Bull. Math. Soc. Sci. Math. Roumanie
Tome 58(106) No. 3, 2015, 271–282

Acute Triangulations of Archimedean Surfaces. The Truncated Tetrahedron

by
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To Bazil Brînzănescu on the occasion of his 70th birthday

Abstract

In this paper we prove that the surface of the regular truncated tetrahedron can be triangulated into 10 non-obtuse geodesic triangles, and also into 12 acute geodesic triangles. Furthermore, we show that both triangulations have minimal size.

Key Words: Archimedean surface, regular truncated tetrahedron, acute triangulation, cut locus.
2010 Mathematics Subject Classification: Primary 52B10,
Secondary 52B05, 53C45.

1 Introduction

We call the boundary of a compact convex set in $\mathbb{R}^3$ with non-empty interior a *convex surface*. A shortest path between two points of such a surface will be called a *segment*. A triangle with three segments as sides is called a *geodesic triangle*. By a *triangulation* of a convex surface S we mean a set of geodesic triangles such that every point of S is in some triangle, and the intersection of any two triangles is either empty, or consists of a vertex, or of an edge of both triangles. This makes sense for any compact surface admitting segments, such as connected polyhedral surfaces or Alexandrov surfaces, including Riemannian and convex surfaces. For all these surfaces, two geodesics starting at the same point determine a well defined angle. An *acute* (*non-obtuse*) triangulation is a triangulation such that the angles of all geodesic triangles are smaller (respectively, not greater) than $\frac{\pi}{2}$. The number of triangles in a triangulation is called its *size*.

In 1960, Gardner reported in his "Mathematical Games" section of the *Scientific American* (see [5], [6], [7]) a problem of Stover asking whether an obtuse triangle can be cut into smaller acute triangles. Also in 1960, the same problem has also been independently proposed by Goldberg in [8] and solved by Manheimer [17].

In the same year, Burago and Zalgaller [1] proved the existence of acute triangulations of arbitrary two-dimensional polyhedral surfaces, accidentally also solving the above problem.

Burago and Zalgaller's deep work concentrated on the existence, not on the size of the triangulations. Later, more attention was paid to the size.

In 1980, Cassidy and Lord [2] considered acute triangulations of the square. More recently, acute triangulations of quadrilaterals [14], [3], trapezoids [25], pentagons [24] and arbitrary convex polygons [15, 23, 30] have also been considered, all of them containing size estimates, in several cases proven to be optimal.

Several compact convex or flat surfaces have also been investigated: the surfaces of all Platonic solids in [9], [11], [12], and [13], double triangles in [29], double quadrilaterals in [26], double planar convex bodies in [28], flat Möbius strips in [27], and flat tori in [10].

In 2009, the existence of acute triangulations of polyhedral surfaces was considered again by Saraf [21], who gave a shorter proof of their existence, still without estimates on the size. In 2011, Maehara [16] provided for the first time bounds (depending on some natural geometric parameters) for the size of acute triangulations of an arbitrary polyhedral surface. See also Zamfirescu's survey [31].

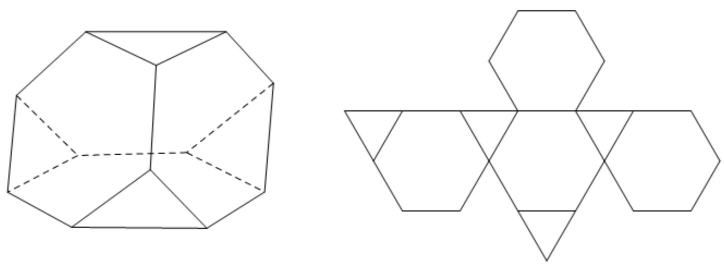

Figure 1: The surface $\mathcal{S}$ of the regular truncated tetrahedron

Motivated by the now complete study of the acute (and non-obtuse) triangulations of all Platonic surfaces, including optimal bounds for their size, we want to discuss here the case of the surfaces of Archimedean solids, and start with perhaps the most important of them, the surface of the regular truncated tetrahedron, shown in Figure 1. This Archimedean solid has four regular hexagons and four equilateral triangles as faces.

The *cut locus* of a point on a complete, simply connected and real analytic Riemannian 2-manifold, was introduced and first investigated in 1905 by Poincaré [20]. Later, Myers [18, 19], Whitehead [22] and many more continued its study. The notion can be easily extended to any Alexandrov surfaces, including all polytopal surfaces. (Remember that a *polytope* is the convex hull of a finite set, and a *polytopal surface* is the boundary of a polytope.)

