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Stanley depth on five generated, squarefree, monomial ideals

by
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Abstract

Let $I \supsetneq J$ be two squarefree monomial ideals of a polynomial algebra over a field generated in degree $\geq d$, resp. $\geq d+1$. Suppose that I is either generated by four squarefree monomials of degrees d and others of degrees $\geq d+1$, or by five special monomials of degrees d. If the Stanley depth of I/J is $\leq d+1$ then the usual depth of I/J is $\leq d+1$ too.

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Introduction

Let K be a field and $S = K[x_1, \ldots, x_n]$ be the polynomial K-algebra in n variables. Let $I \supsetneq J$ be two squarefree monomial ideals of S and suppose that I is generated by squarefree monomials of degrees $\geq d$ for some positive integer d. After a multigraded isomorphism we may assume either that $J = 0$, or J is generated in degrees $\geq d+1$.

Let $P_{I\setminus J}$ be the poset of all squarefree monomials of $I \setminus J$ with the order given by the divisibility. Let P be a partition of $P_{I\setminus J}$ in intervals $[u, v] = \{w \in P_{I\setminus J} : u|w, w|v\}$, let us say $P_{I\setminus J} = \cup_i [u_i, v_i]$, the union being disjoint. Define $\operatorname{sdepth} P = \min_i \deg v_i$ and the *Stanley depth* of I/J given by $\operatorname{sdepth}_S I/J = \max_P \operatorname{sdepth} P$, where P runs in the set of all partitions of $P_{I\setminus J}$ (see [3], [19]). Stanley's Conjecture says that $\operatorname{sdepth}_S I/J \geq \operatorname{depth}_S I/J$.

In spite of so many papers on this subject (see [3], [10], [17], [1], [4], [18], [11], [7], [2], [12], [16]) Stanley's Conjecture remains open after more than thirty years. Meanwhile, new concepts as for example the Hilbert depth (see [1], [20], [5]) proved to be helpful in this area (see for instance [18, Theorem 2.4]). Using

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a Theorem of Uliczka [20] it was shown in [8] that for $n = 6$ the Hilbert depth of $S \oplus m$ is strictly bigger than the Hilbert depth of m, where m is the maximal graded ideal of S. Thus for $n = 6$ one could also expect $\operatorname{sdepth}_S(S \oplus m) > \operatorname{sdepth}_S m$, that is a negative answer for a Herzog's question. This was stated later by Ichim and Zarojanu [6].

Suppose that $I \subset S$ is minimally generated by some squarefree monomials $f_1, \ldots, f_r$ of degrees d, and a set E of squarefree monomials of degree $\geq d+1$. By [3, Proposition 3.1] (see [12, Lemma 1.1]) we have $\operatorname{depth}_S I/J \geq d$. Thus if $\operatorname{sdepth}_S I/J = d$ then Stanley's Conjecture says that $\operatorname{depth}_S I/J = d$. This is exactly what [12, Theorem 4.3]) states. Next step in studying Stanley's Conjecture is to prove the following weaker one.

Conjecture 1. *Suppose that $I \subset S$ is minimally generated by some squarefree monomials $f_1, \ldots, f_r$ of degrees d, and a set E of squarefree monomials of degree $\geq d+1$. If $\operatorname{sdepth}_S I/J = d+1$ then $\operatorname{depth}_S I/J \leq d+1$.*

This conjecture is studied in [14], [15], [16] either when $r = 1$, or when $E = \emptyset$ and $r \leq 3$. Recently, these results were improved in the next theorem.

Theorem 1. *(A. Popescu, D.Popescu [9, Theorem 0.6]) Let C be the set of the squarefree monomials of degree $d+2$ of $I \setminus J$. Conjecture 1 holds in each of the following two cases:*

1. $r \leq 3$,
2. $r = 4$, $E = \emptyset$ *and there exists* $c \in C$ *such that* $\operatorname{supp} c \not\subset \cup_{i\in[4]} \operatorname{supp} f_i$.

The purpose of this paper is to extend the above theorem in the following form.

Theorem 2. *Let B be the set of the squarefree monomials of degree $d+1$ of $I \setminus J$. Conjecture 1 holds in each of the following two cases:*

1. $r \leq 4$,
2. $r = 5$, *and there exists* $t \notin \cup_{i\in[5]} \operatorname{supp} f_i$, $t \in [n]$ *such that* $(B \setminus E) \cap (x_t) \neq \emptyset$ *and* $E \subset (x_t)$.

The above theorem follows from Theorems 3, 4 (the case $r = 4$, $E = \emptyset$ is given already in Proposition 2). It is worth to mention that the idea of the proof of Proposition 2, and Theorem 1 started already in the proof of [16, Lemma 4.1] when $r = 1$. Here *path* is a more general notion, the reason being to suit better the exposition. However, the case $r = 4$, $E \neq \emptyset$ is more complicated (see Remark 8) and we have to study separately the special case when $f_i \in (v)$, $i \in [4]$ for some monomial v of degree $d-1$ (see the proof of Theorem 3).

What can be done next? We believe that Conjecture 1 holds, but the proofs will become harder with increasing r. Perhaps for each $r \geq 5$ the proof could be done in more or less a common form but leaving some "pathological" cases which should be done separately. Thus to get a proof of Conjecture 1 seems to be a difficult aim.

We owe thanks to a Referee, who noticed some mistakes in a previous version of this paper, especially in the proof of Lemma 3.

1 Depth and Stanley depth

Suppose that I is minimally generated by some squarefree monomials $f_1, \ldots, f_r$ of degrees d for some $d \in \mathbb{N}$ and a set of squarefree monomials E of degree $\geq d+1$. Let B (resp. C) be the set of the squarefree monomials of degrees $d+1$ (resp. $d+2$) of $I \setminus J$. Set $s = |B|$, $q = |C|$. Let w_{ij} be the least common multiple of f_i and f_j and set W to be the set of all w_{ij}. Let C_3 be the set of all $c \in C \cap (f_1, \ldots, f_r)$ having all divisors from $B \setminus E$ in W. In particular each monomial of C_3 is the least common multiple of three of the f_i. The converse is not true as shown by [9, Example 1.6]. Let C_2 be the set of all $c \in C$, which are the least common multiple of two f_i, that is $C_2 = C \cap W$. Then $C_{23} = C_2 \cup C_3$ is the set of all $c \in C$, which are the least common multiple of two or three f_i. We may have $C_2 \cap C_3 \neq \emptyset$ as shows the following example.

Example 1. Let $n \geq 4$, $f_i = x_i x_{i+1}$, $i \in [3]$, $f_4 = x_1 x_4$ and $I = (f_1, \ldots, f_4)$, $J = 0$. Note that $m = x_1 x_2 x_3 x_4$ is a least common multiple of every three monomials f_j and the divisors of m with degree 3 are $w_{12}, w_{23}, w_{34}, w_{14}$. Thus $m \in C_3$. But $m \in C_2$ because $m = w_{13} = w_{24}$.

We start with a lemma, which slightly extends [9, Theorem 2.1].

Lemma 1. *Suppose that there exists* $t \in [n]$, $t \notin \cup_{i \in [r]} \operatorname{supp} f_i$ *such that* $(B \setminus E) \cap (x_t) \neq \emptyset$ *and* $E \subset (x_t)$. *If Conjecture 1 holds for* $r' < r$ *and* $\operatorname{sdepth}_S I/J = d+1$, *then* $\operatorname{depth}_S I/J \leq d+1$.

Proof: We follow the proof of [9, Theorem 2.1]. Apply induction on $|E|$, the case $|E| = 0$ being done in the quoted theorem. We may suppose that E contains only monomials of degrees $d+1$ by [14, Lemma 1.6]. Since Conjecture 1 holds for $r' < r$ we see that $C \not\subset (f_2, \ldots, f_r, E)$ implies $\operatorname{depth}_S I/J \leq d+1$ by [16, Lemma 1.1]. If Conjecture 1 holds for r and $E \setminus \{a\}$ with some $a \in E$ then $C \not\subset (f_1, \ldots, f_r, E \setminus \{a\})$ implies again $\operatorname{depth}_S I/J \leq d+1$ by the quoted lemma. Thus using the induction hypothesis on $|E|$ we may assume that $C \subset (W) \cup ((E) \cap (f_1, \ldots, f_r)) \cup (\cup_{a, a' \in E, a \neq a'} (a) \cap (a'))$. Let $I_t = I \cap (x_t)$, $J_t = J \cap (x_t)$, $B_t = (B \setminus E) \cap (x_t) = \{x_t f_1, \ldots, x_t f_e\}$, for some $1 \leq e \leq r$. If $\operatorname{sdepth}_S I_t/J_t \leq d+1$ then $\operatorname{depth}_S I_t/J_t \leq d+1$ by [12, Theorem 4.3] because I_t is generated only by monomials of degree $d+1$. Thus $\operatorname{depth}_S I/J \leq \operatorname{depth}_S I_t/J_t \leq d+1$ by [9, Lemma 1.1].

Suppose that $\operatorname{sdepth}_S I_t/J_t \geq d+2$. Then there exists a partition on I_t/J_t with sdepth $d+2$ having some disjoint intervals $[x_t f_i, c_i]$, $i \in [e]$ and $[a, c_a]$, $a \in E$. We may assume that c_i, c_a have degrees $d+2$. We have either $c_i \in (W)$, or $c_i \in ((E) \cap (f_1, \ldots, f_r)) \setminus (W)$. In the first case $c_i = x_t w_{ik_i}$ for some $1 \leq k_i \leq r$,

$k_i \neq i$. Note that $x_t f_{k_i} \in B$ and so $k_i \leq e$. We consider the intervals $[f_i, c_i]$. These intervals contain $x_t f_i$ and possible a w_{ik_i}. If $w_{ik_i} = w_{jk_j}$ for $i \neq j$ then we get $c_i = c_j$ which is false. Thus these intervals are disjoint.

Let I' be the ideal generated by f_j for $e < j \leq r$ and $B \setminus (E \cup (\cup_{i=1}^{e} [f_i, c_i]))$. Set $J' = I' \cap J$. Note that $I' \neq I$ because $e \geq 1$. As we showed already $c_i \notin I'$ for any $i \in [e]$. Also $c_a \notin I'$ because otherwise $c_a = x_t x_k f_j$ for some $e < j \leq r$ and we get $x_t f_j \in B$, which is false. In the following exact sequence

$$0 \to I'/J' \to I/J \to I/(J + I') \to 0$$

the last term has a partition of sdepth $d+2$ given by the intervals $[f_i, c_i]$ for $1 \leq i \leq e$ and $[a, c_a]$ for $a \in E$. It follows that $I' \neq J'$ because $\operatorname{sdepth}_S I/J = d + 1$. Then $\operatorname{sdepth}_S I'/J' \leq d+1$ using [17, Lemma 2.2] and so $\operatorname{depth}_S I'/J' \leq d+1$ by Conjecture 1 applied for $r - e < r$. But the last term of the above sequence has depth $> d$ because x_t does not annihilate f_i for $i \in [e]$. With the Depth Lemma we get $\operatorname{depth}_S I/J \leq d + 1$. $\square$

Next we give a variant of the above lemma.

Lemma 2. *Suppose that $r > 2$, $E = \emptyset$, $C \subset (W)$ and there exists $t \in [n]$, $t \notin \cup_{i \in [r]} \operatorname{supp} f_i$ such that $x_t w_{ij} \in C$ for some $1 \leq i < j \leq r$. If Conjecture 1 holds for $r' \leq r - 2$ and $\operatorname{sdepth}_S I/J = d + 1$, then $\operatorname{depth}_S I/J \leq d + 1$.*

Proof: We follow the proof of the above lemma, skipping the first part since we have already $C \subset (W)$. Note that in our case $x_t f_i, x_t f_j \in B$ and so $e \geq 2$. Thus I' is generated by at most $(r - 2)$ monomials of degrees d and some others of degrees $\geq d + 1$. Therefore, Conjecture 1 holds for I'/J' and so the above proof works in our case. $\square$

For $r \leq 3$ the following lemma is part from the proof of [9, Lemma 3.2] but not in an explicit way. Here we try to formalize better the arguments in order to apply them when $r = 4$.

Lemma 3. *Suppose that $r \leq 4$ and for each $i \in [r]$ there exists $c_i \in C \cap (f_i)$ such that the intervals $[f_i, c_i]$, $i \in [r]$ are disjoint. Then $\operatorname{depth}_S I/J \geq d + 1$.*

Proof: The proof consists of an induction part dealing with the case $C \not\subset (W)$ followed by a case analysis covering the case $C \subset (W)$.

Case 1, $C \not\subset (W)$

Suppose that there exists $c \in C \setminus (W)$, let us say $c \in (f_1) \setminus (f_2, \ldots, f_r)$. Then $[f_1, c]$ is disjoint with respect to $[f_i, c_i]$, $1 < i \leq r$ and we may change c_1 by c, that is we may suppose that $c_1 \in (f_1) \setminus (f_2, \ldots, f_r)$. Let $B \cap [f_1, c_1] = \{b, b'\}$ and $L = (f_2, \ldots, f_r, B \setminus \{b, b', E\})$. In the following exact sequence

$$0 \to L/(J \cap L) \to I/J \to I/(J, L) \to 0$$

the first term has depth $\geq d+1$ by induction hypothesis and the last term is isomorphic with $(f_1)/((J,L)\cap(f_1))$ and has depth $\geq d+1$ because $b \notin (J,L)$. Thus $\operatorname{depth}_S I/J \geq d+1$ by the Depth Lemma.

