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Homogenization cases of heat transfer in structures with interfacial barriers

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Abstract

The paper study the asymptotic behaviour of the heat transfer in a bounded domain formed by two interwoven connected components separated by an interface on which the heat flux is continuous and the temperature subjects to a first-order jump condition. The macroscopic laws and their effective coefficients are obtained by means of the two-scale convergence technique of the periodic homogenization theory for several orders of magnitude of the conductivities and of the jump transmission coefficient.

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1 Introduction

This paper deals with the asymptotic behaviour (for $\varepsilon \to 0$) of the heat transfer problem in the framework introduced by [10], a realistic ε-periodic structure composed of two connected components. The reference conductor (where the conductivity is of unity order with respect to ε) is set in the ambient component, the only one which is reaching the boundary of the domain. The second component contains the core material, where the conductivity is of $\varepsilon^{2\beta}$-order, with $\beta \in (0, 1]$. The jump transfer coefficient of the interface has ε^r-order, with $r \in (-1, 1]$.

Since now, this problem has been treated only for $\beta = 0$ and when the core material is composed of isolated grains (see [2], [3], [5], [7] and [8]). For a structure with connected core material, only the case $\beta = 0$ and $r = 1$ has been rigorously studied (see [6]).

In order to derive the macroscopic behaviour we obtain the two-scale homogenized systems by applying the two-scale convergence technique of the periodic homogenization theory (see [1] and [9]). In each distinct case we uncouple the local-periodic problems and determine the specific effective coefficients of the macroscopic problems, which, luckily here, are well-posed and therefore, uniquely defining the asymptotic behaviour of the temperature.

2 The heat conduction problem

Let Ω be an open connected bounded set in $\mathbb{R}^N$ ($N \geq 3$), locally located on one side of the boundary $\partial\Omega$, a Lipschitz manifold composed of a finite number of connected components.

Let Y_a be a Lipschitz open connected subset of the unit cube $Y = (0,1)^N$. We assume that $Y_b = Y \setminus \overline{Y}_a$ has a locally Lipschitz boundary, that the intersections of ∂Y_b with ∂Y are reproduced identically on the opposite faces of Y,

$$\Sigma^{+i} = \{y \in \partial Y : \ y_i = 1\}, \ \ \Sigma^{-i} = \{y \in \partial Y : \ y_i = 0\}, \ \ \forall i \in \{1, 2, \ldots, N\}, \quad (2.1)$$

and that $\overline{Y}_b \cap \Sigma^{\pm i} \subset\subset \Sigma^{\pm i}$. We assume also that repeating Y by periodicity, the reunion of all the $\overline{Y}_a$ parts is a connected domain in $\mathbb{R}^N$ with a locally C^2 boundary; we denote it by $\mathbb{R}^N_a$ and we set the origin of the coordinate system such that there exists $R > 0$ with the property $B(0,R) \subseteq \mathbb{R}^N_a$. Moreover, we denote $\Gamma := \partial Y_a \cap \partial Y_b$ and ν the normal on Γ (exterior to Y_a).

If e_i stands for the unit vector of the canonical basis in $\mathbb{R}^N$ then, for any $\varepsilon \in (0,1)$, we introduce

$$\mathbb{Z}_\varepsilon = \{k \in \mathbb{Z}^N : \ \varepsilon k + \varepsilon Y \subseteq \Omega\}, \quad (2.2)$$

$$I_\varepsilon = \{k \in \mathbb{Z}_\varepsilon : \ \varepsilon k \pm \varepsilon e_i + \varepsilon Y \subseteq \Omega, \ \forall i \in \{1, ..., N\}\}. \quad (2.3)$$

The core component of our structure is defined by

$$\Omega_{\varepsilon b} = \text{int} \left(\bigcup_{k \in I_\varepsilon} (\varepsilon k + \varepsilon \overline{Y}_b) \right) \quad (2.4)$$

and the reference conductor by

$$\Omega_{\varepsilon a} = \Omega \setminus \overline{\Omega}_{\varepsilon b}. \quad (2.5)$$

The interface between the two components is denoted by

$$\Gamma_\varepsilon = \partial\Omega_{\varepsilon a} \cap \partial\Omega_{\varepsilon b} = \partial\Omega_{\varepsilon b}, \quad (2.6)$$

and we have to remark that all the boundaries are at least locally Lipschitz, that $\Omega_{\varepsilon a}$ is connected and that $\Omega_{\varepsilon b}$ can be also connected.