Let S be a polytopal surface. For each point $p \in S$ and each direction τ at p, there exist many segments starting at p in direction τ, including each other. If px is the maximal one, the point x is called the *cut point* of p in direction τ. The *cut locus* $C(p)$ of p is the set of cut points of p in all directions.

For example, let p be a vertex of a cube. Then, on the surface of the cube, $C(p)$ is the union of six line-segments starting from the vertex antipodal to p, among which three line-segments are edges of the cube, and the other three are diagonals of the incident faces.

The cut locus will be one of our main tools. Some well-known properties of the cut-locus on a polytopal surface S follow.

The set $S \setminus C(p)$ is homeomorphic to the open disc, and each of its points is joined with p by a single segment. The cut-locus is topologically a tree. If $x \in C(p)$ is not an endpoint of the tree $C(p)$, then at least two segments join x to p. If the number of segments is 2, then they form equal angles with $C(p)$. In our case of a polytopal surface, $C(p)$ is always a finite union of line-segments. If a point x is joined by two segments with p, then $x \in C(p)$.

Now, let $\mathcal{S}$ be the boundary and $\mathscr{S}$ the 1-skeleton of the regular truncated tetrahedron with side length 1. The set of vertices of $\mathcal{S}$ will be denoted by $V(\mathcal{S})$. The graph-theoretic distance $d_{\mathscr{S}}(v, w)$ between vertices $v, w \in V(\mathcal{S})$ of the graph $\mathscr{S}$ is called the *$\mathscr{S}$-distance*.

Let uv denote the segment between two points u and v on the surface $\mathcal{S}$, if the segment is unique. If there is more than one segment from u to v, we still use the notation uv, after specifying which one of the segments we have in mind. Its length will be denoted by $|uv|$.

The size of a triangulation $\mathscr{T}$ of $\mathcal{S}$ will similarly be denoted by $|\mathscr{T}|$, no confusion being possible.

Now, let $\mathscr{T}$ be an acute triangulation of $\mathcal{S}$, and $\mathscr{T}_0$ a non-obtuse one. We shall prove here that $|\mathscr{T}_0| \geq 10$ and $|\mathscr{T}| \geq 12$.

2 Non-obtuse triangulations

Let u, v be two vertices of $\mathcal{S}$. If a segment passes through the interiors of two hexagonal faces of $\mathcal{S}$, then we call it an *h-segment*. Let $n(u, v)$ denote the maximal number of vertices of $\mathcal{S}$ lying in the union of the interiors of any two non-obtuse triangles with common edge uv. If uv is an edge of a non-obtuse triangulation, let $\triangle uv$ denote the triangle with one side uv and lying on the left side of uv (when looking from u to v). Clearly, $\triangle vu$ is the other triangle of the triangulation sharing the side uv.

Lemma 1. *Let u, v be two vertices of $\mathcal{S}$. If uv is not an h-segment, then $n(u, v) \leq 1$.*

Proof: The total angle around each vertex of $\mathcal{S}$ is $\frac{5\pi}{3}$. By the Gauß-Bonnet formula, no non-obtuse triangle on $\mathcal{S}$ can contain two vertices of $\mathcal{S}$ in its interior. Thus, we only need to show that one of the triangles $\triangle uv$, $\triangle vu$ contains no vertex of $\mathcal{S}$ in its interior. Denote by w the third vertex of the non-obtuse triangle $\triangle uv$. Let G^u_{uv}, G^v_{uv} be the maximal segments starting from u, respectively v, in a direction orthogonal to uv, to the left (same side as $\triangle uv$).

Our strategy will be the following. For various $u, v \in V(\mathcal{S})$, we consider G^u_{uv} and G^v_{uv}. Let $G^u_{uv} = uu^\star$ and $G^v_{uv} = vv^\star$. Then $u^\star \in C(u)$ and $v^\star \in C(v)$. We shall look for a point $z \in C(u) \cap C(v)$ and for the arcs $\widetilde{u^\star z} \subset C(u)$ and $\widetilde{v^\star z} \subset C(v)$. It is inside the region $uvv^\star zu^\star u$ that we can possibly find positions for the third vertex of $\triangle uv$.

