f_3 we see that at least,let us say, $w_{12} \in B$. Then $f_i \in (u_1)$, $i \in [2]$ for some monomial $u_1 = (f_1 f_2)/w_{12}$ of degree $d-1$.

We may assume that $f_i \in (u_1)$ if and only if $i \in [k_1]$ for some $2 \leq k_1 \leq r$. Set $U_1 = \{f_1, \ldots, f_{k_1}\}$. We also assume that

$$\{u_i : i \in [e]\} = \{u : u = gcd(f_i, f_j), \deg u = d-1, i \neq j \in [r]\},$$

and define

$$U_i = \{f_j : f_j \in (u_i), j \in [r]\}$$

for each $i \in [e]$. Since each $f_t \in U_i$ divides a certain $c \in C$ we see from our construction that there exist $f_p, f_l \in U_i$ such that $w_{tp}, w_{tl} \in B$. Note that if $|U_i \cap U_j| \geq 2$ then we get $u_i = u_j$ and so $i = j$. Thus $|U_i \cap U_j| \leq 1$ for all $i, j \in [e]$, $i \neq j$.

Suppose that $w_{ij} \in J$ for all $i \in [k_1]$ and for all $j > k_1$, let us say $f_i = u_1 x_i$ for $i \in [k_1]$. Set $I' = (f_1, \ldots, f_{k_1})$, $J' = I' \cap J$ and $\hat{S} = K[\{x_i : i \in [k_1] \cup \operatorname{supp} u_1\}]$. Then $\operatorname{depth}_S I'/J' = \operatorname{depth}_{\hat{S}} I' \cap \hat{S}/J' \cap \hat{S} = \deg u_1 + \operatorname{depth}_{\hat{S}}(x_1, \ldots, x_{k_1})\hat{S}/(J' : u_1) \cap \hat{S} = d$. Thus $\operatorname{depth}_S I/J = d$ by the Depth Lemma applied to the exact sequence

$$0 \to I'/J' \to I/J \to I/(I', J) \to 0,$$

since the last term has depth $\geq d$ being generated by squarefree monomials of degree $\geq d$. In particular, $\operatorname{depth}_S I/J = d$ if $e = 1$. If $e > 1$ we may assume that for each $i \in [e]$ there exists $j \in [e]$ with $U_i \cap U_j \neq \emptyset$.

Example 4. Back to Example 3 note that we may take $u_1 = x_{12}$, $u_2 = x_6$ and $U_1 = \{f_1, \ldots, f_6\}$, $U_2 = \{f_6, \ldots, f_{11}\}$.

Lemma 4. *Suppose that $e \geq 2$ and $f_r \in U_e$. Let I' be the ideal generated by all $f_k \in U_e \setminus \{f_r\}$. If $\operatorname{depth}_S I/(J, I', f_r) \geq d+1$ and there exists t with $f_t \in (\cup_{i=1}^{e-1} U_i) \setminus U_e$ such that $w_{rt} \in B \setminus I'$ then $\operatorname{depth}_S I/(J, I') = d+1$.*

Proof: Using the Depth Lemma applied to the exact sequence

$$0 \to (f_r)/(f_r) \cap (J, I') \to I/(J, I') \to I/(J, I', f_r) \to 0$$

we see that it is enough to show that $\operatorname{depth}_S (f_r)/(f_r) \cap (J, I') = d+1$. By our hypothesis the squarefree monomials from $(f_r) \setminus (J, I')$ have the form $w_{rt'}$ for some t' with $f_{t'} \in (\cup_{i=1}^{e-1} U_i) \setminus U_e$ and $w_{rt'} \in B \setminus I'$.