Next, we introduce the Hilbert space

$$H_\varepsilon = \left\{ v \in L^2(\Omega) : v\Big|_{\Omega_{\varepsilon a}} \in H^1(\Omega_{\varepsilon a}), \ v\Big|_{\Omega_{\varepsilon b}} \in H^1(\Omega_{\varepsilon b}), \ v = 0 \text{ on } \partial\Omega \right\} \quad (2.7)$$

endowed with the scalar product

$$(u,v)_{H_\varepsilon} = \int_{\Omega_{\varepsilon a}} \nabla u \nabla v + \varepsilon^2 \int_{\Omega_{\varepsilon b}} \nabla u \nabla v + \varepsilon \int_{\Gamma_\varepsilon} [u][v], \tag{2.8}$$

where $[u] = \gamma_{\varepsilon b} u - \gamma_{\varepsilon a} u$ and $\gamma_{\varepsilon a} u$, $\gamma_{\varepsilon b} u$ are the traces of u on Γ_ε defined in $H^1(\Omega_{\varepsilon a})$ and $H^1(\Omega_{\varepsilon b})$, respectively.

Our domain has the following well-known properties (see [4], [6]):

Lemma 1. *There exists an extension operator $P_\varepsilon \in \mathcal{L}\left(H^1(\Omega_{\varepsilon a}); H_0^1(\Omega)\right)$ such that*

$$P_\varepsilon v = v \text{ in } \Omega_{\varepsilon a}, \tag{2.9}$$

$$|\nabla P_\varepsilon v|_{L^2(\Omega)} \le C\, |\nabla v|_{L^2(\Omega_{\varepsilon a})}, \ \forall v \in H^1(\Omega_{\varepsilon a}), \tag{2.10}$$

where $C > 0$ is a constant independent of ε.

Lemma 2. *For any $v \in H_\varepsilon$ there exists $C > 0$, independent of ε, such that*

$$|v|_{L^2(\Omega_{\varepsilon a})} \le C\, |\nabla v|_{L^2(\Omega_{\varepsilon a})}, \tag{2.11}$$

$$\varepsilon^{1/2} |\gamma_{\varepsilon a} v|_{L^2(\Gamma_\varepsilon)} \le C \left(|v|_{L^2(\Omega_{\varepsilon a})} + \varepsilon\, |\nabla v|_{L^2(\Omega_{\varepsilon a})} \right), \tag{2.12}$$

$$|v|_{L^2(\Omega_{\varepsilon b})} \le C \left(\varepsilon^{1/2} |\gamma_{\varepsilon b} v|_{L^2(\Gamma_\varepsilon)} + \varepsilon\, |\nabla v|_{L^2(\Omega_{\varepsilon b})} \right). \tag{2.13}$$

Remark 1. *Taking in account the L^2-norm of the jump on Γ_ε the results of the previous Lemma have an important consequence:*

$$|v|_{L^2(\Omega_{\varepsilon b})} \le C\, |v|_{H_\varepsilon}, \forall v \in H_\varepsilon. \tag{2.14}$$

For $\varepsilon \in (0,1)$ we introduce the transmission factor $h^\varepsilon(x) = h(x/\varepsilon)$, where $h \in C(\overline{Y})$, and the symmetric conductivities $a_{ij}^\varepsilon(x) = a_{ij}(x/\varepsilon)$, $b_{ij}^\varepsilon(x) = b_{ij}(x/\varepsilon)$, where $a_{ij},\ b_{ij} \in L_{per}^\infty(Y)$, with the property that there exists $\delta > 0$ such that

$$h \ge \delta, \ \ a.e. \ \ on \ \ Y, \tag{2.15}$$

$$a_{ij}\xi_i\xi_j \ge \delta\xi_i\xi_i \ \text{ and } \ b_{ij}\xi_i\xi_j \ge \delta\xi_i\xi_i, \ \ \forall \xi \in \mathbb{R}^N, \ \ a.e. \ \ on \ \ Y. \tag{2.16}$$

