Bull. Math. Soc. Sci. Math. Roumanie
Tome 58(106) No. 4, 2015, 451–461

Inequalities for a polynomial with prescribed zeros

by
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Abstract

For a polynomial $p(z)$ of degree n with a zero of order $k(\geq 1)$ at β, it is known that

$$\max_{|z|=1} \left|\frac{p(z)}{(z-\beta)^k}\right| \leq \left(\frac{n-k+1}{1+|\beta|}\right)^k \max_{1\leq l\leq n-k+1} |p(\gamma_l')|,$$

$\gamma_1', \gamma_2', \ldots, \gamma_{n-k+1}'$ being the roots of $z^{n-k+1} + e^{i\gamma(n-k+1)} = 0$, with $\gamma = \arg\beta$ ($\gamma = 0$ for $\beta = 0$). By considering a polynomial $p(z)$ of degree n with zeros $\beta_1, \beta_2, \ldots, \beta_k$ we have obtained certain inequalities thereby giving a refinement of the known result.

Key Words: Polynomial, prescribed zeros, inequalities, refinement.
2010 Mathematics Subject Classification: Primary 30C10, Secondary 30A10.

1 Introduction and statement of results

Famous chemist Mendeleev [5] while making a study of the specific gravity of a solution as a function of the percentage of the dissolved substance, obtained a pretty mathematical result for polynomials of degree 2 and told it to contemporary famous mathematician A. A. Markov who [4] naturally investigated the corresponding problem for polynomials of degree n and proved what has come to be known as Markov's Theorem:

Markov's Theorem. *If $P(x)$ is a real polynomial of degree n and $|P(x)| \leq 1$ on $[-1,1]$ then $|P'(x)| \leq n^2$ on $[-1,1]$, with equality attainable only at ± 1 and only when $P(x) = \pm T_n(x)$, where $T_n(x)$ (the so called Chebyshev polynomial) is $\cos n \cos^{-1} x$.*

After about 20 years S. Bernstein wanted, for applications in the theory of approximation of functions by polynomials, the analogue of Markov's theorem for the unit disk in the complex plane instead of for the interval $[-1,1]$. He asked, if $P(z)$ is a polynomial of degree n and $|P(z)| \leq 1$ for $|z| \leq 1$, how large can $|P'(z)|$ be for $|z| \leq 1$. Using maximum modulus principle we can say that he asked, if $P(z)$ is a polynomial of degree n and $|P(z)| \leq 1$ for $|z| = 1$, how large can $|P'(z)|$ be for $|z| = 1$. The answer [2] is that $|P'(z)| \leq n$ for $|z| = 1$, with equality attained for $P(z) = z^n$. Entire result can be restated as: If $P(z)$ is a polynomial of degree n such that $\max_{|z|=1} |P(z)| \leq 1$ then $\max_{|z|=1} |P'(z)| \leq n$ (i.e. $\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|$). And this result itself has come to be known as Bernstein's Theorem. Bernstein's theorem has been generalized in many ways with applications in Theory of Approximation. Thinking similarly for results with applications in Theory of Approximation by using Schwarz's lemma and its various implications Rahman and Mohammad [6] thought of obtaining a bound for

$$\max_{|z|=1} \left|\frac{p(z)}{z-a}\right|,$$

$p(z)$ being a polynomial of degree at most n, with $\max_{|z|=1} |p(z)| = 1$ and $p(a) = 0$ for a fixed a on the unit circle and proved

Theorem A. *If $p(z)$ is a polynomial of degree n such that $|p(z)| \leq 1$ on the unit circle and $p(1) = 0$ then for $|z| \leq 1$*

$$\left|\frac{p(z)}{z-1}\right| \leq \frac{n}{2}.$$

The example $\frac{1}{2}(z^n - 1)$ shows that the result is best possible.

