Bull. Math. Soc. Sci. Math. Roumanie
Tome 59(107) No. 2, 2016, 151–158

A Note On Smale Manifolds and Lorentzian Sasaki-Einstein Geometry

by
Ralph R. Gomez*

Abstract

In this note, we construct new examples of Lorentzian Sasaki-Einstein (LSE) metrics on Smale manifolds M. It has already been established by the author that such metrics exist on the so-called torsion free Smale manifolds, i.e. the k-fold connected sum of $S^2 \times S^3$ for all k. Now, we show that LSE metrics exist on Smale manifolds for which $H_2(M,\mathbb{Z})_{tor}$ is nontrivial. In particular, we show that most simply-connected positive Sasakian rational homology 5-spheres are also negative Sasakian (hence LSE). Moreover, we show that for each pair of positive integers (n,s) with $n,s>1$, there exists a Lorentzian Sasaki-Einstein Smale manifold M such that $H_2(M,\mathbb{Z})_{tors} = (\mathbb{Z}/n)^{2s}$. Finally, we are able to construct so-called mixed Smale manifolds (connect sum of torsion free Smale manifolds with rational homology spheres) which admit LSE metrics and have arbitrary second Betti number. This gives infinitely many examples which do not admit positive Sasakian structures. These results partially address open problems formulated by C.Boyer and K.Galicki.

Key Words: Sasakian manifolds, Lorentzian-Einstien metrics, rational homology spheres.
2010 Mathematics Subject Classification: Primary 53C25; Secondary 53D10, 53C50.

1 Introduction

Let M be a compact, simply-connected, 5-manifold with $w_2(M)=0$. By a theorem of Smale [15], $H_2(M,\mathbb{Z})$ uniquely determines M. Following [5], let us call such manifolds M *Smale* manifolds and divide them up into three classes: *i.*) torsion free Smale manifolds with positive second Betti number, that is k-fold

*During the preparation of some of this work, the author was support by James A. Michener Faculty Fellowship at Swarthmore College.

connected sums of $S^2 \times S^3$ for some k *ii.)* rational homology spheres, including S^5 and *iii.)* mixed Smale manifolds, which are the connected sums of torsion free Smale manifolds with rational homology spheres. There has been a flurry of work (see [4] for sweeping survey) dedicated to determining which Smale manifolds admit so-called quasi-regular positive Sasakian structures. A quasi-regular positive (negative) Sasakian manifold can be viewed as a principal circle orbibundle over a complex algebraic orbifold $(X, \Delta = \sum(1 - \frac{1}{m_i})D_i)$ such that the orbifold first Chern class is positive (negative). In [8], it was shown that positive Sasakian structures exist on an arbitrary connected sum of torsion free Smale manifolds. Then in [12] Kollár established a classification theorem for simply-connected rational homology 5-spheres admitting positive Sasakian structures. There are partial results for the mixed Smale manifolds. See [4] for a more complete survey and [5] for more recent results.

Negative Sasakian geometry on Smale manifolds is not as developed. In [11], it was shown that negative Sasakian structures exist on the k-fold connected sum of $S^2 \times S^3$ for all k. It was shown in [10] that the simply-connected rational homology 5-sphere M_k admits both positive and negative Sasakian structures for $k \geq 5$. Until now, negative Sasakian structures on other simply-connected positive Sasakian rational homology 5-spheres have not been constructed. It turns out that a negative Sasakian structure can give rise to negative Sasaki η-Einstein as well as Lorentzian Sasaki-Einstein metrics. Thus, negative Sasakian geometry is a strong tool in constructing such manifolds with these metric properties. In fact, Boyer and Galicki formulated the following open problems on negative Sasakian geometry in dimension five on pages $359 - 360$ in [4]:

a.) Determine which simply-connected rational homology 5-spheres admit negative Sasakian structures.
b.) Determine which torsion groups [torsion subgroups in $H_2(M, \mathbb{Z})$] *correspond to Smale-Barden manifolds admitting negative Sasakian structures.*

The more general Smale-Barden five-manifolds correspond to the non-spin case and that will not be addressed in this paper. The principle aim of this note is to make a significant first step towards solutions to the above problems. The main theorem is:

Theorem 1
(1.) *The following simply-connected positive Sasakian rational homology* 5*-spheres admit a negative Sasakian structure:*

$$M_m, \quad 2M_3, \quad 3M_3, \quad 4M_3, \quad 2M_4, \quad 2M_5$$

where m is a positive integer with $m \geq 5$ and $m \neq 30j$ for some positive integer j. Consequently, these manifolds admit both positive and negative Sasakian structures.