Finally, considering $\beta \in (0,1]$, $r \in (-1,1]$ and $f \in L^2(\Omega)$, we look for the temperature distribution u^ε which satisfies the heat conduction equations

$$-\frac{\partial}{\partial x_i}\left(a_{ij}^\varepsilon \frac{\partial u^\varepsilon}{\partial x_j} \right) = f \text{ in } \Omega_{\varepsilon a}, \tag{2.17}$$

$$-\varepsilon^{2\beta} \frac{\partial}{\partial x_i}\left(b_{ij}^\varepsilon \frac{\partial u^\varepsilon}{\partial x_j} \right) = f \text{ in } \Omega_{\varepsilon b}, \tag{2.18}$$

and the following transmission and boundary conditions

$$a_{ij}^{\varepsilon}\frac{\partial u^{\varepsilon}}{\partial x_j}\nu_i^{\varepsilon}=\varepsilon^{2\beta}b_{ij}^{\varepsilon}\frac{\partial u^{\varepsilon}}{\partial x_j}\nu_i^{\varepsilon}=\varepsilon^{r}h^{\varepsilon}\left(\gamma_{\varepsilon b}u^{\varepsilon}-\gamma_{\varepsilon a}u^{\varepsilon}\right)\ \text{on}\ \Gamma_{\varepsilon}, \tag{2.19}$$

$$u_{\varepsilon}=0\ \text{on}\ \ \partial\Omega. \tag{2.20}$$

The variational formulation of the problem (2.17)-(2.20) is the following:
To find $u^{\varepsilon}\in H_{\varepsilon}$ such that

$$\int_{\Omega_{\varepsilon a}}a_{ij}^{\varepsilon}\frac{\partial u^{\varepsilon}}{\partial x_j}\frac{\partial v}{\partial x_i}+\varepsilon^{2\beta}\int_{\Omega_{\varepsilon b}}b_{ij}^{\varepsilon}\frac{\partial u^{\varepsilon}}{\partial x_j}\frac{\partial v}{\partial x_i}+\varepsilon^{r}\int_{\Gamma_{\varepsilon}}h^{\varepsilon}[u^{\varepsilon}][v]=\int_{\Omega}fv,\ \ \forall v\in H_{\varepsilon}. \tag{2.21}$$

Applying Lax-Milgram Theorem and using (2.11)-(2.16), we get:

Theorem 1. *For any $\varepsilon\in(0,1)$ there exists a unique $u^{\varepsilon}\in H_{\varepsilon}$, solution of the problem (2.21).*

3 A priori estimates of the temperature

First, using coerciveness property and the inequalities (2.11)-(2.14), we find some $C>0$, independent of ε, such that

$$|u^{\varepsilon}|_{L^2(\Omega)}\leq C,\ |\nabla u^{\varepsilon}|_{L^2(\Omega_{\varepsilon a})}\leq C,\ \varepsilon^{\beta}\,|\nabla u^{\varepsilon}|_{L^2(\Omega_{\varepsilon b})}\leq C,\ \varepsilon^{r/2}\,|[u^{\varepsilon}]|_{L^2(\Gamma_{\varepsilon})}\leq C. \tag{3.1}$$

Next, using the notations

$$\widehat{u}_{\alpha}^{\varepsilon}=\begin{cases}u & \text{in }\Omega_{\varepsilon\alpha}\\ 0 & \text{in }\Omega-\Omega_{\varepsilon\alpha}\end{cases}\qquad \widehat{\nabla u}_{\alpha}^{\varepsilon}=\begin{cases}\nabla u & \text{in }\Omega_{\varepsilon\alpha}\\ 0 & \text{in }\Omega-\Omega_{\varepsilon\alpha},\end{cases} \tag{3.2}$$

$\forall u\in H^1(\Omega_{\varepsilon\alpha}),\alpha\in\{a,b\}$, and introducing the Hilbert spaces

$$H_{per}^1\left(Y_a\right)=\left\{\varphi\in H_{loc}^1\left(\mathbb{R}_a^N\right):\ \varphi\text{ is }Y\text{-periodic}\right\}, \tag{3.3}$$

$$\widetilde{H}_{per}^1\left(Y_a\right)=\left\{\varphi\in H_{loc}^1\left(\mathbb{R}_a^N\right):\ \int_{Y_a}\varphi=0\ \text{ and }\ \varphi\text{ is }Y\text{-periodic}\right\}, \tag{3.4}$$

we can present the main compactness result:

Theorem 2. *For every $\beta\in(0,1]$ and $r\in(-1,1]$ there exists $u_a\in H_0^1(\Omega)$, $\eta_a\in L^2\left(\Omega;\widetilde{H}_{per}^1(Y_a)\right)$ and $u_b\in L^2(\Omega,L_{per}^2(Y_b))$ such that the following convergences hold on some subsequence*

$$\widehat{u}_a^{\varepsilon}\overset{2s}{\rightharpoonup}\chi_a u_a, \tag{3.5}$$

$$\widehat{\nabla u}_a^{\varepsilon}\overset{2s}{\rightharpoonup}\chi_a\left(\nabla_x u_a+\nabla_y\eta_a(\cdot,y)\right), \tag{3.6}$$

$$\widehat{u}_b^{\varepsilon}\overset{2s}{\rightharpoonup}\chi_b u_b, \tag{3.7}$$

where $\chi_\alpha : L^2(\Omega \times Y_\alpha) \to L^2(\Omega \times Y),\ \alpha \in \{a, b\}$, *denotes the straight prolongation with zero; sometimes it can be identified with the characteristic value of* Y_α.

When $\beta \in (0,1)$ *we find that* u_b *is independent of* y, *with* $u_b \in L^2(\Omega)$.

When $\beta = 1$ *it holds*

$$\varepsilon \widehat{\nabla u_b^\varepsilon} \overset{2s}{\rightharpoonup} \chi_b \nabla_y u_b. \tag{3.8}$$

Proof: The properties (3.5)-(3.7) follow from the a priori estimates. They can be proved by adapting the methods of [1], except the fact that u_a has to vanish on $\partial\Omega$. For this, as the estimations (3.1) imply that $\left\{ |\nabla u_a^\varepsilon|_{L^2(\Omega_{\varepsilon a})} \right\}_\varepsilon$ is bounded, then using the Poincaré-Friedrichs inequality and the extension operator (2.9)-(2.10) we obtain

$$|P_\varepsilon u_a^\varepsilon|_{H_0^1(\Omega)} \le C\, |\nabla P_\varepsilon u_a^\varepsilon|_{L^2(\Omega)} \le C\, |\nabla u_a^\varepsilon|_{L^2(\Omega_{\varepsilon a})} \le C,$$

which shows that $\{P_\varepsilon u_a^\varepsilon\}_\varepsilon$ is bounded in $H_0^1(\Omega)$. Hence, there exists $u_a' \in H_0^1(\Omega)$ such that $P_\varepsilon u_a^\varepsilon \rightharpoonup u_a'$ in $H_0^1(\Omega)$ and consequently $\chi_a(\{\frac{x}{\varepsilon}\}) P_\varepsilon u_a^\varepsilon \overset{2s}{\rightharpoonup} \chi_a(y) u_a'$. On the other hand, as $\chi_a(\{\frac{x}{\varepsilon}\}) P_\varepsilon u_a^\varepsilon = \widehat{u}_a^\varepsilon$ and $\widehat{u}_a^\varepsilon \overset{2s}{\rightharpoonup} \chi_a(y) u_a$, then, by identifying the limits, we get $u_a = u_a'$ in Ω.

When $\beta \in (0,1)$, we have to prove that u_b is independent of y. Using the a priori estimates (3.1), for any $\Psi \in \left[\mathcal{D}(\Omega; C_{per}^\infty(Y))\right]^N$ it holds

$$\varepsilon \int_\Omega \widehat{\nabla u_b^\varepsilon}(x) \Psi\left(x, \frac{x}{\varepsilon}\right) dx = \varepsilon^{1-\beta} \varepsilon^\beta \int_\Omega \widehat{\nabla u_b^\varepsilon}(x) \Psi\left(x, \frac{x}{\varepsilon}\right) dx \longrightarrow 0, \tag{3.9}$$

which is identical to that from which the same property follows in the classical way (see [1]).

When $\beta = 1$, the estimations (3.1) imply that $\left\{\varepsilon \widehat{\nabla u_b^\varepsilon}\right\}_\varepsilon$ is bounded in $L^2(\Omega)$ and hence we can assume that it has a two-scale limit on the same subsequence as $\left\{\varepsilon \widehat{u_b^\varepsilon}\right\}_\varepsilon$ (see the main compactness theorem of [1] or [9]). The form of this limit, that is (3.8), can still be found by adapting the standard methods (see Proposition 1.14 of [1]).