Case 2, $r=2$

In this case, note that one from c_1, c_2 is not in $(W)=(w_{12})$, that is we are in the above case. Indeed, if $c_1 \in (W)$ then either $c_1 = w_{12}$ and so c_2 cannot be in (W), or $c_1 = x_j w_{12}$ and then $w_{12} \in [f_1, c_1]$ cannot divide c_2 since the intervals are disjoint.

From now on assume that $r>2$.

Case 3, $c_1 \in (w_{12})$, $f_i \not| c_1$ for $i>2$ and $c_i \notin (w_{12})$ for $1<i\leq r$.

First suppose that $w_{12} \in B$. We have $c_1 = x_j w_{12}$ for some j and we see that $b = f_1 x_j \notin (f_2,\ldots,f_r)$. Set $T=(f_2,\ldots,f_r, B\setminus\{b,E\})$. In the following exact sequences

$$0 \to T/(J\cap T) \to I/J \to I/(J,T) \to 0$$

$$0 \to (w_{12})/(J\cap(w_{12})) \to T/(J\cap T) \to T/((J,w_{12})\cap T) \to 0$$

the last terms have depth $\geq d+1$ since $b \notin (J,T)$ and using the induction hypothesis in the second situation. As the first term of the second sequence has depth $\geq d+1$ we get $\operatorname{depth}_S T/(J\cap T) \geq d+1$ and so $\operatorname{depth}_S I/J \geq d+1$ using the Depth Lemma in both exact sequences.

If $w_{12} \in C$ then both monomials b, b' from $B \cap [f_1, c_1]$ are not in $(f_2,\ldots,f_r)$ and the above proof goes with b' instead w_{12}.

Case 4, $r=3$.

By Case 1 we may suppose that $C \subset (W)$. Then w_{12}, w_{13}, w_{23} are different because otherwise only one c_i can be in (W). We may suppose that $c_1 \in (w_{12})$, $c_2 \in (w_{23})$, $c_3 \in (w_{13})$, because each c_i is a multiple of one w_{ij} which can be present just in one interval since these are disjoint. If $f_3|c_1$ then w_{13} is present in both intervals $[f_1,c_1]$, $[f_3,c_3]$. If let us say $w_{12} \in C$, then $c_2, c_3 \notin (w_{12})$ because $c_3 \neq c_1 \neq c_2$. Thus we are in Case 3.

If $w_{12} \in B$ and $c_2, c_3 \notin (w_{12})$ then we are in Case 3. Otherwise, we may suppose that either $c_2 \in (w_{12})$, or $c_3 \in (w_{12})$. In the first case, we have w_{12} in both intervals $[f_1,c_1]$, $[f_2,c_2]$, which is false. In the second case, we have also w_{23} present in both intervals $[f_2,c_2]$, $[f_3,c_3]$, again false.

Case 5, $r=4$, $c_1 \in (w_{12})$, $w_{12} \in B$, $f_i \not| c_1$ for $2<i\leq 4$, $c_3 \in (w_{12})$.

It follows that $c_3 \in (w_{23})$. Thus $c_2 \notin (w_{23})$, that is $f_3 \not| c_2$, because otherwise the intervals $[f_2,c_2]$, $[f_3,c_3]$ will contain w_{23}, which is false. If $c_2 \in (w_{12})$ then the intervals $[f_1,c_1]$, $[f_2,c_2]$ will contain w_{12}. It follows that $c_2 \in (w_{24})$. Note that $c_4 \notin (w_{24})$ because otherwise w_{24} belongs to $[f_2,c_2]\cap[f_4,c_4]$. If $c_3 \notin (w_{24})$ then we are in Case 3 with w_{24} instead w_{12} and c_2 instead c_1.

Remains to see the case when $c_3 \in (f_1)\cap(f_2)\cap(f_3)\cap(f_4)$. Then $c_4 \notin (f_3)$ because otherwise w_{34} is in $[f_3,c_3]\cap[f_4,c_4]$. In the exact sequence

$$0 \to (f_3)/(J\cap(f_3)) \to I/J \to I/(J,f_3) \to 0$$

the last term has depth $\geq d+1$ by induction hypothesis. The first term has depth $\geq d+1$ since for example $w_{23} \notin J$. By the Depth Lemma we get $\operatorname{depth}_S I/J \geq d+1$.

Case 6, $r=4$, the general case.

Since $|W| \leq 6$ there exist an interval, let us say $[f_1, c_1]$, containing just one w_{ij}, let us say w_{12}. Thus no f_i, $2 < i \leq 4$ divides c_1. If $w_{12} \in C$ then no c_i, $i > 1$ belongs to (w_{12}) because otherwise $c_i = c_1$. If $w_{12} \in B$ and one $c_i \in (w_{12})$, $i > 1$ then we must have $i = 2$ because otherwise we are in Case 5. But if $c_2 \in (w_{12})$ then w_{12} is present in both intervals $[f_1, c_1]$, $[f_2, c_2]$, which is false. Thus $c_i \notin (w_{12})$ for all $1 < i \leq 4$, that is Case 3. $\square$

Remark 1. When $r > 4$ the statement of the above lemma is not valid anymore, as shows the following example.

Example 2. Let $n = 5$, $d = 1$, $I = (x_1, \ldots, x_5)$,

$$J = (x_1x_3x_4, x_1x_2x_4, x_1x_3x_5, x_2x_3x_5, x_2x_4x_5).$$

Set $c_1 = x_1x_2x_3$, $c_2 = x_2x_3x_4$, $c_3 = x_3x_4x_5$, $c_4 = x_1x_4x_5$, $c_5 = x_1x_2x_5$. We have $C = \{c_1, \ldots, c_5\}$ and $B = W$. Thus $s = 2r$ and $\operatorname{sdepth}_S I/J = 3$ because we have a partition on I/J given by the intervals $[x_i, c_i]$, $i \in [5]$. But $\operatorname{depth}_S I/J = 1$ because of the following exact sequence

$$0 \to I/J \to S/J \to S/I \to 0$$

where the last term has depth 0 and the middle ≥ 2.

The proposition below is an extension of [9, Lemma 3.2], its proof is given in the next section.

Proposition 1. *Suppose that the following conditions hold:*

1. $r = 4$, $8 \leq s \leq q+4$,
2. $C \subset (\cup_{i,j\in[4], i\neq j}(f_i) \cap (f_j)) \cup ((E) \cap (f_1, \ldots, f_4)) \cup (\cup_{a,a'\in E, a\neq a'}(a) \cap (a'))$,
3. *there exists* $b \in (B \cap (f_1)) \setminus (f_2, f_3, f_4)$ *such that* $\operatorname{sdepth}_S I_b/J_b \geq d+2$ *for* $I_b = (f_2, \ldots, f_r, B \setminus \{b\})$, $J_b = J \cap I_b$,
4. *the least common multiple* ω_1 *of* f_2, f_3, f_4 *is not in* $(C_3 \setminus W) \cap (E)$ *(see Example 1).*

Then either $\operatorname{sdepth}_S I/J \geq d+2$, *or there exists a nonzero ideal* $I' \subsetneq I$ *generated by a subset of* $\{f_1, \ldots, f_4\} \cup B$ *such that* $\operatorname{depth}_S I/(J, I') \geq d+1$ *and either* $\operatorname{sdepth}_S I'/J' \leq d+1$ *for* $J' = J \cap I'$ *or* $\operatorname{depth}_S I'/J' \leq d+1$.

Proposition 2. *Conjecture 1 holds for* $r = 4$ *when the least common multiples* ω_i *of* $f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_4$, $i \in [4]$ *are not in* $(C_3 \setminus W) \cap (E)$. *In particular, Conjecture 1 holds when* $r = 4$ *and* $E = \emptyset$.

Proof: By Theorems [13, Theorem 1.3], [18, Theorem 2.4] (more precisely the particular forms given in [9, Theorems 0.3, 0.4]) we may suppose that $8 = 2r \leq s \leq q + 4$ and we may assume that E contains only monomials of degrees $d+1$ by [14, Lemma 1.6]. We may assume that there exists $b \in B \cap (f_1, \ldots, f_4)$ which is not in W because otherwise $B \cap (f_1, \ldots, f_4) \subset B \cap W$ and therefore $|B \cap (f_1, \ldots, f_4)| \leq |B \cap W| \leq 6$. By [18, Theorem 2.4] this implies the depth $\leq d+1$ of the first term of the exact sequence

$$0 \to (f_1, \ldots, f_r)/(J \cap (f_1, \ldots, f_r)) \to I/J \to (E)/((J, f_1, \ldots, f_r) \cap (E)) \to 0$$

and then the middle has depth $\leq d+1$ too using the Depth Lemma.

Renumbering f_i we may suppose that there exists $b \in (f_1) \setminus (f_2, \ldots, f_4)$. As in the proof of [9, Theorem 1.7] we may suppose that the first term of the exact sequence

$$0 \to I_b/J_b \to I/J \to I/(J, I_b) \to 0$$

has sdepth $\geq d+2$. Otherwise it has depth $\leq d+1$ by Theorem 1. Note that the last term is isomorphic with $(f_1)/((f_1) \cap (J, I_b))$ and it has depth $\geq d+1$ because $b \notin (J, I_b)$. Then the middle term of the above exact sequence has depth $\leq d+1$ by the Depth Lemma.

Thus we may assume that the condition (3) of Proposition 1 holds. Also we may apply [16, Lemma 1.1] and see that the condition (2) of Proposition 1 holds. Applying Proposition 1 we get either $\operatorname{sdepth}_S I/J \geq d+2$ contradicting our assumption, or there exists a nonzero ideal $I' \subsetneq I$ generated by a subset G of B, or by G and a subset of $\{f_1, \ldots, f_4\}$ such that $\operatorname{sdepth}_S I'/J' \leq d+1$ for $J' = J \cap I'$ and $\operatorname{depth}_S I/(J, I') \geq d+1$. In the last case we see that $\operatorname{depth}_S I'/J' \leq d+1$ by Theorem 1, or by induction on s, and so $\operatorname{depth}_S I/J \leq d+1$ applying in the following exact sequence

$$0 \to I'/J' \to I/J \to I/(J, I') \to 0$$

the Depth Lemma. □

2 Proof of Proposition 1

Since $\operatorname{sdepth}_S I_b/J_b \geq d+2$ by (3), there exists a partition P_b on I_b/J_b with sdepth $d+2$. We may choose P_b such that each interval starting with a squarefree monomial of degree d, $d+1$ ends with a monomial of C. In P_b we have three disjoint intervals $[f_2, c_2']$, $[f_3, c_3']$, $[f_4, c_4']$. Suppose that $B \cap [f_i, c_i'] = \{u_i, u_i'\}$, $1 < i \leq 4$. For all $b' \in B \setminus \{b, u_2, u_2', \ldots, u_4, u_4'\}$ we have an interval $[b', c_{b'}]$. We define $h : B \setminus \{b, u_2, u_2', \ldots, u_4, u_4'\} \to C$ by $b' \longmapsto c_{b'}$. Then h is an injection and $|\operatorname{Im} h| = s - 7 \leq q - 3$.

We follow the proofs of [9, Lemmas 3.1, 3.2]. A sequence $a_1, \ldots, a_k$ is called a *path* from a_1 to a_k if the following statements hold:

(i) $a_l \in B \setminus \{b, u_2, u_2', \ldots, u_4, u_4'\}$, $l \in [k]$,

(ii) $a_l \neq a_j$ for $1 \leq l < j \leq k$,

(iii) $a_{l+1}|h(a_l)$ for all $1 \leq l < k$.

This path is *weak* if $h(a_j) \in (b, u_2, u_2', \ldots, u_4, u_4')$ for some $j \in [k]$. It is *bad* if $h(a_j) \in (b)$ for some $j \in [k]$ and it is *maximal* if all divisors from B of $h(a_k)$ are in $\{b, u_2, u_2', \ldots, u_4, u_4', a_1, \ldots, a_k\}$. We say that the above path *starts with* a_1. Note that here the notion of path is more general than the notion of path used in [16] and [9].

By hypothesis $s \geq 8$ and there exists $a_1 \in B \setminus \{b, u_2, u_2', \ldots, u_4, u_4'\}$. We construct below, as an example, a path with $k > 1$. By recurrence choose if possible a_{p+1} to be a divisor from $B \setminus \{b, u_2, u_2', \ldots, u_4, u_4', a_1, \ldots, a_p\}$ of $m_p = h(a_p)$, $p \geq 1$. This construction ends at step $p = e$ if all divisors from B of m_e are in $\{b, u_2, u_2', \ldots, u_4, u_4', a_1, \ldots, a_e\}$. This is a maximal path. If one $m_p \in (u_2, u_2', \ldots, u_4, u_4')$ then the constructed path is weak. If one $m_p \in (b)$ then this path is bad.

We start the proof with some helpful lemmas.