(2.) *For every pair of positive integers (n, s) with $n, s > 1$ there exists a Smale* *manifold M which admits a negative Sasakian structure such that $H_2(M,\mathbb{Z})_{tors} = (\mathbb{Z}/n)^{2s}$.*

(3.) *There exists mixed Smale manifolds M which are negative Sasakian and have arbitrary second Betti number but do not admit any positive Sasakian structure. Moreover, all the manifolds listed in (1.) - (3.) can be realized as a links of an isolated hypersurface singularity at the origin and admit negative Saskaki η-Einstein, hence Lorentzian Sasaki-Einstein metrics.*

The only manifolds from the classification list of positive Sasakian simply connected rational homology 5-spheres missing in the above theorem is nM_2 and M_m where $m = 2, 3, 4$. It is interesting to note that generally, it is not known for which n the manifolds nM_2 are Sasaki-Einstein of positive scalar curvature (See [5] for some recent known values of n.)

It is worthwhile to contrast the second part of the above theorem with 5-dimensional simply-connected positive Sasakian geometry. Let M be a positive quasi-regular Sasakian manifold which is simply connected. Then the torsion group in $H_2(M,\mathbb{Z})$ is constrained to be one of the following [12]:

$$(\mathbb{Z}/m)^2, (\mathbb{Z}/5)^4, (\mathbb{Z}/4)^4, (\mathbb{Z}/3)^4, (\mathbb{Z}/3)^6, (\mathbb{Z}/3)^8, (\mathbb{Z}/2)^{2n} \tag{1.1}$$

for any $n, m \in \mathbb{Z}^+$.

Continuing the comparison with positive Sasaki-Einstein geometry of Smale manifolds, there are no examples of Sasaki-Einstein structures on mixed Smale manifolds with second Betti number larger than nine. The third part of our theorem illustrates that is far from the case in Lorentzian Sasaki-Einstein geometry.

The organization of the note is as follows. The second sections recalls some basic definitions of hypersurfaces in weighted projective spaces and Sasakian structures on links. Finally, the third section gives the proof of Theorem 1.

2 Preliminaries

The plan of the proof in Theorem 1, given in the next section, uses the Boyer-Galicki method in producing examples of Sasakian manifolds. The idea is as follows: First, construct a particular hypersurface in weighted projective space which possesses a certain branch divisor. Then determine the diffeomorphism type of the manifold that arises as the total space of the principal circle orbibundle over this hypersurface by using theorems of Smale [15] and Kollár [12]. In this section, we collect some useful definitions needed for the proof in the next section.

The weighted $\mathbb{C}^*$-action, denoted by $\mathbb{C}(\mathbf{w})$, on $\mathbb{C}^{n+1}$ is given by

$$(z_0, \dots, z_n) \mapsto (\lambda^{w_0} z_0, \dots, \lambda^{w_n} z_n)$$

where $w = (w_0, \dots, w_n)$ is a sequence of positive integers, $\lambda \in \mathbb{C}^*$ and $gcd(w_0, \dots, w_n) = 1$. So we have

Definition 1. *The quotient space* $(\mathbb{C}^{n+1}\backslash\{\mathbf{0}\})/\mathbb{C}^*(\mathbf{w})$ *is defined as weighted projective space* $\mathbb{P}(\mathbf{w})$.

We are interested in hypersurfaces in weighted projective space defined by zero sets of particular polynomials.