Aziz [1] obtained a refinement of Theorem A and proved

Theorem B. *Let $p(z)$ be a polynomial of degree n such that $p(\beta) = 0$ where β is an arbitrary non-negative real number. If $z_1, z_2, \ldots, z_n$ are the zeros of $z^n + 1$ then*

$$\max_{|z|=1} \left|\frac{p(z)}{z-\beta}\right| \leq \frac{n}{1+\beta} \max_{1 \leq i \leq n} |p(z_i)|.$$

We [3] obtained the following generalization of Theorem B.

Theorem C. *Let $p(z)$ be a polynomial of degree n such that*

$$p(z) = (z-\beta)^k q(z),\ k \geq 1 \text{ and } \beta \text{ is arbitrary.}$$

Then

$$\max_{|z|=1} \left|\frac{p(z)}{(z-\beta)^k}\right| \leq \Big(\frac{n-k+1}{1+|\beta|}\Big)^k \max_{1 \leq l \leq n-k+1} |p(\gamma_l')|,$$

where $\gamma_1', \gamma_2', \ldots, \gamma_{n-k+1}'$ are the roots of

$$z^{n-k+1} + e^{i\gamma(n-k+1)} = 0$$

and

$$\gamma = \begin{cases} \arg\beta & , \beta \neq 0, \\ 0 & , \beta = 0. \end{cases}$$

In this paper we consider a polynomial $p(z)$ of degree n with zeros $\beta_1, \beta_2, \ldots, \beta_k$ and prove certain inequalities which help us to obtain a refinement of Theorem C. More precisely we prove

Theorem 1. *Let $p(z)$ be a polynomial of degree n such that*

$$p(z) = \big\{(z-\beta_1)(z-\beta_2)\ldots(z-\beta_k)\big\}q(z),\ \ k > 1. \tag{1.1}$$

Further let

$$\gamma_k = \begin{cases} \arg\beta_k & ,\beta_k \neq 0, \\ 0 & ,\beta_k = 0, \end{cases}$$

with $v_1^{(k)}, v_2^{(k)}, \ldots, v_n^{(k)}$ being the roots of

$$z^n + e^{in\gamma_k} = 0.$$

Then

$$\max_{|z|=1}\left|\tfrac{p(z)}{(z-\beta_1)(z-\beta_2)\ldots(z-\beta_k)}\right| \leq \tfrac{n(n-1)\ldots(n-k+1)}{(1+|\beta_1|)(1+|\beta_2|)\ldots(1+|\beta_k|)}\max_{1\leq l\leq n}|p(v_l^{(k)})|.$$

As the order of $\beta_1, \beta_2, \ldots, \beta_k$ is immaterial, we can obtain

Theorem 2. *Let $p(z)$ be a polynomial of degree n such that*

$$p(z) = \big\{(z-\beta_1)(z-\beta_2)\ldots(z-\beta_k)\big\}q(z),\ \ k > 1.$$

Further for $1 \leq j \leq k$ let

$$\gamma_j = \begin{cases} \arg\beta_j & ,\beta_j \neq 0, \\ 0 & ,\beta_j = 0, \end{cases}$$

with $v_1^{(j)}, v_2^{(j)}, \ldots, v_n^{(j)}$ being the roots of

$$z^n + e^{in\gamma_j} = 0.$$

Then for $1 \leq j \leq k$

$$\max_{|z|=1}\left|\tfrac{p(z)}{(z-\beta_1)(z-\beta_2)\ldots(z-\beta_k)}\right| \leq \tfrac{n(n-1)\ldots(n-k+1)}{(1+|\beta_1|)(1+|\beta_2|)\ldots(1+|\beta_k|)}\max_{1\leq l\leq n}|p(v_l^{(j)})|.$$

Using Theorem 2 we obtain

Corollary 1. *Under the same hypotheses as in Theorem 2*

$$\max_{|z|=1}\left|\tfrac{p(z)}{(z-\beta_1)(z-\beta_2)\ldots(z-\beta_k)}\right| \leq \tfrac{n(n-1)\ldots(n-k+1)}{(1+|\beta_1|)(1+|\beta_2|)\ldots(1+|\beta_k|)}\min_{1\leq j\leq k}\big(\max_{1\leq m\leq n}|p(v_m^{(j)})|\big).$$