Lemma 4. *P_b could be changed in order to have the following properties:*

1. *For all $1 < i < j \leq 4$ with $u_i, u_j \notin W$ and $w_{ij} \in B \setminus \{u_2, u_2', \ldots, u_4, u_4'\}$ it holds that $h(w_{ij}) \notin (u_i) \cap (u_j)$,*

2. *For each $1 \leq i < j \leq 4$ with $u_j \in W$, $u_j' \notin W$, $w_{ij} \in B \setminus \{u_2, u_2', \ldots, u_4, u_4'\}$ it holds that $h(w_{ij}) \notin (u_j)$ and if $h(w_{ij}) \in (u_j')$ then $i > 1$,*

3. *For each $1 \leq i < j \leq 4$ with $u_j, u_j' \notin W$ and $w_{ij} \in B \setminus \{u_2, u_2', \ldots, u_4, u_4'\}$ it holds that $h(w_{ij}) \notin (u_j, u_j')$.*

Proof: Suppose that $w_{ij} \in B \setminus \{u_2, u_2', \ldots, u_4, u_4'\}$ and $h(w_{ij}) \in (u_i)$ for some $2 \leq i \leq 4$ and $j \in [4]$, $j \neq i$. We have $h(w_{ij}) = x_l w_{ij}$ for some $l \notin \operatorname{supp} w_{ij}$ and it follows that $u_i = x_l f_i$. Changing in P_b the intervals $[f_i, c_i']$, $[w_{ij}, h(w_{ij})]$ with $[f_i, h(w_{ij})]$, $[u_i', c_i']$ we may assume that the new $u_i' = w_{ij}$. We will apply this procedure several times eventually obtaining a partition P_b with the above properties. In case (1) we change in this way u_i' by w_{ij}. Note that the number of elements among $\{u_2, u_2', \ldots, u_4, u_4'\}$ which are from $B \cap W$ is either preserved or increases by one. Applying this procedure several time we get (1) fulfilled.

In case (3) the above procedure preserves among $\{u_2, u_2', \ldots, u_4, u_4'\}$ the former elements which were from $B \cap W$ and includes a new one w_{ij}. After several steps we get fulfilled (3).

For case (2) if $u_j \in W$, $u_j' \notin W$ and $h(w_{ij}) \in (u_j)$ we change as above u_j' by w_{ij}. Note that the number of elements among $\{u_2, u_2', \ldots, u_4, u_4'\}$ which are from $B \cap W$ increases by one. If $h(w_{ij}) \in (u_j')$ then we may change in this way u_j by w_{ij}. We do this only if $i = 1$. Note that the number of elements among $\{u_2, u_2', \ldots, u_4, u_4'\}$ which are from $B \cap W$ is preserved. Our procedure does not affect those c_i' with $u_i, u_i' \in W$ and does not affect the property (1). After several such procedures we get also (2) fulfilled. $\square$

From now on we suppose that P_b has the properties mentioned in the above lemma. Moreover, we fix $a_1 \in B \setminus \{b, u_2, u_2', \dots, u_4, u_4'\}$ and let $a_1, \dots, a_p$ be a path which is not bad. For an $a' \in B \setminus \{b, u_2, u_2', \dots, u_4, u_4'\}$ set

$$T_{a'} = \{b' \in B : \text{there exists a path } a_1' = a', \dots, a_e' \text{ not bad with } a_e' = b'\},$$

$U_{a'} = h(T_{a'})$, $G_{a'} = B \setminus T_{a'}$. If $a' = a_1$ we write simply T_1 instead T_{a_1} and similarly U_1, G_1.

Remark 2. Any divisor from B of a monomial of U_1 is in $T_1 \cup \{u_2, u_2', \dots, u_4, u_4'\}$.

Lemma 5. *If no weak path and no bad path starts with a_1 then the conclusion of Proposition 1 holds.*

Proof: Assume that $[r] \setminus \{j \in [r] : U_1 \cap (f_j) \neq \emptyset\} = \{k_1, \dots, k_\nu\}$ for some $1 \le k_1 < \dots < k_\nu \le 4$, $0 \le \nu \le 4$. Set $k = (k_1, \dots, k_\nu)$, $I_k' = (f_{k_1}, \dots, f_{k_\nu}, G_1)$, $J_k' = I_k' \cap J$, and $I_0' = (G_1)$, $J_0' = I_0' \cap J$ for $\nu = 0$. Note that all divisors from B of a monomial $c \in U_1$ belong to T_1, and $I_0' \neq 0$ because $b \in I_0'$. Consider the following exact sequence

$$0 \to I_k'/J_k' \to I/J \to I/(J, I_k') \to 0.$$

If $U_1 \cap (f_1, \dots, f_4) = \emptyset$ then the last term of the above exact sequence given for $k = (1, \dots, 4)$ has depth $\ge d+1$ and sdepth $\ge d+2$ because P_b can be restricted to $(T_1) \setminus (J, I_k')$ since $h(b) \notin I_k'$, for all $b \in T_1$ (see Remark 2). If the first term has sdepth $\ge d+2$ then by [17, Lemma 2.2] the middle term has sdepth $\ge d+2$. Otherwise, take $I' = I_k'$.

If $U_1 \cap (f_1, f_2, f_3) = \emptyset$, but there exists $b_4 \in T_1 \cap (f_4)$, then set $k = (1, 2, 3)$. In the following exact sequence

$$0 \to I_k'/J_k' \to I/J \to I/(J, I_k') \to 0$$

the last term has sdepth $\ge d+2$ since $h(b') \notin I_k'$ for all $b' \in T_1$ and we may substitute the interval $[b_4, h(b_4)]$ from the restriction of P_b by $[f_4, h(b_4)]$, the second monomial from $[f_4, h(b_4)] \cap B$ being also in T_1. As above we get either $\operatorname{sdepth}_S I/J \ge d+2$, or $\operatorname{sdepth}_S I_k'/J_k' \le d+1$, $\operatorname{depth}_S I/(J, I_k') \ge d+1$.

Suppose that $U_1 \cap (f_j) \neq \emptyset$ if and only if $\nu < j \le 4$, for some $0 \le \nu \le 4$ and set $k = (1, \dots, \nu)$. We omit the subcases $0 < \nu < 3$, since they go as in [9, Lemma 3.2], and consider only the worst subcase $\nu = 0$. Let $b_j \in T_1 \cap (f_j)$, $j \in [4]$ and set $c_j = h(b_j)$. For $1 \le l < j \le 4$ we claim that we may choose $b_l \neq b_j$ and such that one from c_l, c_j is not in (w_{lj}). Indeed, if $w_{lj} \notin B$ and $c_l, c_j \in (w_{lj})$ then necessarily $c_l = c_j$ and it follows $b_l = b_j = w_{lj}$, which is false. Suppose that $w_{lj} \in B$ and $c_j = x_p w_{lj}$. Then choose $b_l = x_p f_l \in T_1$. If $c_l = h(b_l) \in (w_{lj})$ then we get $c_l = c_j$ and so $b_l = b_j = w_{lj}$ which is impossible.

We show that we may choose $b_j \in T_1 \cap (f_j)$, $j \in [4]$ such that the intervals $[f_j, c_j]$, $j \in [4]$ are disjoint. Let C_2, C_3 be as in the beginning of the previous section. Set $C'_2 = U_1 \cap C_2$, $C'_3 = U_1 \cap C_3$, $C'_{23} = C'_2 \cup C'_3$. Let $\tilde{c} \in C'_2$, let us say $\tilde{c}$ is the least common multiple of f_1, f_2. Then $\tilde{c}$ has as divisors two multiples g_1, g_2 of f_1 and two multiples of f_2. If $\hat{c} \in C'_2$ is also a multiple of g_1, let us say $\hat{c}$ is the least common multiple of f_1, f_3 then g_2 does not divide $\hat{c}$ and the least common multiple of f_2, f_3 is not in C. Thus the divisors from $B \setminus E$ of $\tilde{c}$, $\hat{c}$ are at least 7. Since the divisors from $B \setminus E$ of $\tilde{c}$, $\hat{c}$ are in $T_1 \setminus E$ we see in this way that $|T_1 \setminus E| \geq |C'_2| + 3$. If $|C'_2| \neq 0$ then $|C'_3| \leq 1$ and so $|T_1 \setminus E| \geq |C'_{23}| + 2$. Assume that $|C'_2| = 0$. Then $|C'_3| \leq 4$. Let $\tilde{c} \in C'_3$ be the least common multiple of f_1, f_2, f_3 then w_{12}, w_{23}, w_{13} are the only divisors from $T_1 \setminus E$ of $\tilde{c}$ (this could be not true when $|C'_2| \neq 0$ as shows Example 1). If $\hat{c} \in C'_3$ is the least common multiple of f_1, f_2, f_4 we have also w_{14}, w_{24} in $T_1 \setminus E$. Similarly, if $|C'_3| \geq 3$ we get also $w_{34} \in T_1 \setminus E$. Thus $|T_1 \setminus E| \geq |C'_3| + 2 = |C'_{23}| + 2$ also when $|C'_2| = 0$.

Then there exist two different $b_j \in T_1 \cap (f_j)$ such that $c_j = h(b_j) \notin C'_{23}$ for let us say $j = 1, 2$ and so each of the intervals $[f_j, c_j]$, $j = 1, 2$ has at most one monomial from $T_1 \cap W$. Suppose the worst subcase when $[f_1, c_1]$ contains $w_{12} \in B$, and $[f_2, c_2]$ contains $w_{2j} \in B$ for some $j \neq 2$. First assume that $j \geq 3$, let us say $j = 3$. Then choose as above $b_3 \in T_1 \cap (f_3)$, $b_4 \in T_1 \cap (f_4)$ such that $c_3 \notin (w_{23})$, $c_4 \notin (w_{34})$. Then $[f_3, c_3]$ has from $T_1 \cap W$ at most w_{13}, w_{34} and $[f_4, c_4]$ has from $T_1 \cap W$ at most w_{14}, w_{24}. Thus the corresponding intervals are disjoint.

Otherwise, $j = 1$ and we have $c_j = x_{p_j} w_{12}$, $j \in [2]$, for some $p_j \notin \operatorname{supp} w_{12}$, $p_1 \neq p_2$. Take $b'_1 = x_{p_2} f_1$, $b'_2 = x_{p_1} f_2$ and $v_1 = h(b'_1)$, $v_2 = h(b'_2)$. Then v_1, v_2 are not in C'_3 because otherwise b'_1, respectively b'_2 is in W, which is false. Note that $v_2 \notin (w_{12})$, because otherwise $v_2 = x_{p_1} w_{12} = c_1$ which is false since $b_1 \neq b'_2$. Similarly $v_1 \notin (w_{12})$. If let us say $v_2 \notin C'_2$ then we may take $b_2 = b'_2$ and we see that for the new c_2 (namely v_2) the interval $[f_2, c_2]$ contains at most a monomial from W, which we assume to be w_{23} and we proceed as above. If $v_1, v_2 \in C'_2$, we may assume that $v_1 = w_{13} \in C$ and either $v_2 = w_{23} \in C$, or $v_2 = w_{24} \in C$. In the first case we choose b_3, b_4 such that $c_3 \notin (w_{34})$, $c_4 \notin (w_{24})$ and we see that $[f_3, c_3]$ has no monomial from W. Indeed, if $c_3 \in (w_{23})$ (the case $c_3 \in (w_{13})$ is similar) then $c_3 = v_2$, which is false since then $h(b'_2) = v_2 = c_3 = h(b_3)$ and so $b'_2 = b_3 \in (w_{23})$, h being injective. Also $[f_4, c_4]$ has at most w_{14}, w_{34}. Thus taking $b_i = b'_i$, $c_i = v_i$ for $i \in [2]$ we have again the intervals $[f_j, c_j]$, $j \in [4]$ disjoint. Similarly in the second case choose b_3, b_4 such that $c_3 \notin (w_{23})$, $c_4 \notin (w_{34})$ and we see that $[f_3, c_3]$ have at most w_{34} and $[f_4, c_4]$ have at most w_{14}, which is enough, because as above $c_3 \neq w_{13}$ and $c_4 \neq w_{24}$.

Next we replace the intervals $[b_j, c_j]$, $1 \leq j \leq 4$ from the restriction of P_b to $(T_1) \setminus (J, I'_0)$ with $[f_j, c_j]$, the second monomial from $[f_j, c_j] \cap B$ being also in T_1. Note that $I/(J, I'_0)$ has depth $\geq d + 1$ by Lemma 3. Thus, as above we get either $\operatorname{sdepth}_S I/J \geq d + 2$, or $\operatorname{sdepth}_S I'_0/J'_0 \leq d + 1$, $\operatorname{depth}_S I/(J, I'_0) \geq d + 1$. □

Lemma 6. *Let $a_1, \ldots, a_{e_1}$ be a bad path, $m_j = h(a_j)$, $j \in [e_1]$ and $m_{e_1} = bx_i$. Suppose that $m_{e_1} \notin (u_2, u'_2, \ldots, u_4, u'_4)$. Then one of the following statements holds:*

1. $\operatorname{sdepth}_S I/J \geq d+2$,

2. *there exists* $a_{e_1+1} \in (B \cap (f_1)) \setminus \{b, u_2, u_2', \ldots, u_4, u_4'\}$ *dividing* m_{e_1} *such that every path* $a_{e_1+1}, \ldots, a_{e_2}$ *satisfies* $\{a_1, \ldots, a_{e_1}\} \cap \{a_{e_1+1}, \ldots, a_{e_2}\} = \emptyset$.

Proof: If $a_{e_1} = f_1 x_i$ then changing in P_b the interval $[a_{e_1}, m_{e_1}]$ by $[f_1, m_{e_1}]$ we get a partition on I/J with sdepth $d+2$. If $f_1 x_i \in \{a_1, \ldots, a_{e_1-1}\}$, let us say $f_1 x_i = a_v$, $1 \leq v < e_1$ then we may replace in P_b the intervals $[a_k, m_k], v \leq k \leq e_1$ with the intervals $[a_v, m_{e_1}], [a_{k+1}, m_k], v \leq k \leq e_1 - 1$. Now we see that we have in P_b the interval $[a_v, m_v]$ (the new m_v is the old m_{e_1}) and switching it with the interval $[f_1, m_v]$ we get a partition with sdepth $\geq d+2$ for I/J. Thus we may assume that $f_1 x_i \notin \{a_1, ..., a_{e_1}\}$. Note that e_1 could be also 1 as in Example 3 when we take $a_1 = x_5 x_6$, in this case we take $f_1 x_i = x_1 x_5$ and $\{x_1 x_5, x_2 x_5\}$ is a maximal path which is weak but not bad.