Definition 2. *Let* f *be a polynomial in* $\mathbb{C}[z_0, \ldots, z_n]$. *The polynomial is said to be weighted homogeneous of degree* d *if for any* $\lambda \in \mathbb{C}^*$ *we have*

$$f(\lambda^{w_0} z_0, ..., \lambda^{w_n} z_n) = \lambda^d f(z_0, ..., z_n).$$

We say $\mathcal{Z}_f$ *is a weighted hypersruface of* $\mathbb{P}(\mathbf{w})$ *if the hypersurface* $\mathcal{Z}_f$ *is the zero locus of the weighted homogenous polynomial* f.

The hypersurfaces $\mathcal{Z}_f$ in weighted projective space in this article are all Kähler orbifolds. These orbifolds will have branch divisors. Recall, a branch divisor $\triangle$ is a $\mathbb{Q}$-divisor of the form $\triangle = \sum_i (1 - \frac{1}{m_i}) D_i$ where D_i are Weil divisors and m_i is the ramification index, which is the gcd of the orders of the local uniformizing groups of the orbifold $\mathcal{Z}_f$ at all points D_i. We will denote the orbifold together with this branch divisor as $(\mathcal{Z}_f, \triangle)$.

We will use a well-known theorem of Boyer and Galicki (see for example [1]) which asserts that one can construct a principle circle orbibundle over a weighted hypersurface $\mathcal{Z}_f$ and this total space is a link of an isolated hypersurface singularity.

Definition 3. *[13] The link* L_f *of an isolated hypersurface singularity at the origin defined by the weighted homogenous polynomial of degree* d *is the smooth* $n-2$ *connected manifold of dimension* $2n-1$ *given by* $L_f = C_f \cap S^{2n+1}$ *where* C_f *is the weighted affine cone and* S^{2n+1} *is viewed as the unit sphere in* $\mathbb{C}^{n+1}$.

A link of an isolated hypersurface singularity admits a Sasakian structure (see [1] or [4]). Let us review the definition of a Sasakian manifold. A *Sasakian* manifold is a smooth Riemannian manifold M^{2n+1} with some structure tensors $\mathcal{S} = (\xi, \eta, \Phi, g)$ that makes M^{2n+1} into a *normal contact metric structure*. (We may sometimes refer to $\mathcal{S}$ as a Sasakian structure on the manifold.) The Reeb vector field ξ is a Killing vector field. The Reeb vector field foliates M by one-dimensional leaves. The 1-form η is a contact form and Φ is an endomorphism of the tangent bundle such that Φ restricted to $\ker\eta = \mathcal{D}$ is an integrable almost complex structure. Moreover, if the leaves are compact, we say the Sasakian manifold is *quasi-regular*. Furthermore, the Sasakian manifold is called Sasaki η-Einstein if the metric g satisfies

$$Ric_g = \lambda g + \nu \eta \otimes \eta,$$

where $\lambda + \nu = 2n$ and we assume dim $M > 3$. If $Ric_g = \lambda g$ for a constant λ, we say the Sasakian manifold is Sasaki-Einstein. Finally, if the metric g is Einstein and indefinite, we say the Sasakian structure is Lorentzian Sasaki-Einstein.

The last definition we need is a notion of negativity. This will allow us to use theorems which ensure the constructed links L_f are Lorentzian Sasaki-Einstein.

Definition 4. *A link L_f of degree d is a negative Sasakian manifold if $d - \sum_i w_i > 0$.*

3 Proof of Theorem 1

To prove Theorem 1, the problem is reduced to constructing particular weighted hypersurfaces which possess certain branch divisors. These branch divisors will influence the type of topology on the links over the hypersurfaces.
Consider the following hypersurface in weighted projective space

$$\mathcal{Z}_f \subset \mathbb{P}(2qp, 2\alpha p, 2\alpha q, \alpha(p-1))$$

where the hypersurface $\mathcal{Z}_f$ is defined as the zero locus of the weighted homogenous polynomial of degree $d = 2qp\alpha$ (assume q,p,α are all distinct)