Remark 1. *For $k = 1$ Corollary 1 is Theorem C with $k = 1$ and is therefore true.*

Theorem 3. *Let $p(z)$ be a polynomial of degree n such that*

$$p(z) = \{(z-\beta_1)(z-\beta_2)\dots(z-\beta_k)\}q(z), k \geq 1. \tag{1.2}$$

Further let $v_1, v_2, \dots, v_{n-k+1}$ be the roots of

$$z^{n-k+1} + 1 = 0$$

and for $1 \leq j \leq k$ let

$$S_j = \begin{cases} \left|\frac{1-\beta_j}{1-|\beta_j|^2}\right| & , |\beta_j| \neq 1, \\ \frac{1}{2} & , \beta_j = 1, \\ \infty, (\textit{with the understanding that for such a possibility, the expression } \{(\prod_{j=1}^k S_j)\max_{1\leq s\leq n-k+1}|p(v_s)|\} \textit{ will also take the value } \infty) & , |\beta_j| = 1 \textit{ with } \beta_j \neq 1. \end{cases} \tag{1.3}$$

Then

$$\max_{|z|=1}\left|\frac{p(z)}{(z-\beta_1)(z-\beta_2)\dots(z-\beta_k)}\right| \leq (n-k+1)^k\Big(\prod_{j=1}^k S_j\Big)\max_{1\leq s\leq n-k+1}|p(v_s)|.$$

Theorem 4. *Let $p(z)$ be a polynomial of degree n such that*

$$p(z) = \{(z-\beta_1)(z-\beta_2)\dots(z-\beta_k)\}q(z), \ k > 1.$$

Further for $1 \leq j \leq k$ let

$$\gamma_j = \begin{cases} \arg\beta_j & , \beta_j \neq 0, \\ 0 & , \beta_j = 0, \end{cases}$$

with $t_1^{(j)}, t_2^{(j)}, \dots, t_{n-k+1}^{(j)}$ being the roots of

$$z^{n-k+1} + e^{i(n-k+1)\gamma_j} = 0$$

and for $1 \leq l \leq k$, with $l \neq j$ let

$$T_l^{(j)} = \begin{cases} 1 & , \beta_l = \beta_j, \\ 2\left|\frac{1-|\beta_l|e^{i(\gamma_l-\gamma_j)}}{1-|\beta_l|^2}\right| + \left|\frac{1-|\beta_j|}{1-|\beta_l|}\right| & , \beta_l \neq \beta_j \textit{ with } |\beta_l| \neq 1, \\ \infty, (\textit{with the understanding that for such a possibility, the expression } \{(\prod_{l=1, l\neq j}^k T_l^{(j)})\max_{1\leq s\leq n-k+1}|p(t_s^{(j)})|\} \textit{ will also take the value } \infty) & , \beta_l \neq \beta_j \textit{ with } |\beta_l| = 1. \end{cases} \tag{1.4}$$

Then for $1 \leq j \leq k$

$$\max_{|z|=1} \left|\frac{p(z)}{(z-\beta_1)(z-\beta_2)\ldots(z-\beta_k)}\right| \leq \frac{(n-k+1)^k}{(1+|\beta_j|)^k}\left\{\left(\prod_{\substack{l=1\\ l\neq j}}^{k} T_l^{(j)}\right)\max_{1\leq s\leq n-k+1}|p(t_s^{(j)})|\right\}.$$

Using Theorem 4 we obtain

Corollary 2. *Under the same hypotheses as in Theorem 4*

$$\max_{|z|=1} \left|\frac{p(z)}{(z-\beta_1)(z-\beta_2)\ldots(z-\beta_k)}\right| \leq (n-k+1)^k \min_{1\leq j\leq k}\left(\frac{\left\{\left(\prod_{\substack{l=1\\ l\neq j}}^{k} T_l^{(j)}\right)\max_{1\leq s\leq n-k+1}|p(t_s^{(j)})|\right\}}{(1+|\beta_j|)^k}\right).$$

On combining Corollary 1, Theorem 3 and Corollary 2 we obtain the following refinement as well as a generalization of Theorem C with $k > 1$.