By hypothesis $m_{e_1} \notin (u_2, u_2', \ldots, u_4, u_4')$ and so $f_1 x_i \notin \{u_2, u_2', \ldots, u_4, u_4'\}$. Then set $a_{e_1+1} = f_1 x_i$ and let $a_{e_1+1}, \ldots, a_{e_2}$ be a path starting with a_{e_1+1} and set $m_p = h(a_p), p > e_1$. If $a_p = a_v$ for $v \leq e_1$, $p > e_1$ then change in P_b the intervals $[a_k, m_k], v \leq k \leq p-1$ with the intervals $[a_v, m_{p-1}], [a_{k+1}, m_k], v \leq k \leq p-2$. We have in the new P_b an interval $[f_1 x_i, m_{e_1}]$ and switching it to $[f_1, m_{e_1}]$ we get a partition with sdepth $\geq d+2$ for I/J. Thus we may suppose that $a_{p+1} \notin \{b, u_2, u_2', \ldots, u_4, u_4', a_1, \ldots, a_p\}$ and so (2) holds. $\square$

Example 3. Let $n = 7$, $r = 4$, $d = 1$, $f_i = x_i$ for $i \in [4]$, $E = \{x_5 x_6, x_5 x_7\}$, $I = (x_1, \ldots, x_4, E)$ and

$$J = (x_1x_7, x_2x_7, x_3x_7, x_4x_7, x_1x_2x_4, x_1x_2x_6, x_1x_3x_4, x_1x_3x_6, x_2x_3x_4, x_2x_4x_5,$$

$$x_2x_5x_6, x_3x_5x_6, x_4x_5x_6).$$

Then

$$B = \{x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_1x_6, x_2x_3, x_2x_4, x_2x_5, x_2x_6,$$

$$x_3x_4, x_3x_5, x_3x_6, x_4x_5, x_4x_6\} \cup E$$

and

$$C = \{x_1x_2x_3, x_1x_2x_5, x_1x_3x_5, x_1x_4x_5, x_1x_4x_6, x_1x_5x_6, x_2x_3x_5, x_2x_3x_6, x_2x_4x_6,$$

$$x_3x_4x_5, x_3x_4x_6, x_5x_6x_7\}.$$

We have $q = 12$ and $s = q + r = 16$. Take $b = x_1 x_6$ and $I_b = (x_2, x_3, x_4, B \setminus \{b\}, E)$, $J_b = I_b \cap J$. There exists a partition P_b with sdepth 3 on I_b/J_b given by the intervals $[x_2, x_1x_2x_3]$, $[x_3, x_1x_3x_5]$, $[x_4, x_1x_4x_6]$, $[x_1x_5, x_1x_2x_5]$, $[x_2x_4, x_2x_4x_6]$, $[x_2x_5, x_2x_3x_5]$, $[x_2x_6, x_2x_3x_6]$, $[x_3x_4, x_3x_4x_5]$, $[x_3x_6, x_3x_4x_6]$, $[x_4x_5, x_1x_4x_5]$, $[x_5x_6, x_1x_5x_6]$, $[x_5x_7, x_5x_6x_7]$. We have $c_2' = x_1x_2x_3$, $c_3' = x_1x_3x_5$, $c_4' = x_1x_4x_6$ and $u_2 = x_2x_3$, $u_2' = x_1x_2$, $u_3 = x_3x_5$, $u_3' = x_1x_3$, $u_4 = x_1x_4$, $u_4' = x_4x_6$. Take $a_1 = x_2x_4$, $m_1 = x_2x_4x_6$. This is a weak path

but not bad. It can be extended to a maximal one $x_2x_4, x_2x_6, x_3x_6, x_3x_4, x_4x_5, x_1x_5, x_2x_5$ which is not bad.
Bad paths are for example $\{x_5x_6\}$, $\{x_5x_7, x_5x_6\}$, $\{x_5x_7, x_5x_6, x_1x_5, x_2x_5\}$, the last one being maximal. Replacing in P_b the intervals $[x_4, x_1x_4x_6]$, $[x_2x_4, x_2x_4x_6]$ with $[x_4, x_2x_4x_6]$, $[x_1, x_1x_4x_6]$ we get a partition on I/J with sdepth 3.

Lemma 7. *Let $a_1, \ldots, a_{e_1}$ be a bad path, $m_j = h(a_j)$, $j \in [e_1]$ and $m_{e_1} = bx_i$. Suppose that $a_{e_1} \in E$ and $m_{e_1} \in (u_2, u_2', \ldots, u_4, u_4')$. Then one of the following statements holds:*

1. *there exists $a_{e_1+1} \in B \setminus (\{b, u_2, u_2', \ldots, u_4, u_4'\} \cup E)$ dividing m_{e_1} such that every path $a_{e_1+1}, \ldots, a_{e_2}$ satisfies $\{a_1, \ldots, a_{e_1}\} \cap \{a_{e_1+1}, \ldots, a_{e_2}\} = \emptyset$,*

2. *there exist j, $2 \le j \le 4$ and a new partition P_b of I_b/J_b for which T_1 is preserved such that $a_{e_1} \in (f_j)$ and $m_{e_1} \in (u_j, u_j')$.*

Proof: Assume that $m_{e_1} = x_ib$ for some i and let us say $m_{e_1} \in (u_2')$. Then $f_1x_i = u_2' = w_{12}$ and so there exists another divisor $\tilde{a}$ of m_{e_1} from $B \cap (f_2)$ different of w_{12}. If $\tilde{a} \in [f_2, c_2']$ then we get $m_{e_1} = c_2'$, which is false. If $\tilde{a}$ is not in $\{b, u_2, u_2', \ldots, u_4, u_4'\}$ then set $a_{e_1+1} = \tilde{a}$. If let us say $\tilde{a} = u_3$ then $\tilde{a} = w_{23}$ and so m_{e_1} is the least common multiple of f_1, f_2, f_3. Clearly, $m_{e_1} \notin C_3$ because otherwise $b \in W$, which is false. Then $m_{e_1} = w_{13} \in C$ and we may find, let us say another divisor $\hat{a}$ of m_{e_1} from $B \cap (f_3)$ which is not u_3' because $m_{e_1} \neq c_3'$. If $\hat{a}$ is in $\{u_4, u_4'\}$ then we may find an a' in $B \cap (f_4)$ which is not in $\{u_4, u_4'\}$ because $m_{e_1} \neq c_4'$. Thus in general we may find an a'' in $B \cap (f_j)$ for some $2 \le j \le 4$ which is not in $\{b, u_2, u_2', \ldots, u_4, u_4'\}$ and $m_{e_1} \in (u_j, u_j')$. Set $a_{e_1+1} = a''$. Let $a_{e_1+1}, \ldots, a_{e_2}$ be a path. If we are not in the case (1) then $a_p = a_v$ for $v \le e_1$, $p > e_1$ and change in P_b the intervals $[a_k, m_k], v \le k \le p-1$ with the intervals $[a_v, m_{p-1}], [a_{k+1}, m_k], v \le k \le p-2$. Note that the new a_{e_1} is the old $a_{e_1+1} \in (f_j)$, that is the case (2). □

Lemma 8. *Suppose that* $\operatorname{sdepth}_S I/J \le d+1$. *Then there exists a partition P_b of I_b/J_b such that for any $a_1 \in B \setminus \{b, u_2, u_2', \ldots, u_4, u_4'\}$ and any bad path $a_1, \ldots, a_{e_1}$, $m_j = h(a_j)$, $j \in [e_1]$ with $m_{e_1} = bx_i$ the following statements holds:*

1. *$m_{e_1} \notin (u_2, u_2', \ldots, u_4, u_4')$,*

2. *there exists $a_{e_1+1} \in B \setminus (\{b, u_2, u_2', \ldots, u_4, u_4'\} \cup E)$ dividing m_{e_1} such that every path $a_{e_1+1}, \ldots, a_{e_2}$ satisfies $\{a_1, \ldots, a_{e_1}\} \cap \{a_{e_1+1}, \ldots, a_{e_2}\} = \emptyset$.*

Proof: If for any $a_1 \in B \setminus \{b, u_2, u_2', \ldots, u_4, u_4'\}$ there exist no bad path starting with a_1 there exists nothing to show. If for any such a_1 for each bad path $a_1, \ldots, a_{e_1}$, $m_j = h(a_j)$, $j \in [e_1]$ with $m_{e_1} \in (b)$ it holds $m_{e_1} \notin (u_2, u_2', \ldots, u_4, u_4')$ then then to get (2) apply Lemma 6. Now suppose that there exists a_1 and a bad path $a_1, \ldots, a_{e_1}$, $m_j = h(a_j)$, $j \in [e_1]$ with let us say $m_{e_1} \in (b) \cap (u_2)$. If we are not in case (2) then by Lemma 7 we may change P_b such that T_1 is preserved,

$a_{e_1} \in (f_j)$ and $m_{e_1} \in (u_j, u'_j)$ for some $2 \leq j \leq 4$. Assume that $j = 2$ and so $m_{e_1} \in (w_{12})$, let us say $u'_2 = w_{12}$. Replacing in P_b the intervals $[f_2, c'_2]$, $[a_{e_1}, m_{e_1}]$ with $[f_2, m_{e_1}]$, $[u_2, c'_2]$ the new c'_2 is the least common multiple of b and f_2. Thus there exists no path $a_1, \dots, a_{e_1}$ with $h(a_{e_1}) \in (b) \cap (u_2, u'_2)$ because $h(a_{e_1}) \neq c'_2$. Applying this procedure several time we see that there exists no path $a_1, \dots, a_{e_1}$ with $h(a_{e_1}) \in (b) \cap (u_2, u'_2, \dots, u_4, u'_4)$. Then we may apply Lemma 6 as above. □

Example 4. Let $n = 5$, $I = (x_1, \dots, x_4)$, $J = (x_2x_3x_4, x_2x_3x_5, x_2x_4x_5, x_3x_4x_5)$. So

$$C = \{x_1x_2x_3, x_1x_2x_4, x_1x_2x_5, x_1x_3x_4, x_1x_3x_5, x_1x_4x_5\},$$

$$B = \{x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_2x_3, x_2x_4, x_2x_5, x_3x_4, x_3x_5, x_4x_5\}.$$

Then $q = 6$, $s = 10 = q + r$. Set $b = x_1x_5$, $a_1 = x_2x_5$, $a_2 = x_3x_5$, $a_4 = x_4x_5$, $m_1 = x_1x_2x_5$, $m_2 = x_1x_3x_5$, $m_3 = x_1x_4x_5$, $c'_2 = x_1x_2x_3$, $c'_3 = x_1x_3x_4$, $c'_4 = x_1x_2x_4$. We have on I_b/J_b the partition P_b given by the intervals $[x_i, c'_i]$, $2 \leq i \leq 4$ and $[a_j, m_j]$, $j \in [3]$. Clearly, P_b has sdepth 3 and $m_i = bx_i$, $2 \leq i \leq 4$. Using the above lemma we change in P_b the intervals $[a_{i-1}, m_{i-1}]$, $[x_i, c'_i]$ with $[f_i, m_{i-1}]$, $[x_ix_5, c'_i]$ for $2 \leq i \leq 4$. Now we see that all m from the new U_1 are not in $(b) \cap (u_2, u'_2, \dots, u_4, u'_4)$.

We have $\operatorname{sdepth}_S I/J \leq 2$. If $\operatorname{sdepth}_S I/J = 3$ then there exists an interval $[x_1, c]$ with $c \in \{m_1, m_2, m_3\}$. If $c = m_i$ for some $2 \leq i \leq 4$ then for any interval $[x_i, c']$ it holds $[x_1, c] \cap [x_i, c'] = \{x_1x_i\}$, which is impossible. Also we have $\operatorname{depth}_S I/J \leq 2$ by Lemma 12.

Remark 3. Suppose that $\operatorname{sdepth}_S I/J \leq d + 1$. We change P_b as in Lemma 8. Moreover assume that there exists a bad path $a_{e_1+1}, \dots, a_{e_2}$. Using the same lemma we find a_{e_2+1} such that for each path $a_{e_2+1}, \dots, a_{e_3}$ one has $\{a_{e_1+1}, \dots, a_{e_2}\} \cap \{a_{e_{i_2}+1}, \dots, a_{e_3}\} = \emptyset$. The same argument gives also $\{a_1, \dots, a_{e_1}\} \cap \{a_{e_{i_2}+1}, \dots, a_{e_3}\} = \emptyset$. Thus we may find some disjoint sets of elements $\{a_{e_j+1}, \dots, a_{e_{j+1}}\}$, $j \geq 0$, where $e_0 = 0$. It follows that after some steps we arrive in the case when for some l there exist no bad path starting with a_{l+1}.