$$f = z_0^\alpha + z_1^q + z_2^p + z_2 z_3^{2q}.$$

(These polynomials are of Type II in the Yau-Yu list [16].) Furthermore, let us assume $gcd(q, 2\alpha p(p-1)) = 1$ and $(p, \alpha) = 1$. With these assumptions together with the assumption that the gcd of all the weights is one, it follows that α, q are odd and p is even. Therefore, we have $gcd(\alpha, 2qp) = 1$ and so the link of L_f is a simply-connected rational homology 5-sphere by Proposition 2.1 of [3] (or see Theorem 9.3.17 in [4]). There are two curves in the branch divisor: C_0 of degree $2qp$ corresponding to $z_0 = 0$ and C_1 of degree $qp\alpha$ corresponding to $z_3 = 0$. More precisely, $C_0 = z_1^q + z_2^p + z_2 z_3^{2q}$ and $C_1 = z_0^\alpha + z_1^q + z_2^p$ with ramification indices α and 2 respectively. Hence, on the hypersurface $\mathcal{Z}_f$, we have the branch divisor $\Delta = \frac{\alpha-1}{\alpha} C_0 + \frac{1}{2} C_1$. A negative Sasakian structure exists on L_f as long as

$$2qp(\alpha-1) + \alpha(1 - 3p - 2q) > 0. \tag{3.1}$$

To determine the topology of L_f, we can apply theorem 50 in [12]. In particular, we must compute the genus of the curves C_0 and C_1. Recall the genus formula (see for example [9]) which we shall use

$$2g(C) = \frac{d^2}{w_0 w_1 w_2} - d \sum_{i<j} \frac{gcd(w_i, w_j)}{w_i w_j} + \sum_i \frac{gcd(d, w_i)}{w_i} - 1. \tag{3.2}$$

where C is a curve of degree in d in the weighted projective space $\mathbb{P}(w_0, w_1, w_2)$ By applying this formula (and using the gcd conditions defined above) to $C_0 \subset \mathbb{P}(2p, 2q, p-1)$ and $C_1 \subset \mathbb{P}(qp, \alpha p, \alpha q)$, we find

$$2g(C_0) = \frac{(2qp)^2}{(2p)(p-1)(2q)} - 2qp\left(\frac{2}{4qp} + \frac{1}{2p(p-1)} + \frac{1}{2q(p-1)}\right) + \left(\frac{1}{p-1} + 1\right) = q - 1.$$

Similarly, we can calculate the genus of $C_1 \subset \mathbb{P}(qp, \alpha p, \alpha q)$ and here we find

$$2g(C_1) = 1 - qp\alpha\left(\frac{p}{q\alpha p^2} + \frac{\alpha}{\alpha^2 pq} + \frac{q}{q^2 \alpha p}\right) + 2 = 0.$$

This shows that C_1 is a rational curve which means that this curve will not contribute to torsion in $H_2(L_f,\mathbb{Z})$, by Theorem 5.7 in [12]. So we have that L_f is a negative Sasakian rational homology 5 sphere and by Smale's Theorem A in [15], we have L_f is diffeomorphic to $\frac{q-1}{2}M_\alpha$ and hence $H_2(L_f,\mathbb{Z}) = (\mathbb{Z}/\alpha)^{q-1}$. Note that q is assumed to be odd.

By choosing the appropriate weights, we can obtain some of the positive rational homology spheres in part 1 of Theorem 1 which, due to our construction now admit negative Sasakian structures as well. We can get $2M_3$ by letting $q = 5$, $\alpha = 3$ and $p = 4$. Let $q = 7$, $\alpha = 3$ and $p = 4$, getting the rational homology sphere with torsion $(\mathbb{Z}/3)^6$. It should be noted that particular choices of p will exhibit other negative Sasakian structures on a given rational homology sphere. Again using the Type II polynomials as described in [16], we can obtain the other rational homology spheres in the list. Since the computation is nearly identical to the above, the remaining cases are the first three entries in Table 1 at the end of this section. The condition that $m \neq 30j$ ensures that M_m is positive Sasakian, as established in [12].