Theorem 5. *Let* $p(z)$ *be a polynomial of degree* n *such that*

$$p(z) = \{(z-\beta_1)(z-\beta_2)\ldots(z-\beta_k)\}q(z), \;\; k > 1.$$

Further let $v_1, v_2, \ldots, v_{n-k+1}$ *be the roots of*

$$z^{n-k+1} + 1 = 0$$

and for $1 \leq j \leq k$ *let*

$$S_j = \begin{cases} \left|\frac{1-\beta_j}{1-|\beta_j|^2}\right| & , |\beta_j| \neq 1, \\ \frac{1}{2} & , \beta_j = 1, \\ \infty, (\textit{with the understanding} & , |\beta_j| = 1 \textit{ with } \beta_j \neq 1, \\ \;\textit{that for such a possibility,} & \\ \;\textit{the expression} & \\ \left\{\left(\prod_{j=1}^{k} S_j\right)\max_{1\leq s\leq n-k+1}|p(v_s)|\right\} & \\ \;\textit{will also take the value } \infty) & \end{cases}$$

$$\gamma_j = \begin{cases} \arg\beta_j & , \beta_j \neq 0, \\ 0 & , \beta_j = 0, \end{cases}$$

with $v_1^{(j)}, v_2^{(j)}, \ldots, v_n^{(j)}$ *being the roots of*

$$z^n + e^{in\gamma_j} = 0,$$

$t_1^{(j)}, t_2^{(j)}, \ldots, t_{n-k+1}^{(j)}$ *being the roots of*

$$z^{n-k+1} + e^{i(n-k+1)\gamma_j} = 0$$

and for $1 \le l \le k$, with $l \ne j$ let

$$T_l^{(j)} = \begin{cases} 1 & ,\beta_l = \beta_j, \\ 2\left|\frac{1-|\beta_l|e^{i(\gamma_l-\gamma_j)}}{1-|\beta_l|^2}\right| + \left|\frac{1-|\beta_j|}{1-|\beta_l|}\right| & ,\beta_l \ne \beta_j \text{ with } |\beta_l| \ne 1, \\ \infty, (\text{with the understanding that} & ,\beta_l \ne \beta_j \text{ with } |\beta_l| = 1. \\ \text{for such a possibility, the expression} & \\ \Big\{\Big(\prod_{\substack{l=1\\ l\ne j}}^{k} T_l^{(j)}\Big) \max_{1\le s\le n-k+1} |p(t_s^{(j)})|\Big\} & \\ \text{will also take the value } \infty) & \end{cases}$$

Then

$$\begin{aligned} &\max_{|z|=1} \left|\frac{p(z)}{(z-\beta_1)(z-\beta_2)\ldots(z-\beta_k)}\right| \le \\ &\min\Bigg[\frac{n(n-1)\ldots(n-k+1)}{(1+|\beta_1|)(1+|\beta_2|)\ldots(1+|\beta_k|)} \min_{1\le j\le k}(\max_{1\le m\le n} |p(v_m^{(j)})|), \\ &(n-k+1)^k(\Pi_{j=1}^k S_j) \max_{1\le s\le n-k+1} |p(v_s)|, \\ &(n-k+1)^k \min_{1\le j\le k}\Big\{\frac{\Big(\prod_{\substack{l=1\\ l\ne j}}^{k} T_l^{(j)}\Big) \max_{1\le s\le n-k+1} |p(t_s^{(j)})|}{(1+|\beta_j|)^k}\Big\}\Bigg]. \end{aligned}$$

And on combining Theorem 3 with $k = 1$ and Corollary 1 for $k = 1$ along with Remark 1 we obtain the following refinement of Theorem C with $k = 1$.