Lemma 9. *Suppose that* $\operatorname{sdepth}_S I/J \leq d+1$ *and* $\tilde{P}_b$ *is a partition of* I_b/J_b *given by Lemma 8. Assume that no bad path starts with* a_1, $U_1 \cap (u_2) \neq \emptyset$ *and there exists a divisor* $\tilde{a}$ *in* $(B \cap (f_2)) \setminus \{u_2, u'_2, \dots, u_4, u'_4\}$ *of a monomial* $m \in U_1 \cap (u_2)$. *Then there exist a partition* P_b *and a (possible bad) path* $a_1, \dots, a_p$ *such that* $T_{a_p} \cap \{a_1, \dots, a_{p-1}\} = \emptyset$, u_2 *and* c'_i, $i = 3, 4$ *are not changed in* P_b, *no bad path starts with* a_p *and one of the following statements holds:*

1. $U_{a_p} \cap (u_2) = \emptyset$,
2. $U_{a_p} \cap (u_2) \neq \emptyset$ *and there exists* $b_2 \in T_{a_p} \cap (f_2)$ *with* $h(b_2) \in (u_2)$,
3. $U_{a_p} \cap (u_2) \neq \emptyset$ *and every monomial of* $U_{a_p} \cap (u_2)$ *has all its divisors from* $B \cap (f_2)$ *contained in* $\{u_2, u'_2, \dots, u_4, u'_4\}$.

Moreover, if also $U_1 \cap (u_2') \neq \emptyset$*, then we may choose* P_b *and the path* $a_1, \ldots, a_p$ *such that either* $U_{a_p} \cap (u_2') = \emptyset$ *when there exists a bad path starting with a divisor from* $B \setminus \{u_2, u_2', \ldots, u_4, u_4'\}$ *of* c_2'*, or otherwise* $u_2' \in T_{a_p}$ *and* $c_2' = h(u_2')$*.*

Proof: Let $a_1, \ldots, a_e$ be a weak path, $m_j = h(a_j)$, $j \in [e]$ such that $m_e = m$. If $a_e = \tilde{a}$ then take $b_2 = a_e$. If $a_e \neq \tilde{a}$ but there exists $1 \leq v < e$ such that $a_v = \tilde{a}$. Then we may replace in P_b the intervals $[a_p, m_p], v \leq p \leq e$ with the intervals $[a_v, m_e], [a_{p+1}, m_p], v \leq p < e$. The old m_e becomes the new m_v, that is we reduce to the above case when $v = e$.

Now assume that there exist no such v but there exists a path $a_{e+1} = \tilde{a}, \ldots, a_l$ such that $m_l = h(a_l) \in (a_{v'})$ for some $v' \in [e]$. Then we replace in P_b the intervals $[a_j, m_j], v' \leq j \leq l$ with the intervals $[a_{v'}, m_l]$, $[a_{j+1}, m_j]$, $v' \leq j < l$. The new m_{e+1} is the old m_e but the new a_{e+1} is the old a_{e+1} and we may proceed as above.

Finally, suppose that no path starting with a_{e+1} contains an element from $\{a_1, \ldots, a_e\}$. Taking $p = e+1$ we see that $m \notin U_{a_p} \cap (u_2)$. If there exists another monomial m' like m then we repeat this procedure and after a while we may get (2), or (3).

Remains to see what happens when we have also $U_{a_p} \cap (u_2') \neq \emptyset$. Assume that there exist no bad path starting with a divisor of c_2' from $B \setminus \{u_2, u_2', \ldots, u_4, u_4'\}$. Then changing in P_b the intervals $[b_2, h(b_2]$, $[f_2, c_2']$ with $[f_2, h(b_2)]$, $[u_2', c_2']$ we see that there exists a path $a_1, \ldots, a_k$, which is not bad, such that the old $u_2' = a_k$. We may complete T_{a_p} such that $a_k \in T_{a_p}$ and all divisors from B of c_2' which are not in $\{u_2, b_2, u_3, u_3', u_4, u_4'\}$ belong to T_{a_p}. For this aim we complete T_{a_p} with the elements connected by a path with u_2' (see Example 5).

Next suppose that there exists a bad path $a_k = u_2', \ldots, a_l$ with $h(a_l) \in (b)$. We may assume that $\tilde{P}_b$ is given by Lemma 8 and so there exist no multiple of b in $U_1 \cap (u_2, u_2', u_3, u_3', u_4, u_4')$. Note that $u_2'' = b_2$ the new u_2' considered above has no multiple in $U_1 \cap (b)$ because $b_2 \in U_1$. By Lemma 6 there exists $a_{l+1} \in B \setminus \{b, u_2, u_2'', u_3, u_3', u_4, u_4'\}$ dividing $h(a_l)$ such that every path $a_{l+1}, \ldots, a_{l_1}$ satisfies $\{a_1, \ldots, a_l\} \cap \{a_{l+1}, \ldots, a_{l_1}\} = \emptyset$. Using Remark 3 if necessary we have $T_{a_{p'}} \cap \{a_1, \ldots, a_{p'-1}\} = \emptyset$ for some $p' > l$, and the above situation will not appear, that is the old u_2' will not divide anymore a monomial from $U_{a_{p'}} \cap (u_2, u_2'', u_3, u_3', u_4, u_4')$. It is also possible that u_2 will not divide a monomial from $U_{a_{p'}}$. □

The following bad example is similar to [9, Example 3.3].

Example 5. Let $n = 7$, $r = 4$, $d = 1$, $f_i = x_i$ for $i \in [4]$, $E = \{x_5x_6, x_5x_7\}$, $I = (x_1, \ldots, x_4, E)$ and

$$J = (x_1x_7, x_2x_4, x_2x_6, x_2x_7, x_3x_6, x_3x_7, x_4x_6, x_4x_7, x_3x_4x_5).$$

Then $B = \{x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_1x_6, x_2x_3, x_2x_5, x_3x_4, x_3x_5, x_4x_5\} \cup E$ and

$$C = \{x_1x_2x_3, x_1x_2x_5, x_1x_3x_4, x_1x_3x_5, x_1x_4x_5, x_1x_5x_6, x_2x_3x_5, x_5x_6x_7\}.$$

We have $q = 8$ and $s = q + r = 12$. Take $b = x_1x_6$ and $I_b = (x_2, x_3, x_4, B \setminus \{b\}, E)$, $J_b = I_b \cap J$. There exists a partition P_b with sdepth 3 on I_b/J_b given by the intervals $[x_2, x_1x_2x_3]$, $[x_3, x_1x_3x_4]$, $[x_4, x_1x_4x_5]$, $[x_1x_5, x_1x_3x_5]$, $[x_2x_5, x_1x_2x_5]$, $[x_3x_5, x_2x_3x_5]$, $[x_5x_6, x_1x_5x_6]$, $[x_5x_7, x_5x_6x_7]$. We have $c_2' = x_1x_2x_3$, $c_3' = x_1x_3x_4$, $c_4' = x_1x_4x_5$ and $u_2 = x_1x_2$, $u_2' = x_2x_3$, $u_3 = x_3x_4$, $u_3' = x_1x_3$, $u_4 = x_1x_4$, $u_4' = x_4x_5$. Take $a_1 = x_1x_5$, $a_2 = x_3x_5$, $a_3 = x_2x_5$. This gives a maximal weak path but not bad and defines $T_1 = \{x_1x_5, x_3x_5, x_2x_5\}$, $U_1 = \{x_1x_3x_5, x_2x_3x_5, x_1x_2x_5\}$.

As in the above lemma we may change in P_b the intervals $[x_2, x_1x_2x_3]$, $[x_2x_5, x_1x_2x_5]$ with $[x_2, x_1x_2x_5]$, $[x_2x_3, x_1x_2x_3]$. Note that the old u_2' is not anymore in $[f_2, c_2']$ and divides $x_2x_3x_5 \in U_1$. Moreover, we have the path $\{a_1, x_1x_5, x_3x_5, x_2x_3\}$ and so we must take $T_1' = (T_1 \cup \{x_2x_3\}) \setminus \{x_2x_5\}$, $U_1' = (U_1 \cup \{x_1x_2x_3\}) \setminus \{x_1x_2x_5\}$ as it is hinted in the above proof. The new u_2, u_2' are all divisors of $x_1x_2x_5$ - the new c_2', which are not in T_1'. However, this change of P_b was not necessary because the new u_2, u_2', u_3' are all divisors from B of the old c_2' (see Remark 7 and Example 6). The same thing is true for c_3' and c_4' has all divisors from B among $\{a_1, u_4, u_4'\}$.

Remark 4. Suppose that in Lemma 9 the partition $\tilde{P}_b$ satisfies also the property (1) mentioned in Lemma 4. If $\tilde{a} = w_{2i}$ for some $i = 3, 4$ then $m \notin (u_i, u_i')$. In particular $b_2 \neq w_{23}, w_{24}$.

Lemma 10. *Assume that $U_{a_p} \cap (u_2) \neq \emptyset$ and a monomial m of $U_{a_p} \cap (u_2)$ has all its divisors from $B \cap (f_2)$ contained in $\{u_2, u_2', \ldots, u_4, u_4'\}$. Then one of the following statements holds:*

1. *m has a divisor $\tilde{a}_i \in (B \cap (f_i)) \setminus \{u_2, u_2', \ldots, u_4, u_4'\}$ for some $i = 3, 4$,*
2. *$m \in C_3 \setminus W$ and it is the least common multiple of f_2, f_3, f_4.*

Proof: There exists a divisor $\hat{a} \notin \{u_2, u_2'\}$ of m from $B \cap (f_2)$, otherwise $m = c_2'$. By our assumption we have let us say $\hat{a} = u_3 = w_{23}$. Then there exists a divisor $a' \neq u_3$ from $B \cap (f_3)$. If $a' \notin \{u_2, u_2', \ldots, u_4, u_4'\}$ then we are in (1). Otherwise, $a' = u_4 = w_{34}$. If $m \in W$ then $m = w_{24} \in C_2$ and there exists a divisor of m from $(B \cap (f_4)) \setminus \{u_2, u_2', \ldots, u_4, u_4'\}$, that is (1) holds. Thus we may suppose that $m \notin W$ and all its divisors from $B \setminus E$ are w_{23}, w_{34}, w_{24}, that is m is in (2). □

Remark 5. Assume that in the above lemma m has the form given in Example 1. Then $m \notin \{c_2', c_3', c_4'\}$ and so necessarily w_{12}, w_{13}, w_{14} are divisors of m from $B \setminus \{u_2, u_2', \ldots, u_4, u_4'\}$, that is m is in case (1).

Lemma 11. *Suppose that* $\operatorname{sdepth}_S I/J \leq d+1$ *and $\tilde{P}_b$ is a partition of I_b/J_b given by Lemma 8. Assume that $\tilde{P}_b$ satisfies also the properties mentioned in Lemma 4 and no bad path starts with a_1. Then there exist a partition P_b which satisfies the properties mentioned in Lemma 4 and a (possible bad) path $a_1, \ldots, a_p$ such that $T_{a_p} \cap \{a_1, \ldots, a_{p-1}\} = \emptyset$, no bad path starts with a_p, and for every $i = 2, 3, 4$*

such that there exists a divisor $\tilde{a}_i$ in $(B\cap(f_i))\setminus\{u_2,u_2',\ldots,u_4,u_4'\}$ of a monomial from $U_1\cap(u_i)$, one of the following statements holds:

1. $U_{a_p}\cap(u_i)=\emptyset$,
2. $U_{a_p}\cap(u_i)\neq\emptyset$ *and there exists* $b_i\in T_{a_p}\cap(f_i)$ *with* $h(b_i)\in(u_i)$,
3. $U_{a_p}\cap(u_i)\neq\emptyset$ *and every monomial of* $U_{a_p}\cap(u_i)$ *has all its divisors from* $B\cap(f_i)$ *contained in* $\{u_2,u_2',\ldots,u_4,u_4'\}$.

Moreover, these possible b_i are different and if for some $i=2,3,4$ it holds also $U_1\cap(u_i')\neq\emptyset$, then we may choose P_b and the path $a_1,\ldots,a_p$ such that either $U_{a_p}\cap(u_i')=\emptyset$ when there exists a bad path starting with a divisor from $B\setminus\{u_2,u_2',\ldots,u_4,u_4'\}$ of c_i', or otherwise $u_i'\in T_{a_p}$ and $h(u_i')$ is the old c_i'.

Proof: Suppose that there exists a divisor $\tilde{a}_2$ in $(B\cap(f_2))\setminus\{u_2,u_2',\ldots,u_4,u_4'\}$ of a monomial from $U_1\cap(u_2)$ with respect of $\tilde{P}_b$. Using Lemma 9 we find a partition P_b and a (possible bad) path $a_1,\ldots,a_{p_1}$ such that $T_{a_{p_1}}\cap\{a_1,\ldots,a_{p_1-1}\}=\emptyset$, no bad path starts with a_{p_1} and one of the following statements holds:

j_2) $U_{a_{p_1}}\cap(u_2)=\emptyset$,

j_2') $U_{a_{p_1}}\cap(u_2)\neq\emptyset$ and there exists $b_2\in T_{a_{p_1}}\cap(f_2)$ with $h(b_2)\in(u_2)$,

j_2'') $U_{a_{p_1}}\cap(u_2)\neq\emptyset$ and every monomial of $U_{a_{p_1}}\cap(u_2)$ has all its divisors from $B\cap(f_2)$ contained in $\{u_2,u_2',\ldots,u_4,u_4'\}$.