For the second part of Theorem 1, we study the following orbifold in weighted projective space cut out by the Brieskorn-Pham polynomial equation $f = z_0^2 + z_1^{2l} + z_2^l + z_3^{2nl} = 0$ of degree $d = 2nl$ in the weighted projective space $\mathbb{P}(nl, n, 2n, 1)$. The negative Sasakian condition is $n(l-3) > 1$ so let $l \geq 4, n \geq 2$. Note that we have a branch divisor for $z_3 = 0$ with ramification index n so this generates the curve $C = z_0^2 + z_1^{2l} + z_2^l \subset \mathbb{P}(nl, n, 2n) = \mathbb{P}(l, 1, 2)$ where this curve has degree $d = 2l$. As before, we compute the genus of this curve $H_2(L_f,\mathbb{Z})$ obtaining:

$$2g(C_f) = l - gcd(l, 2).$$

Since l is an integer greater than or equal to 4, part two of the theorem is established.

To establish the third part of the theorem, we need to construct a link for which the second Betti number is arbitrary. To this end, we construct the orbifold hypersurface $f = z_0^{k+1} + z_1^{k+1} + z_2^{k+1} + z_0 z_3^n$ in the weighted projective space $\mathbb{P}(n, n, n, k)$ of degree $n(k+1)$. This was investigated in an unpublished part of the author's thesis [10]. Assume that $gcd(n, k) = 1$. To calculate the second Betti number of the link, we use the formula devised by Milnor and Orlik [14]. Recall that this involves computing

$$divisor\Delta(t) = \prod_i \big(\frac{\Lambda_{u_i}}{v_i} - 1\big) = 1 + \sum a_j \Lambda_j$$

using the relation $\Lambda_a \Lambda_b = gcd(a, b)\Lambda_{lcm(a,b)}$. The symbol $\frac{u_i}{v_i}$ is the irreducible representation of $\frac{d}{w_i}$, where d is the degree of the polynomial and w_i are the weights. The formula for the second Betti number of the link is

$$b_2(L_f) = 1 + \sum_i a_i.$$

Applying this procedure to our example, we obtain

$$divisor\Delta(t) = \big(\Lambda_{k+1}-1\big)^3\big(\frac{1}{k}\Lambda_{n(k+1)}-1\big) = k^2\Lambda_{n(k+1)}-(k^2-k+1)\Lambda_{k+1}+1 \quad (3.3)$$

and therefore $b_2(L_f) = k$. The negative Sasakian condition is easily seen to be $n(k-2) > 0$.
Of course, there is a branch divisor for $z_3 = 0$, with ramification index n, corresponding to $C = z_0^{k+1} + z_1^{k+1} + z_2^{k+1} \subset \mathbb{P}(1,1,1) = \mathbb{P}^2$ so using the usual genus formula for curves, we find $2g(C) = k(k-1)$. Then we compute $H_2(L_f,\mathbb{Z})$ as above and we find, then, that

$$H_2(L_f,\mathbb{Z}) = \mathbb{Z}^k \oplus (\mathbb{Z}/n)^{k(k-1)}.$$

Hence, for each positive integer $k \geq 3$ we may choose an n such that the resulting torsion subgroup of $H_2(L_f,\mathbb{Z})$ is not on the list in (1.1).

Now, since all of our examples are negative Sasakian links, by Theorem 17 and Corollary 24 in [6] they are negative Sasaki η-Einstein as well as Lorentzian Sasaki-Einstein. The fact that our examples are spin follows from Corollary 11.8.5 of [4]. This concludes the proof. □

Table 1 Negative Sasakian rational homology 5-Spheres

$\mathbf{w} = (w_0, w_1, w_2, w_3)$	Link L_f	degree	manifold
$(15, 12, 4, 28)$	$z_0^4 + z_1^5 + z_2^{15} + z_2z_3^2$	60	$2M_4$
$(42, 35, 15, 65)$	$z_0^5 + z_1^6 + z_2^{14} + z_2z_3^3$	210	$2M_5$
$(68, 51, 6, 33)$	$z_0^3 + z_1^4 + z_2^{34} + z_2z_3^6$	204	$4M_3$
$(p, m, m((p+1)/4), m((p-1)/2)$	$z_0^m + z_1^p + z_2^2z_3 + z_3^2z_1$	mp	$M_m, m > 4$

The entry in the very bottom row was used in [2, 3] to establish Sasaki metrics of positive Ricci curvature on certain simply-connected rational homology 5-spheres and it was also realized in the author's thesis [10] that one could use those links to obtain the existence of negative Sasakian structures on certain rational homology spheres. An assumption is needed on m, p to make things work out and that is for each m choose a prime p of the form $p = 4l - 1$ such that $(m, p) = 1$. The negativity condition is easily seen to be $(m-4)(l-1) > 3$.