Case 1. $d_{\mathscr{S}}(u, v) = 1$

In this case, uv is the common edge of two hexagons or the common edge of a hexagon and a triangle. Interchange u and v if necessary, such that $\triangle uv$ lies on the same side of uv as a hexagon.

Clearly, $G^u_{uv} = uu_1$ and $G^v_{uv} = vv_1$, see Figure 2. From every point of $u_1p \setminus \{u_1\}$ there are two segments to u, one through a triangle (pu_1v_1) and a hexagon, and a second through two hexagons. Hence, $u_1p \subset C(u)$, analogously $v_1p \subset C(v)$, and $z = p$. Thus, the third vertex of

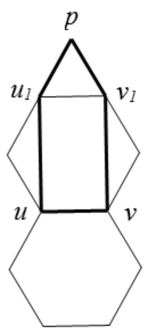

Figure 2: $d_{\mathscr{S}}(u,v)=1$

$\triangle uv$ must be contained in the pentagonal region uvv_1pu_1u. Obviously, there is no vertex of $\mathcal{S}$ there.

Case 2. $d_{\mathscr{S}}(u,v)=2$

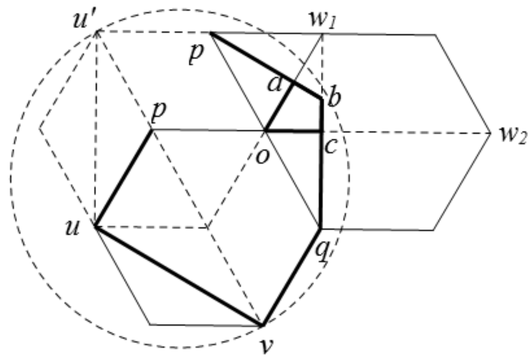

Figure 3: $d_{\mathscr{S}}(u,v)=2$

Then uv is a small diagonal of a hexagon. In Figure 3, it is easy to see that $G^u_{uv}=up$ and $G^v_{uv}=vq$. In the triangular face of $\mathcal{S}$ with a vertex at p, let a be the midpoint of the side opposite to p. We have $pa\subset C(u)$, because each point of $pa\setminus\{p\}$ is joined with u by two segments. Let w_1 be the vertex of $\mathcal{S}$ such that $d_{\mathscr{S}}(w_1,u)=d_{\mathscr{S}}(w_1,q)=2$. Similarly, $qw_1\subset C(v)$. Take $b\in qw_1$ such that $a\in pb$. Then $pb\subset C(u)$. Now we get $z=b$. Thus, if both $\angle uvw$ and $\angle vuw$ are not greater than $\frac{\pi}{2}$, the vertex w must be located inside the pentagonal region $uvqbpu$.

Suppose there is one vertex of $\mathcal{S}$ in the interior of $\triangle uv$. Then this must be the vertex o adjacent to both p and q. Let w_2 be the vertex of $\mathcal{S}$ such that $d_{\mathscr{S}}(w_2,w_1)=d_{\mathscr{S}}(w_2,q)=2$.

Let c be the intersection point of qw_1 and ow_2. Then $oc \subset C(u)$ and $ow_1 \subset C(v)$. Thus, in order to ensure that $\triangle uv$ contains o in its interior, the vertex w must be in the quadrilateral region $oabco$.

However, $oabco$ is contained in the interior of the circle D whose diameter is the dash line-segment $u'v$ considered in an unfolding of $\mathcal{S}$ on the plane of w_1oqw_2. Indeed, u' is the vertex u, unfolded. Let $\{\omega\} = uo \cap vp$. We have $|u'v| = 3$ and $|\omega v| = 3/2$. But $|\omega o| = \sqrt{3}/2$ and $|ob| = 1/\sqrt{3}$. So, $|\omega b| = |\omega o| + |ob| = 5\sqrt{3}/6 < 3/2$, that is, the point b is in the interior of the circle D. Clearly, the points o, a and c are all in the interior of the circle D, and therefore $oabco$ is contained in the interior of the circle D. Thus $\angle uwv > \frac{\pi}{2}$, which is a contradiction.