Moreover, if also $U_1\cap(u_2')\neq\emptyset$, then we may choose P_b and the path $a_1,\ldots,a_{p_1}$ such that either $U_{a_{p_1}}\cap(u_2')=\emptyset$ when there exists a bad path starting with a divisor from $B\setminus\{u_2,u_2',\ldots,u_4,u_4'\}$ of c_2', or otherwise $u_2'\in T_{a_{p_1}}$ and $c_2'=h(u_2')$. After a small change we may suppose that P_b satisfies the properties of Lemma 4 and so $b_2\neq w_{23},w_{24}$.

If $U_{a_{p_1}}\cap(u_3,u_4)=\emptyset$ then we are done. Now assume that there exists a divisor $\tilde{a}_3$ in $B\cap(f_3)\setminus\{u_2,u_2',\ldots,u_4,u_4'\}$ of a monomial $m\in U_{a_{p_1}}\cap(u_3)$, let us say $m=m_e$ for some path $a_{p_1},\ldots,a_e$. If $a_e=\tilde{a}_3$, or $a_e\neq\tilde{a}_3$ but there exists a path $a_{e+1}=\tilde{a}_3,\ldots,a_k$ with $a_k=a_v$ for some $v\leq e$ then we change P_b as in the proof of Lemma 9 to replace c_3' by m. Clearly, c_2',c_3' satisfy (2) for $i=2,3$. Otherwise, if $a_e\neq\tilde{a}_3$ but there exists no path $a_{e+1}=\tilde{a}_3,\ldots,a_k$ with $a_k=a_v$ for some $v\leq e$, apply again the quoted lemma with c_3'. We get a (possible bad) path $a_{p_1},\ldots,a_{p_2}$ with $p_2>p_1$ such that $T_{a_{p_2}}\cap\{a_1,\ldots,a_{p_2-1}\}=\emptyset$, no bad path starts with a_{p_2} and one of the following statements holds:

j_3) $U_{a_{p_2}}\cap(u_3)=\emptyset$,

j_3') $U_{a_{p_2}}\cap(u_3)\neq\emptyset$ and there exists $b_3\in T_{a_{p_2}}\cap(f_3)$ with $h(b_3)\in(u_3)$,

j_3'') $U_{a_{p_2}}\cap(u_3)\neq\emptyset$ and every monomial $m\in U_{a_{p_2}}\cap(u_3)$ has all its divisors from $B\cap(f_3)$ contained in $\{u_2,u_2',\ldots,u_4,u_4'\}$.

If we also have $U_1\cap(u_3')\neq\emptyset$ then it holds a similar statement as in case $i=2$. Note that $b_2\neq b_3$ since $b_2\neq w_{23}$ by Remark 4 and so $h(b_2)\neq h(b_3)$. Very likely meanwhile the corresponding statements of j_2), j_2'), j_2'') do not hold anymore because we could have $b_2\notin T_{a_{p_2}}$. If there exists another $\tilde{a}_2$ we apply again Lemma 9 with c_2' obtaining a new partition P_b and a path $a_{p_2},\ldots,a_{p_3}$ for which this situation is repaired. If now c'_3 does not satisfy (2) then the procedure could continue with c'_3 and so on. However, after a while we must get a path $a_1, \dots, a_{p_{23}}$ such that $T_{a_{p_{23}}} \cap \{a_1, \dots, a_{p_{23}-1}\} = \emptyset$, no bad path starts with $a_{p_{23}}$ and for every $i = 2, 3$ one of the following statements holds:

j_{23}) $U_{a_{p_{23}}} \cap (u_i) = \emptyset$,

j'_{23}) $U_{a_{p_{23}}} \cap (u_i) \neq \emptyset$ there exist $b_i \in T_{a_{p_{23}}} \cap (f_i)$ with $h(b_i) \in (u_i)$,

j''_{23}) $U_{a_{p_{23}}} \cap (u_i) \neq \emptyset$ and every monomial $m \in U_{a_{p_{23}}} \cap (u_i)$ has all its divisors from $B \cap (f_i)$ contained in $\{u_2, u'_2, \dots, u_4, u'_4\}$.

We end the proof applying the same procedure with c'_4 together with c'_2, c'_3 and if necessary Lemma 4. □

Remark 6. Using the properties (2), (3) mentioned in Lemma 4 we may have $b_i = w_{1i}$, for some $2 \leq i \leq 4$ only if $u_i, u'_i \in W$. Thus, let us say $b_2 = w_{12}$ only if $\{u_2, u'_2\} = \{w_{23}, w_{24}\}$. Then $\{u_i, u'_i\} \not\subset W$ for $i = 3, 4$ and so $b_3 \neq w_{13}$, $b_4 \neq w_{14}$, in case b_3, b_4 are given by Lemma 11. Therefore at most one from b_i could be w_{1i}.

The idea of the proof of Proposition 1 fails in a special case hinted by Example 4. This case is solved directly by the following lemma.

Lemma 12. *Suppose that $b = x_j f_1$ and $(B \setminus E) \subset W \cup \{x_j f_1, x_j f_2, x_j f_3, x_j f_4\}$ for some $j \notin \operatorname{supp} f_1$. Then $\operatorname{depth}_S I/J \leq d + 1$.*

Proof: If $|B \setminus E| < 2r = 8$ then $\operatorname{depth}_S I''/J'' \leq 2$ by [18, Theorem 2.4]. Assume that $|B \setminus E| \geq 8$. Our hypothesis gives $|B \cap W| \geq 4$. First assume that $5 \leq |B \cap W| \leq 6$ and we get that let us say $f_i = v x_i$, $1 \leq i \leq 4$ for some monomial v of degree $d - 1$ (see the proof of [16, Lemma 3.2]). Then

$$\operatorname{depth}_S I/J = \deg v + \operatorname{depth}_{S'}((I : v) \cap S')/((J : v) \cap S'),$$

$S' = K[\{x_i : i \in ([n] \setminus \operatorname{supp} v)\}]$ and it is enough to show the case $v = 1$, that is $d = 1$.

We may assume that $f_i = x_i$, $i \in [4]$ and $j = 5$ since $b \notin W$. It follows that $(B \setminus E) \subset W \cup \{b, x_2x_5, x_3x_5, x_4x_5\}$. Set $I'' = (x_1, \dots, x_4)$, $J'' = J \cap I''$. Note that $J \supset (x_1, \dots, x_5)(x_6, \dots, x_n)$ and so $\operatorname{depth}_S I''/J'' = \operatorname{depth}_{S''}(I'' \cap S'')/(J'' \cap S'')$ for $S'' = K[x_1, \dots, x_5]$.

Then $J'' \cap S''$ is generated by at most two monomials and so $\operatorname{depth}_{S''} S''/(J'' \cap S'') \geq 3$. Since $\operatorname{depth}_{S''} S''/(I'' \cap S'') = 1$ it follows that $\operatorname{depth}_S I''/J'' = \operatorname{depth}_{S''}(I'' \cap S'')/(J'' \cap S'') = 2$. Therefore $\operatorname{depth}_S I/J \leq 2$ either when $E = \emptyset$ or by the Depth Lemma since $I/(J, I'')$ is generated by monomials of E which have degrees 2.

Now assume that $|B \cap W| = 4$, let us say $B \cap W = \{w_{14}, w_{23}, w_{24}, w_{34}\}$. Then we may suppose that $f_i = v x_i x_6$, $2 \leq i \leq 4$ and $f_1 = v x_1 x_4$ for some monomial v of degree $d - 2$. As above we may assume that $v = 1$ and $n = 6$. If $j = 6$ then $b = w_{14}$ which is impossible. If let us say $j = 2$ then $(B \setminus E) \subset W \cup \{b, x_2x_3x_6, x_2x_4x_6\}$ and so $|B \setminus E| < 8$, which is false.

Thus $j \not\in \{1,\ldots,4,6\}$ and we may assume that $j=5$. It follows that $J \subset (x_1x_2x_6, x_1x_3x_6, x_1x_2x_4, x_1x_3x_4)$, the inclusion being strict only if $|B\setminus E| < 8$ which is not the case. Thus $J = (x_1x_2x_6, x_1x_3x_6, x_1x_2x_4, x_1x_3x_4)$ and a computation with SINGULAR shows that $\operatorname{depth}_S I/J = 3$ in this case. $\square$

Next we put together the above lemmas to get the proof of Proposition 1. Assume that $\operatorname{sdepth}_S I/J \leq d+1$. We may suppose always that P_b satisfies the properties mentioned in Lemma 4. Applying Lemma 8 and Remark 3 and changing a_1 if necessary we may suppose that no bad path starts from a_1. By Lemma 11 changing a_1 by a_p we may suppose that for every $i=2,3,4$ one of the following statements holds

1) $U_1 \cap (u_i) = \emptyset$,

2) $U_1 \cap (u_i) \neq \emptyset$ and there exists $b_i \in T_1 \cap (f_i)$ with $h(b_i) \in (u_i)$,

3) $U_1 \cap (u_i) \neq \emptyset$ and every monomial of $U_1 \cap (u_i)$ has all its divisors from $B \cap (f_i)$ contained in $\{u_2, u_2', \ldots, u_4, u_4'\}$.

Mainly we study case 3) the other two cases are easier as we will see later. Suppose that $U_1 \cap (u_2) \neq \emptyset$ and every monomial of $U_1 \cap (u_2)$ has all its divisors from $B \cap (f_2)$ contained in $\{u_2, u_2', \ldots, u_4, u_4'\}$. Let $m \in U_1 \cap (u_2)$, let us say $m = h(a_e)$ for some path $a_1, \ldots, a_e$. be as in case 3). We may suppose that $U_1 \cap (u_2') = \emptyset$ because otherwise we may assume as in Lemma 9 that all divisors of c_2' are in the enlarged T_1' of T_1 and so c_2' is preserved. As in the proof of Lemma 10 one of the following statements holds:

1′) $U_1 \cap (u_2) = \{m\}$, $m \in (u_2) \cap (u_3)$, $u_3 = w_{23}$, $m \not\in (u_4, u_4')$ and there exists $\tilde{a}_3 \in T_1 \cap (f_3)$ dividing m with $\tilde{a}_3 = a_e$,

2′) $U_1 \cap (u_2) = \{m\}$, $m \in (u_2) \cap (u_3)$, $u_3 = w_{23}$, $m \not\in (u_4, u_4')$ and there exists $\tilde{a}_3 \in T_1 \cap (f_3)$ dividing m with $\tilde{a}_3 \neq a_e$,

3′) $U_1 \cap (u_2) = \{m\}$, $m \in (u_2) \cap (u_4)$, $u_4 = w_{24}$, $m \not\in (u_3, u_3')$ and there exists $\tilde{a}_4 \in T_1 \cap (f_4)$ dividing m with $\tilde{a}_4 = a_e$,

4′) $U_1 \cap (u_2) = \{m\}$, $m \in (u_2) \cap (u_4)$, $u_4 = w_{24}$, $m \not\in (u_3, u_3')$ and there exists $\tilde{a}_4 \in T_1 \cap (f_4)$ dividing m with $\tilde{a}_4 \neq a_e$,

5′) $m = w_{24} \in (u_2) \cap (u_3) \cap (u_4)$, $u_3 = w_{23}$, $u_4 = w_{34}$ and there exists $\tilde{a}_4 \in T_1 \cap (f_4)$ dividing m with $h(\tilde{a}_4) = m$,

6′) $m = w_{24} \in (u_2) \cap (u_3) \cap (u_4)$, $u_3 = w_{23}$, $u_4 = w_{34}$ and there exists $\tilde{a}_4 \in T_1 \cap (f_4)$ dividing m with $h(\tilde{a}_4) \neq m$,

7′) $m = \omega_1 \in C_3$, $u_2 = w_{24}$, $u_3 = w_{23}$.

In subcase 1′) change in P_b the intervals $[f_3, c_3']$, $[\tilde{a}_3, m]$ with $[f_3, m]$, $[u_3', c_3']$. The new $T_1'' = T_1 \setminus \{\tilde{a}_3\}$ corresponds to $U_1'' = U_1 \setminus \{m\}$ which has empty intersection with (u_2) by our assumption. If T_1'' is not empty then we may go on with T_1'' instead T_1, the advantage being that now we have no problem with u_2. If $T_1'' = \emptyset$ then $e=1$ and the path a_1 is maximal. Since $m \not\in (u_4, u_4')$ we must have $u_2 = x_k f_2$ for some k (we can also have $w_{12} = x_k f_2$) and so $m = x_k w_{23}$, $\tilde{a}_3 = x_k f_3$. If $E \neq \emptyset$ then we may change a_1 by a monomial of E. Assume that $E = \emptyset$. If $c_3' = x_t w_{23}$ for some t then $x_t f_2 \in B$ since it divides c_3'. If $t = k$ then $m = c_3'$. Thus $t \neq k$, $x_t f_2 \not\in \{b, u_2, u_2', \ldots, u_4, u_4'\}$ and we may change a_1

by $x_t f_2$ and the new T_1'' will be not empty. If $c_3' \in C_2$ we may find also a divisor $b' \in B \setminus \{b, u_2, u_2', \ldots, u_4, u_4'\}$ dividing c_3' and changing a_1 by b' we will get the new T_1'' not empty. Remains to assume that $c_3' \in C_3$. Then $u_3' = w_{34}$ and $b'' = w_{24}$ is either in $\{u_2', u_4, u_4'\}$, or we may change a_1 by b'' as above. Suppose that $u_2' = w_{24}$. Then $x_k f_4 \in B$. If $x_k f_4 \notin \{b, u_2, u_2', \ldots, u_4, u_4'\}$ we may change a_1 by $x_k f_4$. Otherwise, let us say $u_4 = x_k f_4$ and $c_4' = x_k w_{14}$. We get $x_k f_1 \in B \setminus \{u_2, u_2', \ldots, u_4, u_4'\}$ and if $b \neq x_k f_1$ then we may change as above a_1 by $x_k f_1$. If $b = x_k f_1$ then note that $B \supset \{w_{23}, w_{24}.w_{34}, w_{14}, b, x_k f_2, x_k f_3, x_k f_4\}$. If there exists a monomial $b' \in B \setminus (W \cup \{b, x_k f_2, x_k f_3, x_k f_4\})$ then change a_1 by b'. Otherwise $B \subset W \cup \{b, x_k f_2, x_k f_3, x_k f_4\}$ and we apply Lemma 12.