Next we will describe the above set of monomials. If there exists no U_l containing f_t, $f_{t'}$ then their contributions to $(f_r)/(f_r) \cap (J, I')$ consist in two different monomials w_{rt}, $w_{rt'}$. Otherwise, we must have $f_r = x_k x_p v$, $f_t = x_k x_m v$ and $f_{t'} = x_p x_m v$ for some different $k, m, p \in [n]$ and one monomial v of degree $d-2$. Thus $w_{rt} = w_{rt'}$ and the contributions of f_t, $f_{t'}$ consist in just one monomial. Let A be the set of all $f_k \in (\cup_{i=1}^{e-1} U_i) \setminus U_e$ such that $w_{rk} \in B \setminus I'$ and define an equivalence relation on A by $f_t \sim f_{t'}$ if $f_t, f_{t'} \in U_i$ for some $i \in [e-1]$. For some f_t from an equivalence class of $A/\sim$ we have $w_{rt} = x_{\gamma_t} f_r$ for one $\gamma_t \in [n]$. Let Γ be the set of all these variables x_{γ_t} for which $w_{rt} \notin (J, I')$. For two x_{γ_t}, $x_{\gamma_{t'}}$ corresponding to different classes we have $w_{tt'} = x_{\gamma_t} x_{\gamma_{t'}} f_r$ since f_t, $f_{t'}$ are not in the same equivalence class. Thus $w_{tt'} \in J$ because otherwise $w_{tt'} \in C$ which is impossible by our hypothesis. Let $Q \subset K[\Gamma]$ be the ideal generated by all squarefree quadratic monomials. The multiplication by f_r gives a bijection between $K[\Gamma]/Q$ and $(f_r)/(f_r) \cap (J, I')$ because each squarefree monomial of $(B \cap (f_r)) \setminus I'$ has the form $w_{rt} = f_r x_{\gamma_t}$ for some t, x_{γ_t} being in Γ. Then $\mathrm{depth}_S (f_r)/(f_r) \cap (J, I') = d + \mathrm{depth}_{K[\Gamma]} K[\Gamma]/Q = d + 1$ since the variables of f_r form a regular sequence for $f_r)/(f_r) \cap (J, I')$ (in the squarefree frame). Note that if $A/\sim$ has just one class of equivalence containing some f_t with $w_{tr} \in B$ then $|\Gamma| = 1$, $Q = 0$ and also it holds $\mathrm{depth}_{K[\Gamma]} K[\Gamma]/Q = 1$. □

Remark 2. In the notations of the above lemma suppose that $w_{rt} \in (J, I')$ for all t with $f_t \in (\cup_{i=1}^{e-1} U_i) \setminus U_e$. Then there exists no squarefree monomial of degree $d+1$ in $(f_r) \setminus (J, I')$ and so $\mathrm{depth}_S (f_r)/(f_r) \cap (J, I') = d$. It follows that $\mathrm{depth}_S I/(J, I') = d$ too.

Lemma 5. *Suppose that* $e \geq 2$. *If* $\mathrm{depth}_S I/(J, (U_e)) \leq d+1$ *then* $\mathrm{depth}_S I/J \leq d+1$.

Proof: Suppose that $U_e \supset \{f_{k+1}, \ldots, f_r\}$ for some $k \leq r$. Let I'_k be the ideal generated by all $f_t \in U_e \setminus \{f_{k+1}, \ldots, f_r\}$. We claim that $\mathrm{depth}_S I/(J, I'_k) \leq d+1$. Apply induction on $r-k$, the case $k = r-1$ being done in Lemma 4 and Remark 2. Assume that $r-k > 1$ and note that $I'_{k+1} = (I'_k, f_{k+1})$. By induction hypothesis we have $\mathrm{depth}_S I/(J, I'_{k+1}) \leq d+1$. If $\mathrm{depth}_S I/(J, I'_{k+1}) = d$ then we get $\mathrm{depth}_S I/(J, I'_k) \leq d+1$ by Lemma 2. If $\mathrm{depth}_S I/(J, I'_{k+1}) = d+1$ we get again $\mathrm{depth}_S I/(J, I'_k) \leq d+1$ by Lemma 4 and Remark 2. This proves our claim.