Theorem 6. *Let $p(z)$ be a polynomial of degree n such that*

$$p(z) = (z - \beta_1)q(z).$$

Let $v_1, v_2, \ldots, v_n$ be the roots of

$$z^n + 1 = 0$$

and

$$S_1 = \begin{cases} \left|\frac{1-\beta_1}{1-|\beta_1|^2}\right| & ,|\beta_1| \ne 1, \\ \frac{1}{2} & ,\beta_1 = 1, \\ \infty, (\text{with the understanding} & ,|\beta_1| = 1 \text{ with } \beta_1 \ne 1, \\ \text{that for such a possibility,} & \\ \text{the expression } \{S_1 \max_{1\le s\le n} |p(v_s)|\} & \\ \text{will also take the value } \infty) & \end{cases}$$

$$\gamma_1 = \begin{cases} \arg\beta_1 & ,\beta_1 \ne 0, \\ 0 & ,\beta_1 = 0, \end{cases}$$

with $v_1^{(1)}, v_2^{(1)}, \ldots, v_n^{(1)}$ being the roots of

$$z^n + e^{in\gamma_1} = 0.$$

Then

$$\max_{|z|=1}\left|\frac{p(z)}{z-\beta_1}\right| \le \min\Big[\frac{n}{1+|\beta_1|}\max_{1\le m\le n}|p(v_m^{(1)})|,\ nS_1\max_{1\le s\le n}|p(v_s)|\Big].$$

Remark 2. *For* $p(z)=(z-1)^2(z+1)(z+2)$ *with*

$$\beta_1=\beta_2=\beta=1,\ k=2 \text{ and } n=4,$$

the bound for

$$\max_{|z|=1}\left|\frac{p(z)}{(z-1)^2}\right|$$

is 20.7 by Theorem C and 11.7 by Theorem 5. And for $p(z)=(z+100)(z+1)^2$ *with*

$$k=1,\ \beta_1=\beta=-100 \text{ and } n=3,$$

the bound for

$$\max_{|z|=1}\left|\frac{p(z)}{z+100}\right|$$

is 12 by Theorem C and 9.14 by Theorem 6.

2 Lemmas

For the proofs of the theorems we require the following lemmas.

Lemma 1. *Let* $z_1, z_2, \ldots, z_n$ *be the zeros of* z^n+1. *Then*

$$\sum_{l=1}^{n}\frac{1}{|z_l-1|^2}=\frac{n^2}{4}.$$

This lemma is due to Aziz [1, relation 11].

Lemma 2. *Under the same hypothesis as in Lemma 1*

$$\frac{1}{|z_l-1|}<\frac{n}{2}, 1\le l\le n.$$

Proof of Lemma 2. If follows easily from Lemma 1.

Lemma 3.

$$\left|\frac{e^{i\theta}-1}{e^{i\theta}-\beta}\right|\le 2\left|\frac{1-\beta}{1-|\beta|^2}\right|,\ |\beta|\ne 1 \ \&\ -\pi\le\theta\le\pi.$$

Proof of Lemma 3. It follows by using usual method for finding maximum value of a function of one variable.

Lemma 4. *Let γ and δ be two complex numbers such that*

$$\gamma \neq \delta \text{ and } |\delta| \neq 1,$$

with

$$\phi_1 = \begin{cases} \arg\gamma & ,\gamma \neq 0, \\ 0 & ,\gamma = 0, \end{cases}$$

$$\phi_2 = \begin{cases} \arg\delta & ,\delta \neq 0, \\ 0 & ,\delta = 0, \end{cases}$$

and

$$\phi = \phi_2 - \phi_1.$$

Then

$$|\frac{e^{i\theta}-\gamma}{e^{i\theta}-\delta}| \leq 2|\frac{1-|\delta|e^{i\phi}}{1-|\delta|^2}| + |\frac{1-|\gamma|}{1-|\delta|}|, \quad -\pi \leq \theta \leq \pi.$$

Proof of Lemma 4.

$$\begin{aligned} |\frac{e^{i\theta}-\gamma}{e^{i\theta}-\delta}| &= |\frac{e^{i\psi}-|\gamma|}{e^{i\psi}-|\delta|e^{i\phi}}|, \ (\psi = \theta - \phi_1), \\ &\leq 2|\frac{1-|\delta|e^{i\phi}}{1-|\delta|^2}| + |\frac{1-|\gamma|}{1-|\delta|}|, \ (\text{by Lemma 3}). \end{aligned}$$

This completes the proof of Lemma 4.