Therefore in this subcase changing P_b (u_3 is preserved and the new u_3' is b_3) and passing from T_1 to T_1'' there exist no problem with u_2. As in Lemma 9 we may suppose that only one from $U_1'' \cap (u_3)$, $U_1'' \cap (u_3')$ is nonempty because otherwise we preserve the new c_3', that is m. If let us say $U_1'' \cap (u_3) = \{m'\}$, and all divisors of m' from $B \cap (f_3)$ are contained in $\{u_3, u_3', u_4, u_4'\}$ then $m' \in (u_3) \cap (u_4)$, $u_4 = w_{34}$ and there exists $\tilde{a}_4 \in T_1'' \cap (f_4)$ dividing m'. If $h(\tilde{a}_4) = m'$ then as above change in P_b the intervals $[f_4, c_4']$, $[\tilde{a}_4, m']$ with $[f_4, m']$, $[u_4', c_4']$. Clearly $\tilde{T}_1 = T_1'' \setminus \{\tilde{a}_4\}$ has empty intersection with (u_3) and similarly to above we may suppose that $\tilde{T}_1 \neq \emptyset$. In this way we arrive to the situation when we will not meet case 3) for $2 \leq i \leq 4$.

In subcase $2'$) we have $a_e \in E$ and $a_{e+1} = \tilde{a}_3 \in T_1$. Take $T_{a_{e+1}}$ instead T_1. If a_e will not appear anymore in $T_{a_{e+1}}$ then $U_{a_{e+1}} \cap (u_2) = \emptyset$ and the problem is solved. Otherwise, if $a_v = a_e$ for some $v > e+1$ then change in P_b the intervals $[a_i, h(a_i)]$, $e \leq i \leq v$ with $[a_{i+1}, h(a_i)]$, $e \leq i < v$, $[a_e, m_v]$ we see that the new a_e is the old a_{e+1}, that is we reduced to the subcase $1'$). Subcases $3'$), $4'$) are similar to $1'$), $2'$).

Change in subcase $5'$) (as in subcase $1'$)) the intervals $[f_4, c_4']$, $[\tilde{a}_4, m]$ of P_b with $[f_4, m]$, $[u_4', c_4']$. The new $T_1'' = T_1 \setminus \{\tilde{a}_4\}$ corresponds to $U_1'' = U_1 \setminus \{m\}$ which has empty intersection with (u_2) by our assumption. The proof continues as in $1'$). Similarly, $6'$) goes as $2'$).

In subcase $7'$) if $\omega_1 \in W$ (see Example 1) then it has 4 divisors from $B \setminus E$ and so one of them is not in $\{u_2, u_2', \ldots, u_4, u_4'\}$ and we may proceed as in subcases $5'$), $6'$). So we may assume that $\omega_1 \notin W$. Then either $u_4 = w_{34}$ and then $a_e \in E$ which is false by our assumption, or $w_{34} \in T_1$. Set $a_{e+1} = w_{34}$. We proceed as in $2'$) taking $T_{a_{e+1}}$ if $a_e \notin T_{a_{e+1}}$ or otherwise changing P_b we reduce to the situation when $h(a_{e+1}) = m$. Then change in P_b the intervals $[f_4, c_4']$, $[a_{e+1}, m]$ with $[f_4, m]$, $[u_4', c_4']$ and as usual the new $U_1'' = U_1 \setminus \{m\}$ has empty intersection with (u_2).

Thus we may assume that for all $2 \leq i \leq 4$ we are in cases 1), 2). When we are in case 2) there exists $b_i \in T_1 \cap (f_i)$ with $h(b_i) \in (u_i)$ and we may consider the intervals $[f_i, c_i']$, which are disjoint since b_i are different by Lemma 11. Moreover, they contain at most one monomial from w_{12}, w_{13}, w_{14} by Remark 6, which is useful next. Remains to study those i with $U_1 \cap (f_i) \neq \emptyset$ but $U_1 \cap (u_i, u_i') = \emptyset$. If $U_1 \cap (u_2, u_2', \ldots, u_4, u_4') = \emptyset$ then we apply Lemma 5. Suppose that $U_1 \cap (f_2) \neq \emptyset$

and $U_1 \cap (u_2, u_2') = \emptyset$ but we found already b_3 and possible b_4 as in 2). If $h(b_3) \notin (f_2)$ then choosing $b' \in B \cap (f_2)$ we see that the intervals $[f_2, h(b')]$, $[f_3, h(b_3)]$ are disjoint. A similar result holds if there exists b_4 and $h(b_4) \notin (f_2)$.

Assume that $h(b_3) \in (f_2)$. Then we may suppose that $u_3 = w_{23}$ and $h(b_3) = x_k w_{23}$ for some $k \in [n] \setminus \operatorname{supp} w_{23}$. We claim that $b'' = x_k f_2 \notin \{u_2, u_2', \ldots, u_4, u_4'\}$. It is clear that $b'' \notin \{u_2, u_2', u_3, u_3'\}$. If $b'' \in \{u_4, u_4'\}$ then $b'' = w_{24} = u_4$, let us say. Thus $h(b_3) \in (u_3, u_4)$ but $h(b_3) \notin (u_2, u_2')$. This means that the monomial $h(b_3) \in U_1 \cap (u_4)$ is in the situation 3) (similarly to $1'$)) which is not possible as we assumed. This shows our claim.

Therefore, $b'' \in T_1 \cap (f_2)$ because it divides $h(b_3)$. If $h(b'') \in (f_3)$ then $h(b'') = kw_{23} = h(b_3)$ which is impossible. If $h(b'') \in (f_4)$ then $h(b'') = x_t w_{24}$ for some t. As we saw above $b'' \neq w_{24}$ and so $t = k$. If b_4 is not done by 2) then it is enough to note that the intervals $[f_2, h(b'')]$, $[f_3, h(b_3)]$ are disjoint. Assume that b_4 is given already from 2) and $u_4 = w_{24}$. Then $\tilde{b} = x_k f_4 \neq u_4'$ because otherwise $h(b'') = h(b_4)$. We see that $\tilde{b} \notin \{u_2, u_2', \ldots, u_4, u_4'\}$ and so $\tilde{b}$ is in $T_1 \cap (f_4)$. But $h(\tilde{b}) \notin (u_4)$ because it is different of $h(b_4)$. Then the intervals $[f_2, h(b'')]$, $[b_3, h(b_3)]$, $[f_4, h(\tilde{b})]$ are disjoint. As in Lemma 5 we find if necessary an interval $[f_1, c]$ disjoint of the rest.

Suppose as in Lemma 5 that $[r] \setminus \{j \in [r] : U_1 \cap (f_j) \neq \emptyset\} = \{k_1, \ldots, k_\nu\}$ for some $1 \leq k_1 < \ldots < k_\nu \leq 4$, $0 \leq \nu \leq 4$. Set $I' = (f_{k_1}, \ldots, f_{k_\nu}, G_1)$, $J' = I' \cap J$, With the help of the above disjoint intervals, P_b induces on $I/(I', J)$ a partition P_b' with sdepth $d+2$. It follows that $\operatorname{sdepth}_S I'/J' \leq d+1$ using [17, Lemma 2.2]. By Lemma 3 we get $\operatorname{depth}_S I/(J, I') \leq d+1$ and we are done. □

Remark 7. Note that in P_b', all divisors from B of the new c_i' are in $T_1 \cup \{u_2, u_2', \ldots, u_4, u_4'\}$. If one old c_i' has already this property then we may keep it.

Remark 8. If $\omega_1 \in (C_3 \setminus W) \cap (E)$ then we may have indeed a problem. For example, if $u_2 = w_{24}$, $u_3 = w_{23}$, $u_4 = w_{34}$, $\omega_1 = h(a_1)$ for some $a_1 \in E$ but $\omega_1 \notin h(E \setminus \{a_1\})$ then the path a_1 is maximal, $T_1 = \{a_1\}$ and our theory fails to solve this case if we cannot change P_b in order to have $\{u_2, u_3, u_4\} \neq \{w_{24}, w_{23}, w_{34}\}$.

Example 6. We continue Example 5. If we take as in the above proof $I' = (b, x_5x_6, x_5x_7)$ and $J' = I' \cap J$ we have the disjoint intervals $[x_i, c_i']$, $2 \leq i \leq 4$ and to conclude that h induces a partition on $I/(I', J)$, which has sdepth 3 we need an interval $[x_1, c_1']$ disjoint of the other ones. But this is hard because there are too many w_{1i} among $\{u_2, u_2', \ldots, u_4, u_4'\}$. We must change one c_i' with one $m \in (U_1 \cap (x_i)) \setminus (x_1)$. The only possibility is to take $m_2 = x_2x_3x_5$. Since $m \in (u_2') \setminus (u_3, u_3', u_4, u_4')$ we may change somehow c_2' with m. This is not easy since $m_2 = h(a_2)$, $a_2 = x_3x_5 \notin (x_2)$. As in Lemma 9 note that $a_1 | m_3 = h(a_3)$ and replacing in P_b the intervals $[a_i, m_i]$, $i \in [3]$, $m_1 = h(a_1)$ with the intervals $[a_1, m_3]$, $[a_2, m_1]$, $[a_3, m_2]$ we see that x_2x_5 - the new a_2, belongs to (x_2). Thus we may change in P_b the intervals $[x_2, c_2']$, $[x_2x_5, m_2]$ with $[x_2, m_2]$, $[u_2, c_2']$. The new T_1 is $T_1' = (T_1 \cup \{x_1x_2\}) \setminus \{x_2x_5\}$. Note that all divisors from $B \cap (x_2)$ of the new c_2' which are different from the new u_2, u_2' are contained in the new T_1. As above $[x_i, c_i']$ are disjoint intervals and changing in P_b the intervals $[x_1x_2, x_1x_2x_3]$, $[x_1x_5, x_1x_2x_5]$ with $[x_1, x_1x_2x_5]$ we get a partition with sdepth 3 on $I/(I', J)$.

3 Main results

We start with an elementary lemma closed to Lemma 12.

Lemma 13. *Let r be arbitrarily chosen, $r' \leq r$, $t \in [n] \setminus \cup_{i=1}^{r'} \operatorname{supp} f_i$ and $I' = (f_1, \ldots, f_{r'})$, $J' = J \cap I'$. Suppose that all w_{ij}, $1 \leq i < j \leq r'$ are in B and different. Then the following statements hold*

1. *there exists a monomial v of degree $d-1$ such that $f_i \in (v)$ for all $i \in [r']$,*
2. *if $x_k(f_1, \ldots, f_{r'}) \subset J$ for all $k \in [n] \setminus (\{t\} \cup (\cup_{i=1}^{r'} \operatorname{supp} f_i))$ then $\operatorname{depth}_S I'/J' \leq d+1$.*

Proof: As in the proof of [16, Lemma 3.2] we may suppose that $f_i = vx_i$ for $i \in [r]$ and some monomial v of degree $d-1$, that is (1) holds. It follows that

$$\operatorname{depth}_S I'/J' = d - 1 + \operatorname{depth}_{S''}(x_1, \ldots, x_{r'})S'' = d + 1$$

where $S'' = K[x_1, \ldots, x_{r'}, x_t]$. □

Theorem 3. *Conjecture 1 holds for $r \leq 4$, the case $r \leq 3$ being given in Theorem 1.*

Proof: Suppose that $\operatorname{sdepth}_S I/J = d+1$ and $E \neq \emptyset$, the case $E = \emptyset$ is given in Proposition 2. The proofs of Proposition 1 and Proposition 2 show that we get $\operatorname{depth}_S I/J \leq d+1$, that is Conjecture 1 holds, when we may choose $b_i \in (B \cap (f_i)) \setminus W$ such that $\omega_i \notin (C_3 \setminus W) \cap (E)$. Suppose that we choose $b_1 \in (B \cap (f_1)) \setminus W$ but $\omega_1 \in (C_3 \setminus W) \cap (E)$. In the last part of the proof of Proposition 1 (see 7′) and also Remark 8) a problem appears when $m = \omega_1 \in T_1$ and let us say $u_2 = w_{24}$, $u_3 = w_{23}$, $u_4 = w_{34}$. As in the proof of [16, Lemma 3.2] we may assume that $f_i = vx_i$ for $2 \leq i \leq 4$ and some monomial v of degree $d-1$. If let us say $x_t f_2 \in B$ for some $t \notin \cup_{i=2}^{4} \operatorname{supp} f_i$ then either $tf_2 = w_{12}$, or $tf_2 \notin W$. In the first case we may suppose, as in the proof of Lemma 12, that one of the following statements hold:

1) $f_i = vx_i$, $i \in [4]$ for some monomial v of degree $d-1$,

2) $f_i = px_ix_5$, $2 \leq i \leq 4$, $f_1 = px_1x_2$ for some monomial p of degree $d-2$.