Now choose $r-k$ maxim, let us say $U_e = \{f_{k+1}, \ldots, f_r\}$. Then $I'_k = 0$ and so we get $\mathrm{depth}_S I/J \leq d+1$. □

Example 5. Let $n = 6$, $r = 8$, $d = 2$, $f_1 = x_1x_2$, $f_2 = x_1x_3$, $f_3 = x_1x_4$, $f_4 = x_2x_3$, $f_5 = x_3x_5$, $f_6 = x_2x_6$, $f_7 = x_3x_6$, $f_8 = x_4x_6$, and $I = (f_1, \ldots, f_8)$,

$$J = (x_1x_2x_5, x_1x_3x_6, x_1x_4x_5, x_1x_4x_6, x_2x_3x_4, x_2x_5x_6, x_3x_4x_5, x_3x_5x_6, x_4x_5x_6).$$

It follows that $U_1 = \{f_1, \ldots, f_3\}$, $U_2 = \{f_2, f_4, f_5, f_7\}$, $U_3 = \{f_6, f_7, f_8\}$, $U_4 = \{f_1, f_4, f_6\}$, $U_5 = \{f_3, f_8\}$, $u_1 = x_1$, $u_2 = x_3$, $u_3 = x_6$, $u_4 = x_2$, $u_5 = x_4$. Let $\tilde{S} = K[x_1, \ldots, x_5]$, $\tilde{J} \subset \tilde{I} \subset \tilde{S}$ be the corresponding ideals given in Example 1. For $I' = (U_3)$ we have $I/(J, I') \cong \tilde{I}S/(\tilde{J}, x_6\tilde{I})S$. Thus $\operatorname{depth}_S I/(J, I') = \operatorname{depth}_{\tilde{S}} \tilde{I}/\tilde{J} \leq 3 = d+1$ by Example 1 and so using Lemma 5 we get $\operatorname{depth}_S I/J \leq 3$ too.

Proposition 2. *If* $\cap_{i\in[e]} U_i \neq \emptyset$ *then* $\operatorname{depth}_S I/J \leq d + 1$.

Proof: As we have seen $\operatorname{depth}_S I/J = d$ if $e = 1$. Assume that $e > 1$ and let us say $f_r \in \cap_{i\in[e]} U_i$. Set $I' = (U_e)$. In $I/(J, I')$ we have $(e-1)$ disjoint $U_i' = U_i \setminus \{f_r\}$, $i \in [e-1]$. It follows that $\operatorname{depth}_S I/(J, I') = d$ and so $\operatorname{depth}_S I/J \leq d + 1$ by Lemma 5. $\square$

Proof of Lemma 3.

If $\cap_{i\in[e]} U_i \neq \emptyset$ then apply the above proposition. Otherwise, suppose that $f_r \in U_j$ if and only if $1 \leq j < k$ for some $1 < k \leq e$. Set $I' = (U_k, \ldots, U_e)$. Applying again the above proposition we get $\operatorname{depth}_S I/(J, I') \leq d + 1$. Set $L_i = \cup_{j\geq i} U_j$. Since $I' = (L_k)$ and $\operatorname{depth}_S I/(J, L_k) \leq d + 1$ we see that $\operatorname{depth}_S I/(J, L_{k+1}) \leq d + 1$ by Lemma 5. Using by recurrence Lemma 5 we get $\operatorname{depth}_S I/J = \operatorname{depth}_S I/(J, L_{e+1}) \leq d + 1$ since $L_{e+1} = \emptyset$. $\square$

It is not necessary to assume in Proposition 2 that $C \cap W = \emptyset$ because anyway this follows as shows the following lemma.

Lemma 6. *If* $\cap_{i\in[e]} U_i \neq \emptyset$ *then* $C \cap W = \emptyset$.