3 Proofs of the theorems

Proof of Theorem 1. The polynomial

$$T_1(z) = (z-\beta_1)q(z) \tag{3.1}$$

is of degree $n-k+1$ and by Theorem C with $k=1$ we have

$$\max_{|z|=1}|q(z)| = \max_{|z|=1}|\frac{T_1(z)}{z-\beta_1}| \leq \frac{n-k+1}{1+|\beta_1|} \max_{1\leq l_1\leq n-k+1}|T_1(v_{l_1}^{(1)})|, \tag{3.2}$$

with $v_1^{(1)}, v_2^{(1)}, \ldots, v_{n-k+1}^{(1)}$ being the roots of

$$z^{n-k+1} + e^{i\gamma_1(n-k+1)} = 0 \tag{3.3}$$

and

$$\gamma_1 = \begin{cases} \arg\beta_1 & ,\beta_1 \neq 0, \\ 0 & ,\beta_1 = 0. \end{cases}$$

Further the polynomial

$$\begin{aligned} T_2(z) &= (z-\beta_2)T_1(z), \\ &= (z-\beta_1)(z-\beta_2)q(z), \ (\text{by (3.1)}), \end{aligned} \tag{3.4}$$

is of degree $n-k+2$ and by Theorem C with $k=1$ we have

$$\max_{|z|=1}|T_1(z)| = \max_{|z|=1}|\frac{T_2(z)}{z-\beta_2}| \leq \frac{n-k+2}{1+|\beta_2|}\max_{1\leq l_2\leq n-k+2}|T_2(v_{l_2}^{(2)})|, \tag{3.5}$$

with $v_1^{(2)}, v_2^{(2)}, \ldots, v_{n-k+2}^{(2)}$ being the roots of

$$z^{n-k+2} + e^{i\gamma_2(n-k+2)} = 0$$

and

$$\gamma_2 = \begin{cases} \arg\beta_2 & ,\beta_2 \neq 0, \\ 0 & ,\beta_2 = 0. \end{cases}$$

Now as

$$|v_{l_1}^{(1)}| = 1,\ 1 \leq l_1 \leq n-k+1,\ (\text{by } (3.3)),$$

we can combine (3.2) and (3.5) and obtain

$$\max_{|z|=1}|q(z)| \leq \frac{(n-k+1)(n-k+2)}{(1+|\beta_1|)(1+|\beta_2|)}\max_{1\leq l_2\leq n-k+2}|T_2(v_{l_2}^{(2)})|.$$

We can now continue and obtain similarly

$$\max_{|z|=1}|q(z)| \leq \frac{(n-k+1)(n-k+2)(n-k+3)}{(1+|\beta_1|)(1+|\beta_2|)(1+|\beta_3|)}\max_{1\leq l_3\leq n-k+3}|T_3(v_{l_3}^{(3)})|,$$

(with

$$\begin{aligned} T_3(z) &= (z-\beta_3)T_2(z), \\ &= (z-\beta_1)(z-\beta_2)(z-\beta_3)q(z),\ (\text{by } (3.4)), \end{aligned} \tag{3.6}$$

$v_1^{(3)}, v_2^{(3)}, \ldots, v_{n-k+3}^{(3)}$ being the roots of

$$z^{n-k+3} + e^{i\gamma_3(n-k+3)} = 0$$

and

$$\gamma_3 = \begin{cases} \arg\beta_3 & ,\beta_3 \neq 0, \\ 0 & ,\beta_3 = 0, \end{cases} \qquad),$$

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$$\max_{|z|=1}|q(z)| \leq \frac{(n-k+1)(n-k+2)\ldots(n-k+k)}{(1+|\beta_1|)(1+|\beta_2|)\ldots(1+|\beta_k|)}\max_{1\leq l\leq n-k+k}|T_k(v_l^{(k)})|, \tag{3.7}$$

(with

$$T_k(z) = (z-\beta_k)T_{k-1}(z),$$
$$= (z-\beta_1)(z-\beta_2)\ldots(z-\beta_k)q(z), (\text{similar to (3.4) and (3.6)})) \quad (3.8)$$

Now Theorem 1 follows by using (1.1) and (3.8) in (3.7).