□

Now, for any $k \in \{1, 2, ..., N\}$, we define $\eta_{ak} \in \widetilde{H}_{per}^1(Y_a)$ as the unique solution of the local-periodic problem

$$-\frac{\partial}{\partial y_i}\left(a_{ij} \frac{\partial(\eta_{ak} + y_k)}{\partial y_j}\right) = 0 \quad \text{in } Y_a, \tag{3.10}$$

$$a_{ij} \frac{\partial(\eta_{ak} + y_k)}{\partial y_j} \nu_i = 0 \quad \text{on } \Gamma. \tag{3.11}$$

The symmetric and positively defined effective conductivity A is given by

$$A_{ij} = \int_{Y_a} \left(a_{ij} + a_{ik}\frac{\partial \eta_{aj}}{\partial y_k} \right) dy, \qquad \forall i,j \in \{1,2,...,N\}. \tag{3.12}$$

Finally, we introduce the functions w_0 and w_1, which are the only solutions in $H^1_{per}(Y_b)$ of the following two local-problems:

$$-\frac{\partial}{\partial y_i}\left(b_{ij}\frac{\partial w_0}{\partial y_j}\right) = 1 \quad \text{in } Y_b, \qquad w_0 = 0 \quad \text{on } \Gamma, \tag{3.13}$$

$$-\frac{\partial}{\partial y_i}\left(b_{ij}\frac{\partial w_1}{\partial y_j}\right) = 1 \text{ in } Y_b, \qquad -b_{ij}\frac{\partial w_1}{\partial y_j}\nu_i + hw_1 = 0 \text{ on } \Gamma. \tag{3.14}$$

Due to the existence of the first-order jump interface Γ_ε, there are two effective coefficients describing the microscopic transfer:

$$\widetilde{h} = \int_\Gamma h(y)d\sigma \qquad \text{and} \qquad \widetilde{w_1 h} = \int_\Gamma w_1(y)h(y)d\sigma. \tag{3.15}$$

4 The homogenization process for $\beta \in (0,1)$ and $r = 1$

Remark 2. *Using Theorem 2, we pass (2.21) to the limit, with the following test function*

$$v(x) = \left(\Phi_a(x) + \varepsilon\varphi_a\left(x, \frac{x}{\varepsilon}\right), \Phi_b(x) + \varepsilon\varphi_b\left(x, \frac{x}{\varepsilon}\right)\right), \tag{4.1}$$

where $\Phi_\alpha \in \mathcal{D}(\Omega)$ *and* $\varphi_\alpha \in \mathcal{D}(\Omega; C^\infty_{per}(Y_\alpha))$, $\alpha \in \{a,b\}$. *We get*

$$\int_{\Omega\times Y_a} a_{ij}\left(\frac{\partial u_a}{\partial x_j} + \frac{\partial \eta_a}{\partial y_j}\right)\left(\frac{\partial \Phi_a}{\partial x_i} + \frac{\partial \varphi_a}{\partial y_i}\right) + \int_\Omega \widetilde{h}(u_b - u_a)(\Phi_b - \Phi_a) =$$
$$= \int_{\Omega\times Y} (\chi_a\Phi_a + \chi_b\Phi_b)\, f. \tag{4.2}$$

Introducing the Hilbert space $V_1 := H^1_0(\Omega)\times L^2(\Omega)\times L^2(\Omega, \widetilde{H}^1_{per}(Y_a))$, *endowed with the following scalar product*

$$\langle (u_a,u_b,\eta_a), (\Phi_a,\Phi_b,\varphi_a)\rangle_{V_1} = \int_\Omega \nabla u_a \nabla\Phi_a + \int_\Omega (u_b - u_a)(\Phi_b - \Phi_a) +$$
$$+ \int_{\Omega\times Y_a} \nabla_y \eta_a \nabla_y \varphi_a\,, \tag{4.3}$$

then by density arguments we prove that (u_a,u_b,η_a) *is the only solution of:*
To find $(u_a,u_b,\eta_a) \in V_1$ *satisfying*

$$\int_{\Omega\times Y_a} a_{ij}\left(\frac{\partial u_a}{\partial x_j} + \frac{\partial \eta_a}{\partial y_j}\right)\left(\frac{\partial \Phi_a}{\partial x_i} + \frac{\partial \varphi_a}{\partial y_i}\right) + \widetilde{h}\int_\Omega (u_b - u_a)(\Phi_b - \Phi_a) =$$
$$= \int_{\Omega\times Y} (\chi_a\Phi_a + \chi_b\Phi_b)\, f, \quad \forall\, (\Phi_a,\Phi_b,\varphi_a) \in V_1. \tag{4.4}$$