In both cases we see that if $B \cap (f_2, f_3, f_4) \subset W$ then we have $x_k(f_2, \ldots f_4) \subset J$ for all $k \in [n] \setminus (\{1\} \cup (\cup_{i=2}^{4} \operatorname{supp} f_i))$. By Lemma 13 we get $\operatorname{depth}_S I'/J' \leq d+1$ for $I' = (f_2, f_3, f_4)$, $J' = J \cap I'$ which gives $\operatorname{depth}_S I/J \leq d+1$ since $\operatorname{depth}_S I/(J, I') \geq d+1$, b being not in (J, I'). Thus $B \cap (f_2, f_3, f_4) \not\subset W$ and we may choose, let us say $b_2 \in (B \cap (f_2)) \setminus W$ and again we may get $\operatorname{depth}_S I/J \leq d+1$ if $\omega_2 \notin (C_3 \setminus W) \cap (E)$.

Thus we may assume that $\omega_1, \omega_2 \in (C_3 \setminus W) \cap (E)$. In particular $B \cap W$ consists in at least 5 different monomials and so we may suppose that 1) above holds and

$u'_2 = vx_2x_{k_2}$, $u'_3 = vx_3x_{k_3}$, $u'_4 = vx_4x_{k_4}$ for some $k_i \in ([n] \setminus (\{2,3,4\} \cup \operatorname{supp} v)$. If $k_2 = k_3 = k_4 = 1$ then $c'_2 = \omega_3$, $c'_3 = \omega_4$, $c'_4 = \omega_2$, that is all ω_i are in $C_3 \setminus W$. If let us say $k_3 > 4$ then $b'' = x_{k_3}f_3 \notin W$ and we are ready if $\omega_3 \notin (C_3 \setminus W) \cap (E)$. Thus we may assume that $\omega_3 \in (C_3 \setminus W) \cap (E)$. Consequently in all cases we may assume that 3 from ω_i are in $C_3 \setminus W$. In particular $|B \cap W| = 6$. If $B \cap (f_i) \subset W$ for some $i = 3,4$ then $(J : f_i)$ is generated by x_j with $j \notin (\{1,\dots,4\} \cup \operatorname{supp} v)$. It follows that in the exact sequence

$$0 \to (f_i)/J \cap (f_i) \to I/J \to I/(J,f_i) \to 0$$

the first term has depth $\deg v + 4 = d+3$ and sdepth $\geq d+2$. By [17, Lemma 2.2] we get $\operatorname{sdepth}_S I/(J,f_i) \leq d+1$ and so the last term in the above sequence has depth $\leq d+1$ by Theorem 1. Using the Depth Lemma we get $\operatorname{depth}_S I/J \leq d+1$ too.

Therefore, we may find $b_i \in (B \cap (f_i)) \setminus W$, $i = 3,4$ and as above we may suppose that $\omega_i \in (C_3 \setminus W) \cap (E)$, let us say $\omega_i \in (\tilde{a}_i)$ for some $\tilde{a}_i \in E$. We consider three cases depending on k_i.

Case 1, when $k_i = 1$ and $k_j > 4$ for some $i,j = 2,3,4$, $i \neq j$.

Assume that $k_2 = 1$, that is $c'_2 = \omega_3$ and $k_4 > 1$. Then $a_1 = vx_1x_4 \notin \{u_2,u'_2,\dots,u_4,u'_4\}$ is a divisor of c'_2. Start the usual proof with a_1 and if $\omega_1 \notin U_1$ then we get $\operatorname{depth}_S I/J \leq d+1$. Suppose that there exists a (possible bad) path $a_1,\dots,a_e$, $m_i = h(a_i)$ such that $m_e = \omega_1$. Changing in P_b the intervals $[a_i,m_i]$, $i \in [e]$, $[f_2,c'_2]$, $[f_3,c'_3]$ with $[a_{i+1},m_i]$, $i \in [e-1]$, $[f_1,c'_2]$, $[f_2,m_e]$, $[u'_3,c'_3]$ we see that the new $\tilde{c}'_i$, $i = 1,2,4$ contain two from ω_i. Choose a new a_1 and start to build U_1. This time any monomial from U_1 has at least one divisor from $B \setminus E$ which is not in $\cup_{j=1,2,4}[f_j,\tilde{c}'_j]$ so the usual proof goes.

Case 2, $k_2,k_3,k_4 > 4$.

Then $a_1 = vx_1x_4 \notin \{u_2,u'_2,\dots,u_4,u'_4\}$. Let $m_1 = h(a_1) = a_1x_k$ for some k. If $k = k_4$ then changing in P_b the intervals $[f_4,c'_4]$, $[a_1,m_1]$ with $[f_4,m_1]$, $[u_4,c'_4]$ we see that $u_4 = w_{34}$ does not divide the new c'_4 and so we have no problem with ω_1.

Suppose that $k \neq k_4$ and $k > 4$ then $a_2 = vx_4x_k \notin \{u_2,u'_2,\dots,u_4,u'_4\}$. If there exists no path $a_2,\dots,a_e$, $m_i = h(a_i)$ with $m_e = \omega_1$ then we proceed as usual. Otherwise, let $a_2,\dots,a_e$, $m_i = h(a_i)$ be a (possible bad) path with $m_e = \omega_1$. Changing in P_b the intervals $[a_i,m_i]$, $i \in [e]$, $[f_3,c'_3]$, $[f_4,c'_4]$ with $[a_{i+2},m_{i+1}]$, $i \in [e-2]$, $[f_3,m_e]$, $[f_4,m_1]$, $[u'_3,c'_3]$, $[u'_4,c'_4]$ we see that any monomial from C has at least one divisor from $B \setminus E$ which is not in $\cup_{j=2,3,4}[f_j,\tilde{c}'_j]$ so the usual proof goes, where $\tilde{c}'_j$ denotes the new c'_j for $j = 3,4$ and $\tilde{c}'_2 = c'_2$.

Remains to study the case when $k \neq k_4$ and $k = 2$ or $k = 3$. Assume that $k = 2$, that is $m_1 = \omega_3$. Similarly we may assume that $a_2 = vx_1x_2$, $m_2 = h(a_2) = a_2x_3 = \omega_4$ and $a_3 = vx_1x_3$, $m_3 = h(a_3) = a_3x_4 = \omega_2$. If there exists no path $a_3,\dots,a_e$, $m_i = h(a_i)$ with $m_e = \omega_1$ then we proceed as usual. Otherwise, let $a_3,\dots,a_e$, $m_i = h(a_i)$ be a (possible bad) path with $m_e = \omega_1$. Changing in P_b the intervals $[a_i,m_i]$, $i \in [e]$, $[f_j,c'_j]$, $j = 2,3,4$ with $[a_{i+3},m_{i+2}]$,

$i \in [e-3]$, $[f_1, m_1]$, $[f_3, m_2]$, $[f_4, \omega_1]$, $[u'_2, c'_2]$, $[u'_3, c'_3]$, $[u'_4, c'_4]$ we arrive in a case similar to the next one.

Case 3, $k_2 = k_3 = k_4 = 1$.

Thus $c'_2 = \omega_3 \in (a_1)$ for $a_1 = \tilde{a}_3$. If there exists a path $a_1, \ldots, a_e$, $m_i = h(a_i)$ with $m_e = \omega_1$ then changing in P_b the intervals $[a_i, m_i]$, $i \in [e]$, $[f_2, c'_2]$, $[f_3, c'_3]$ with $[a_{i+1}, m_i]$, $i \in [e-1]$, $[a_1, c'_2]$, $[f_1, c'_3]$, $[f_2, \omega_1]$ we get the new $\tilde{c}'_1 = \omega_4$, $\tilde{c}'_2 = \omega_1$ and $\tilde{c}'_4 = c'_4 = \omega_2$. Thus we may change the three c'_i to be any three monomials from ω_j.

Assume that the above path is bad, let us say $m_p \in (b)$ for $p < e$ and as in Lemma 8 we may suppose that $a_{p+1} \notin E$, $T_{a_{p+1}} \cap \{a_1, \ldots, a_p\} = \emptyset$ and there exists no bad path starting with a_{p+1}. Changing P_b as above we see that the new $\tilde{c}'_i$ are $\omega_1, \omega_2, \omega_4$ and the $\omega_3 \notin U'_{a_{p+1}}$, where $U'_{a_{p+1}}$ corresponds to $T'_{a_{p+1}} = T_{a_{p+1}} \setminus \{a_{p+1}\}$. Set $b' = a_{p+1}$. In fact changing in the new P_b the intervals $[b', m_p]$ with $[b, m_p]$ we get a partition $P_{b'}$ on $I_{b'}/J_{b'}$, where $I_{b'} J_{b'}$ are defined as usually but we could have $b' \in W$. There exists no bad path in $P_{b'}$ because otherwise this induces one in P_b. We may proceed as before since all monomials from $U'_{b'}$ has at least one divisor from $B \setminus E$ which is not in $\cup_{j=1,2,4} [f_j, \tilde{c}'_j]$. Similarly, we do for any $a_1 \in E$ dividing one from c'_2, c'_3, c'_4 and remains to assume that there exists no bad path starting with a divisor from E of any c'_i, $i = 2, 3, 4$.

Now suppose that $a_1 = b_3$ and consider T_1, U_1 as usual and we may suppose that we are still in Case 3 but with $(\tilde{c}'_j)$, $j = 1, 3, 4$. If there exists no bad path starting with a_1 and $m_1 = h(a_1) \in (W)$, let us say $m_1 \in (w_{13})$ then changing in P_b the intervals $[a_1, m_1]$, $[f_1, \tilde{c}'_1]$ with $[f_1, m_1]$, $[\tilde{u}_1, \tilde{c}'_1]$, $\tilde{u}_1 = w_{12}$ we arrive in a case similar to Case 1. If $m_1 \notin (W)$ then assume that in P_b there exist the intervals $[f_1, \omega_2]$, $[f_2, \omega_4]$, $[f_4, \omega_1]$. Then $[f_3, m_1]$ is disjoint of these intervals. Enlarge T_1 to $\tilde{T}_1$ adding all monomials from B connected by a path which is not bad, with the divisors from E of (ω_j), $j = 1, 2, 4$. Thus taking $I' = (B \setminus (\tilde{T}_1 \cup W))$, $J' = J \cap I'$ we get $\operatorname{sdepth}_S I/(J, I') \geq d + 2$ which is enough as usual.

If there exists a bad path $a_1, \ldots, a_e$, $m_i = h(a_i)$, $m_e = \omega_1$, $m_p \in (b)$, $p < e$ then as above we may assume that $a_{p+1} \notin E$, $T_{a_{p+1}} \cap \{a_1, \ldots, a_p\} = \emptyset$ and there exists no bad path starting with a_{p+1}. Moreover, we may choose $a_{p+2} \notin E$ when $e > p + 1$ because $m_{p+1} \neq \omega_1$. Taking as above $b' = a_{p+1}$ and the partition $P_{b'}$ given on $I_{b'}/J_{b'}$ we see that $T_{a_{p+2}} \cap (f_1, \ldots, f_4) \neq \emptyset$ and we reduce to the above situation with $T_{a_{p+2}}$ instead T_1. If $p \geq e - 1$ then $\omega_1 \notin U_{a_{p+2}}$ and so there exists no problem. $\square$

Theorem 4. *Conjecture 1 holds for $r = 5$ if there exists $t \in [n]$ such that $t \notin \cup_{i \in [5]} \operatorname{supp} f_i$, $(B \setminus E) \cap (x_t) \neq \emptyset$ and $E \subset (x_t)$.*

Proof: Apply Lemma 1, since Conjecture 1 holds for $r \leq 4$ by Theorem 3.

$\square$

Example 7. Let $n = 8$, $E = \{x_6 x_7, x_7 x_8\}$, $I = (x_1, x_2, x_3, x_4, x_5, E)$,
$J = (x_1x_6, x_1x_8, x_2x_8, x_3x_6, x_3x_8, x_4x_6, x_4x_7, x_4x_8, x_5x_6, x_5x_7, x_5x_8)$.

We see that we have

$$B = \{x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_1x_7, x_2x_3, x_2x_4, x_2x_5, x_2x_6,$$

$$x_2x_7, x_3x_4, x_3x_5, x_3x_7, x_4x_5\} \cup \{E\},$$

$$C = \{x_1x_2x_3, x_1x_2x_4, x_1x_2x_5, x_1x_2x_7, x_1x_3x_4, x_1x_3x_5, x_1x_3x_7, x_1x_4x_5, x_2x_3x_4,$$

$$x_2x_3x_5, x_2x_3x_7, x_2x_4x_5, x_2x_6x_7, x_3x_4x_5, x_6x_7x_8\}$$

and so $r = 5$, $q = 15$, $s = 16 \leq q+r$. We have $\operatorname{sdepth}_S I/J = 2$, because otherwise the monomial x_2x_6 could enter either in $[x_2, x_2x_6x_7]$, or in $[x_2x_6, x_2x_6x_7]$ and in both cases remain the monomials of E to enter in an interval ending with $x_6x_7x_8$, which is impossible. Then $\operatorname{depth}_S I/J \leq 2$ by the above theorem since $E \subset (x_7)$ and for instance $x_1x_7 \in (B \setminus E) \cap (x_7)$.

Added in Proof: Meanwhile, an example appeared in a paper of Duval et al. (A non-partitionable Cohen-Macaulay simplicial complex), arXiv 1504.04279, which shows in particular that Conjecture 1 is false even when $r = 5$ but there exists no t as in Theorem 2. This says that our result is tight.

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