Proof of Theorem 3. If

$$|\beta_j| = 1 \text{ with } \beta_j \neq 1,$$

for at least one j, $1 \leq j \leq k$ then Theorem 3 follows trivially. Therefore we now assume that

$$|\beta_j| \neq 1 \text{ or } \beta_j = 1, \ 1 \leq j \leq k.$$

Further let

$$T(z) = (z-1)q(z).$$

Then by Theorem B with $\beta = 1$

$$\max_{|z|=1} |q(z)| = \max_{|z|=1} |\frac{T(z)}{z-1}| \leq \frac{n-k+1}{2} \max_{1 \leq s \leq n-k+1} |T(v_s)|. \quad (3.9)$$

Now

$$|T(v_s)| = \frac{1}{|v_s - 1|^{k-1}} \Big(\prod_{j=1}^{k} |\frac{v_s - 1}{v_s - \beta_j}|\Big) |p(v_s)|,$$

which by Lemma 2, Lemma 3 and (1.3) implies that

$$|T(v_s)| \leq 2(n-k+1)^{k-1} \Big(\prod_{j=1}^{k} S_j\Big) |p(v_s)|$$

and therefore by (1.2) and (3.9) we get

$$\max_{|z|=1} |\tfrac{p(z)}{(z-\beta_1)(z-\beta_2)\ldots(z-\beta_k)}| \leq$$
$$(n-k+1)^k \big(\textstyle\prod_{j=1}^{k} S_j\big) \max_{1 \leq s \leq n-k+1} |p(v_s)|.$$

This completes the proof of Theorem 3.

Proof of Theorem 4. If

$$\beta_l \neq \beta_j \text{ with } |\beta_l| = 1,$$

for at least one l, $1 \leq l \leq k$ with $l \neq j$ then Theorem 4 follows trivially. Therefore we now assume that

$$\beta_l = \beta_j \text{ or } \beta_l \neq \beta_j \text{ with } |\beta_l| \neq 1, \ 1 \leq l \leq k \text{ with } l \neq j.$$

Further let

$$p_j(z) = (z-\beta_j)^k q(z).$$

Then by Theorem C

$$\max_{|z|=1}|q(z)| = \max_{|z|=1}\left|\frac{p_j(z)}{(z-\beta_j)^k}\right| \le \left(\frac{n-k+1}{1+|\beta_j|}\right)^k \max_{1\le s\le n-k+1}|p_j(t_s^{(j)})|. \tag{3.10}$$

Now

$$|p_j(t_s^{(j)})| = \Big(\prod_{\substack{l=1\\ l\ne j}}^{k}\left|\frac{t_s^{(j)}-\beta_j}{t_s^{(j)}-\beta_l}\right|\Big)|p(t_s^{(j)})|,$$

which by Lemma 4 and (1.4) implies that

$$|p_j(t_s^{(j)})| \le \Big(\prod_{\substack{l=1\\ l\ne j}}^{k} T_l^{(j)}\Big)|p(t_s^{(j)})|$$

and therefore by (3.10) and (1.1) we get

$$\max_{|z|=1}\left|\frac{p(z)}{(z-\beta_1)(z-\beta_2)\dots(z-\beta_k)}\right| \le \frac{(n-k+1)^k}{(1+|\beta_j|)^k}\Big\{\Big(\prod_{\substack{l=1\\ l\ne j}}^{k} T_l^{(j)}\Big)\max_{1\le s\le n-k+1}|p(t_s^{(j)})|\Big\}.$$

This completes the proof of Theorem 4.

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Received: 14.10.2014
Revised: 02.02.2015
Accepted: 06.02.2015

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