Theorem 3. *If u_ε is the solution of the problem (2.21) the convergences (3.5)-(3.8) hold on the whole sequence and the limit $(u_a, u_b) \in H_0^1(\Omega) \times L^2(\Omega)$ is the unique solution of the homogenized problem*

$$\int_\Omega A_{ij} \frac{\partial u_a}{\partial x_j} \frac{\partial \Phi_a}{\partial x_i} + \int_\Omega \widetilde{h}(u_b - u_a)(\Phi_b - \Phi_a) = \int_\Omega \left(|Y_a|\Phi_a + |Y_b|\Phi_b\right) f,$$

$$\forall (\Phi_a, \Phi_b) \in H_0^1(\Omega) \times L^2(\Omega). \tag{4.5}$$

Consequently, the homogenization process is summarized in this case by:

Theorem 4. *If u^ε is the solution of the problem (2.21) then*

$$u^\varepsilon \overset{2s}{\rightharpoonup} u + \frac{|Y_b|}{\widetilde{h}} \chi_b f, \tag{4.6}$$

where $u \in H_0^1(\Omega)$ is the unique solution of the Dirichlet problem

$$\int_\Omega A \nabla u \nabla \Phi = \int_\Omega f \Phi, \ \ \forall \Phi \in H_0^1(\Omega). \tag{4.7}$$

5 The homogenization process for $\beta \in (0,1)$ and $r \in (-1,1)$

Remark 3. *Multiplying the variational problem (2.21) with ε^{1-r}, setting (4.1) as test function and passing to the limit, we find that:*

$$u_a = u_b \in H_0^1(\Omega) \tag{5.1}$$

Next, passing to the limit with (4.1) as test function in (2.21) with $\Phi_a = \Phi_b = \Phi$, we obtain

$$\int_{\Omega \times Y_a} a_{ij} \left(\frac{\partial u_a}{\partial x_j} + \frac{\partial \eta_a}{\partial y_j} \right) \left(\frac{\partial \Phi}{\partial x_i} + \frac{\partial \varphi_a}{\partial y_i} \right) dxdy = \int_\Omega f \Phi dx. \tag{5.2}$$

By density arguments we remark that $(u, \eta_a) \in V_2 := H_0^1(\Omega) \times L^2(\Omega, \widetilde{H}_{per}^1(Y_a))$ is solution of the problem:

To find $(u, \eta_a) \in V_2$ satisfying

$$\int_{\Omega \times Y_a} a_{ij} \left(\frac{\partial u}{\partial x_j} + \frac{\partial \eta_a}{\partial y_j} \right) \left(\frac{\partial \Phi}{\partial x_i} + \frac{\partial \varphi_a}{\partial y_i} \right) dxdy = \int_\Omega f \Phi dx$$

$$\forall \left(\Phi, \varphi_a\right) \in V_2. \tag{5.3}$$

It easy to verify that (5.3) is a well-posed problem in the Hilbert space V_2, endowed with the scalar product:

$$\langle (u, \eta_a), (\Phi, \varphi_a) \rangle_{V_2} = \int_\Omega \nabla u \nabla \Phi + \int_{\Omega \times Y_a} \nabla_y \eta_a \nabla_y \varphi_a. \tag{5.4}$$

In the present case the asymptotic behavior is summarized by:

Theorem 5. *If u^ε is the solution of the problem (2.21) then,*

$$u^\varepsilon \overset{2s}{\rightharpoonup} u, \tag{5.5}$$

where $u \in H_0^1(\Omega)$ is the unique solution of (4.7).