Remark 1. *In positive Sasaki-Einstein geometry of Smale manifolds, an open problem is the following [4]: Suppose $k = b_2(M) > 9$. Then M admits a Sasaki-Einstein structure if and only if $H_2(M,\mathbb{Z})_{tor} = 0$ i.e. M is diffeomorphic to a k-fold connected sum of $S^2 \times S^3$. Said differently, mixed Smale Sasaki-Einstein 5-manifolds with second Betti number bigger than 9 do not exist. By our Theorem 1, we see that such a statement is not true in the Lorentzian Sasaki-Einstein case.*

Acknowledgement. The author would like to thank the referee for valuable suggestions which improved the presentation and clarity of the article.

References

[1] Boyer, C.P., Galicki, K., New Einstein Metrics in Dimension Five *J. Differential Geom Vol.* **57** (2001) No. 3 pp.443-pp.463

[2] Boyer, C.P., Galicki, K. Rational Homology 5-Spheres with Positive Ricci Curvature *Math. Res. Letters* **9** (2002), pp.521-528.

[3] Boyer, C.P., Galicki, K. Erratum and Addendum for Rational Homology 5-Spheres with Positive Ricci Curvature *Math. Res. Letters* **13** (2006), pp.463-465

[4] Boyer, C.P., Galicki, K. *Sasakian Geometry* Oxford Math. Monographs, Oxford University Press, Oxford 2007.

[5] Boyer, C.P. Nakamaye, M. On Sasaki-Einstein Manifolds in Dimenison Five *Geom. Dedicata*, **144**, no. 1, 2010. pp. 141-156.

[6] Boyer, C.P., Galicki, K., Matzeu, P. On Eta-Einstein Sasakian Geometry *Commun. Math. Phys* **262** (2006) pp177-208.

[7] Boyer, C.P., Galicki K., Nakamaye, M Sasakian Geometry, Homotopy Spheres and Positive Ricci Curvature *Topology* **42** (2003) pp.981- 1002.

[8] Boyer, C.P., Galicki, K., Nakamaye, M. On Positive Sasakian Geometry *Geom. Dedicata* **101** (2003) pp.93-102.

[9] A.R. Fletcher Working with Weighted Complete Intersections in *Explicit Birational Geometry of 3-folds* (A.Corti and M.Reid, eds.), Cambridge Univ. Press, Cambridge, 2000.

[10] Gomez, Ralph.R. On Lorentzian Sasaki-Einstein Geometry Ph.D Diss. University of New Mexico 2008.

[11] Gomez, Ralph R. Lorentzian Sasaki-Einstein metrics on Connected Sums of $S^2 \times S^3$ *Geom. Dedicata* **146** (2010) pp.249-255.

[12] Kollár, J. Einstein Metrics on Five-Dimensional Seifert Bundles *J. Geom. Anal* **15** (2005) pp.445-476.

[13] Milnor, J. Singular Points of Complex Hypersurfaces , *Ann. of Math Stud.* Princeton University Press, 1968.

[14] Milnor, J., Orlik, P. *Isolated Singlarities Defined by Weighted Homogeneous Polynomials*, *Topology* **9** 1970, pp.385-393.

[15] Smale, S. On the Structure of 5-manifolds *Ann. of Math.* **75** (1962), pp.38-46.

[16] Yau, S.S-T., Yu, Y. Classification of 3-Dimensional Isolated Rational Hypersurface Singularities with $\mathbb{C}^*$-Action *Rocky Mountain J. Math* **35** (2005), pp.1795-1809.

Received: 09.05.2015
Revised: 11.06.2015
Accepted: 12.07.2015

Department of Mathematics and Statistics,
Swarthmore College
E-mail: `rgomez1@swarthmore.edu`