Proof: Clearly we may suppose that $e > 1$. Let $f_r \in \cap_{i\in[e]} U_i$. Suppose that $w_{tt'} \in C$ for some $t \in U_i$, $t' \in U_j$. Then $i \neq j$ and let us say $f_t = u_i x_1$, $f_{t'} = u_j x_2$ and $f_r = u_i x_3 = u_j x_4$. It follows that $u_i = x_4 v$, $u_j = x_3 v$ for some monomial v and so $f_t = x_1 x_4 v$, $f_{t'} = x_2 x_3 v$, $f_r = v x_3 x_4$. Note that $f_t x_2 \in B$ and so must be of type $w_{tt''}$ for some $t'' \in [r]$. It follows that $f_t, f_{t''} \in U_k$ for some $k \in [e]$. By hypothesis $f_r \in U_k$ and so $k = i$ because otherwise $|U_k \cap U_i| > 1$ which is impossible. Since $f_t x_2 \in B$ we see that $x_2 \notin \operatorname{supp} u_i$ and it follows that $f_{t''} = x_2 x_4 v$. Therefore, $w_{t't''} = x_2 x_3 x_4 v \in B$ and so $f_{t'}, f_{t''} \in U_p$ for some p. This is not possible because otherwise one of $|U_p \cap U_i|$, $|U_p \cap U_j|$, $|U_i \cap U_j|$ is ≥ 2. Contradiction! $\square$

Added in Proof: Meanwhile, an example appeared in a paper of Duval et al. (A non-partitionable Cohen-Macaulay simplicial complex), arXiv 1504.04279, which shows that the Stanley Conjecture fails. However, our Question 1 is still open.

References

[1] A. Aslam, V. Ene, *Simplicial complexes with rigid depth*, Arch. Math. **99** (2012), 315-325.

[2] W. Bruns, J. Herzog, *Cohen-Macaulay rings*, Revised edition, Cambridge University Press, (1998).

[3] W. Decker, G.-M. Greuel, G. Pfister, H. Schönemann, *Singular 3-1-6, A computer algebra system for polynomial computations*, http://www.singular.uni-kl.de (2012).

[4] J. Herzog, M. Vladoiu, X. Zheng, *How to compute the Stanley depth of a monomial ideal,* J. Algebra, **322** (2009), 3151-3169.

[5] A. Popescu, *Special Stanley Decompositions*, Bull. Math. Soc. Sc. Math. Roumanie, **53(101)**, no 4 (2010), 363-372, arXiv:AC/1008.3680.

[6] A. Popescu, D. Popescu, *Four generated, squarefree, monomial ideals* , in "Bridging Algebra, Geometry, and Topology", Editors Denis Ibadula, Willem Veys, Springer Proceed. in Math., and Statistics, **96**, 2014, 231-248, arXiv:AC/1309.4986v5.

[7] D. Popescu, *Depth of factors of square free monomial ideals*, Proceedings of AMS **142** (2014), 1965-1972,arXiv:AC/1110.1963.

[8] D. Popescu, *Upper bounds of depth of monomial ideals*, J. Commutative Algebra, **5**, 2013, 323-327, arXiv:AC/1206.3977.

[9] D. Popescu, *Stanley depth on five generated, squarefree, monomial ideals*, Bull. Math. Soc. Sci. Math. Roumanie, **59(107)**, (2016), no 1, 75-99, arXiv:AC/1312.0923v5.

[10] D. Popescu, A. Zarojanu, *Depth of some square free monomial ideals*, Bull. Math. Soc. Sci. Math. Roumanie, **56(104)**, 2013,117-124.

[11] D. Popescu, A. Zarojanu, *Three generated, squarefree, monomial ideals*, Bull. Math. Soc. Sci. Math. Roumanie, **58(106)**, (2015), no 3, 359-368, arXiv:AC/1307.8292v6.

[12] Y.H. Shen, *Lexsegment ideals of Hilbert depth 1*, (2012), arxiv:AC/1208.1822v1.

[13] R. P. Stanley, *Linear Diophantine equations and local cohomology*, Invent. Math. **68** (1982) 175-193.

Received: 20.01.2015
Accepted: 28.05.2015

Simion Stoilow
Institute of Mathematics of Romanian Academy,
Research unit 5,
P.O.Box 1-764,
Bucharest 014700, Romania.
E-mail: `dorin.popescu@imar.ro`