6 The homogenization process for $\beta = 1$ and $r = 1$

Remark 4. *For $\Phi \in \mathcal{D}(\Omega)$ and $\varphi_\alpha \in \mathcal{D}\left(\Omega; C^\infty_{per}(Y_\alpha)\right)$, with $\alpha \in \{a, b\}$, we pass (2.21) to the limit, using the test function*

$$v(x) = \left(\Phi(x) + \varepsilon\varphi_a\left(x, \frac{x}{\varepsilon}\right), \varphi_b\left(x, \frac{x}{\varepsilon}\right)\right), \quad x \in \Omega. \tag{6.1}$$

It follows that

$$\int_{\Omega\times Y_a} a_{ij}\left(\frac{\partial u_a}{\partial x_j} + \frac{\partial \eta_a}{\partial y_j}\right)\left(\frac{\partial \Phi}{\partial x_i} + \frac{\partial \varphi_a}{\partial y_i}\right) + \int_{\Omega\times Y_b} b_{ij}\frac{\partial u_b}{\partial y_j}\frac{\partial \varphi_b}{\partial y_i} +$$

$$+ \int_{\Omega\times\Gamma} h\left(u_b - u_a\right)\left(\varphi_b - \Phi\right) = \int_{\Omega\times Y_a} f\Phi + \int_{\Omega\times Y_b} f\varphi_b, \tag{6.2}$$

Denoting $V_3 := H_0^1(\Omega) \times L^2(\Omega; H^1_{per}(Y_b)) \times L^2(\Omega, \widetilde{H}^1_{per}(Y_a))$, we find by density arguments that (u_a, u_b, η_a) is the only solution of the problem:

To find $(u_a, u_b, \eta_a) \in V_3$ satisfying

$$\int_{\Omega\times Y_a} a_{ij}\left(\frac{\partial u_a}{\partial x_j} + \frac{\partial \eta_a}{\partial y_j}\right)\left(\frac{\partial \Phi}{\partial x_i} + \frac{\partial \varphi_a}{\partial y_i}\right) + \int_{\Omega\times\Gamma} h\left(u_b - u_a\right)\left(\varphi_b - \Phi\right) +$$

$$+ \int_{\Omega\times Y_b} b_{ij}\frac{\partial u_b}{\partial y_j}\frac{\partial \varphi_b}{\partial y_i} = \int_{\Omega\times Y_a} f\Phi + \int_{\Omega\times Y_b} f\varphi_b, \ \ \forall\left(\Phi, \varphi_b, \varphi_a\right) \in V_3. \tag{6.3}$$

The problem (6.3) is well-posed in V_3, Hilbert space with the scalar product:

$$\langle\left(u_a, u_b, \eta_a\right), \left(\Phi, \varphi_a, \varphi_b\right)\rangle_{V_3} = \int_\Omega \nabla u_a \nabla\Phi + \int_{\Omega\times Y_b} \nabla u_b \nabla\varphi_b +$$

$$+ \int_{\Omega\times\Gamma} \left(u_b - u_a\right)\left(\varphi_b - \Phi\right) + \int_{\Omega\times Y_a} \nabla_y\varphi_a\nabla_y\eta_a. \tag{6.4}$$

Theorem 6. *If u^ε is the solution of (2.21) then*

$$u^\varepsilon \overset{2s}{\rightharpoonup} \left(\mid Y_a \mid + \widetilde{w_1 h}\right) u + w_1\chi_b f, \tag{6.5}$$

where $u \in H_0^1(\Omega)$ and $w_1 \in H^1_{per}(Y_b)$ are defined by (4.7) and (3.14).

Proof: If $u \in H_0^1(\Omega)$ is the solution of the homogenized system (4.7) then it is easy to verify that the only solution of the problem (6.3) is given by

$$u_a(x) = \left(\mid Y_a \mid + \widetilde{w_1 h} \right) u(x), \quad x \in \Omega, \tag{6.6}$$

$$u_b(x,y) = \left(\mid Y_a \mid + \widetilde{w_1 h} \right) u(x) + w_1(y) f(x), \quad (x,y) \in \Omega \times Y_b, \tag{6.7}$$

$$\eta_a(x,y) = \left(\mid Y_a \mid + \widetilde{w_1 h} \right) \eta_{a_k}(y) \frac{\partial u}{\partial x_k}(x), \quad (x,y) \in \Omega \times Y_a. \tag{6.8}$$

□

7 The homogenization process for $\beta = 1$ and $r \in (-1, 1)$

The preliminary result of this case is the following:

Lemma 3. *For any $\Phi \in \mathcal{D}(\Omega)$ and $\varphi_\alpha \in \mathcal{D}(\Omega; C^\infty_{per}(Y_\alpha))$, $\alpha \in \{a, b\}$ such that*

$$\varphi_b(x,y) = \Phi(x), \ \forall (x,y) \in \Omega \times \Gamma \tag{7.1}$$

we have:

$$\int_{\Omega \times Y_a} a_{ij} \left(\frac{\partial u_a}{\partial x_j} + \frac{\partial \eta_a}{\partial y_j} \right) \left(\frac{\partial \Phi}{\partial x_i} + \frac{\partial \varphi_a}{\partial y_i} \right) + \int_{\Omega \times Y_b} b_{ij} \frac{\partial u_b}{\partial y_j} \frac{\partial \varphi_b}{\partial y_i} =$$
$$= \int_{\Omega \times Y_a} f \Phi + \int_{\Omega \times Y_b} f \varphi_b. \tag{7.2}$$

Moreover,

$$u_a = u_b \ \text{on} \ \Omega \times \Gamma. \tag{7.3}$$

Proof: Multiplying (2.21) with ε^{1-r}, setting the test function (6.1) with $\Phi \in \mathcal{D}(\Omega)$, $\varphi_a \in \mathcal{D}(\Omega; C^\infty_{per}(Y_a))$, $\varphi_b \in \mathcal{D}(\Omega; C^\infty_{per}(Y_b))$ and passing to the limit we get

$$\int_{\Omega \times \Gamma} h(y) \left(u_b(x,y) - u_a(x) \right) \left(\varphi_b(x,y) - \Phi(x) \right) = 0, \tag{7.4}$$

which obviously imply (7.3).

In order to obtain (7.2) we set in (2.21) the test function (6.1) with the supplementary condition (7.1). The proof is completed again by passing to the limit, the term corresponding to the integral on Γ_ε being of order $\varepsilon^{1+r/2}$. □

In the light of the previous result, we introduce the space

$$V := \left\{ (\Phi, \varphi) \in H_0^1(\Omega) \times L^2(\Omega; H^1_{per}(Y_b)), \ \ \varphi = \Phi \text{ on } \Omega \times \Gamma \right\}. \tag{7.5}$$

Remark 5. *Using density arguments it follows that*

$$((u_a, u_b), \eta_a) \in V_4 := V \times L^2(\Omega; \widetilde{H}^1_{per}(Y_a)) \text{ is solution of the problem:}$$

To find $((u_a, u_b), \eta_a) \in V_4$ *satisfying*

$$\int_{\Omega\times Y_a} a_{ij}\left(\frac{\partial u_a}{\partial x_j} + \frac{\partial \eta_a}{\partial y_j}\right)\left(\frac{\partial \Phi}{\partial x_i} + \frac{\partial \varphi_a}{\partial y_i}\right) + \int_{\Omega\times Y_b} b_{ij}\frac{\partial u_b}{\partial y_j}\frac{\partial \varphi_b}{\partial y_i} =$$

$$= \int_{\Omega\times Y_a} f\Phi + \int_{\Omega\times Y_b} f\varphi_b, \quad \forall\,((\Phi, \varphi_b), \varphi_a) \in V_4. \tag{7.6}$$

The problem (7.6) is a well-posed in the Hilbert space V_4, *endowed with the scalar product:*

$$\langle ((u_a, u_b), \eta_a), ((\Phi, \varphi_b), \varphi_a)\rangle_{V_4} = \int_\Omega \nabla u_a \nabla \Phi + \int_{\Omega\times Y_b} \nabla_y u_b \nabla_y \varphi_b + \int_{\Omega\times Y_a} \nabla_y \varphi_a \nabla_y \eta_a.$$

Thus, in the present case the results of the homogenization process can be summarized by:

Theorem 7. *If* u^ε *is the solution of the problem (2.21) then,*

$$u^\varepsilon \overset{2s}{\rightharpoonup} |Y_a| u + w_0 \chi_b f, \tag{7.7}$$

where $u \in H^1_0(\Omega)$ *and* $w_0 \in H^1_{per}(Y_b)$ *are defined by (4.7) and (3.13).*

Proof: If $u \in H^1_0(\Omega)$ is the unique solution of (4.7) then we verify that the unique solution of (7.6) is the following:

$$u_a = |Y_a| u, \quad u_b = |Y_a| u + w_0 f, \quad \eta_a = |Y_a| \eta_{ak} \frac{\partial u}{\partial x_k},$$

where η_{ak} and w_0 are defined by the problems (3.10)-(3.11) and (3.13). □

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