Case 3. $d_{\mathscr{S}}(u,v) = 3$

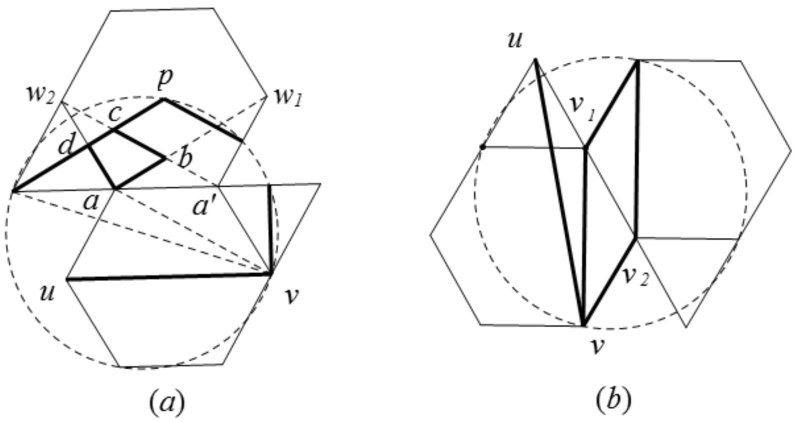

Figure 4: $d_{\mathscr{S}}(u,v) = 3$

Since uv is not an h-segment, there are only two subcases to discuss.

Subcase 1. uv is a (long) diagonal of a hexagon, shown in Figure 4(a).

Let p be the intersection point of G^u_{uv} and G^v_{uv}. In order to ensure that both $\angle uvw$ and $\angle vuw$ are not greater than $\frac{\pi}{2}$, the vertex w must be in the triangular region $uvpu$.

Suppose there is one vertex of $\mathcal{S}$ in the interior of $\triangle uv$. By the symmetry, without loss of generality, we may assume that the vertex a adjacent to u is in the interior of $\triangle uv$, which implies that the vertex a' adjacent to v is outside of $\triangle uv$. Let w_1, w_2 be vertices of $\mathcal{S}$ such that w_1 is adjacent to a' and v, w_2 is adjacent to a and u. It is easy to see that $aw_1 \subset C(u)$, $a'w_2 \subset C(v)$ and $aw_2 \subset C(v)$. Let $\{b\} = aw_1 \cap a'w_2$, $\{c\} = up \cap a'w_2$ and $\{d\} = up \cap aw_2$. Thus, the vertex w must be in the quadrilateral region $abcda$. However, $abcda$ is contained in the interior of the circle D' whose diameter is the dash line-segment $u'v$ considered in an unfolding of $\mathcal{S}$ on the plane of $uva'a$, see Figure 6(a).

Thus, $\angle uwv > \frac{\pi}{2}$, a contradiction.

Subcase 2. uv passes through the interiors of a triangle and a hexagon, see Figure 4(b).

Let v_1, v_2 denote vertices of $\mathcal{S}$ such that $d_{\mathscr{S}}(u, v_1) = d_{\mathscr{S}}(v, v_2) = d_{\mathscr{S}}(v_1, v_2) = 1$. Clearly, $v_1v \cup v_2v \subset C(u)$, $v_1u \cup v_2u \subset C(v)$. It is not difficult to check that if the triangle $\triangle uv$ contains v_2 in its interior, then it must contain v_1. Hence, we suppose that $\triangle uv$ only contains v_1 in its interior. Then w must lie in the region uv_1vv_2u which, unfolded, is a parallelogram with angles $\pi/6$, $5\pi/6$, $\pi/6$, $5\pi/6$. So $\angle uwv \geq \frac{5\pi}{6} > \frac{\pi}{2}$, a contradiction again. □

From the proof of Lemma 1, we extract the following.

Lemma 2. *If uv is the common edge of two hexagons or a long diagonal of a hexagon, then $n(u, v) = 0$. If uv is the common edge of a triangle and a hexagon, or a short diagonal of a hexagon, then $n(u, v) = 1$.*

We call a triangle *basic*, if all its vertices are vertices of $\mathcal{S}$. Let uv be the common edge of a triangle and a hexagon. Let $w \in V(\mathcal{S})$ be such that $d_{\mathscr{S}}(u, w) = d_{\mathscr{S}}(v, w) = 2$. Then the triangle uvw has two angles of $\frac{\pi}{2}$ at u, v and one angle of $\frac{\pi}{3}$ at w, and contains exactly one vertex of $\mathcal{S}$ in its interior, see Figure 5(a). This kind of basic triangle will be called a *simple triangle.*

Lemma 3. *A basic triangle containing exactly one vertex of $\mathcal{S}$ in its interior is non-obtuse if and only if it is simple. Therefore, no acute basic triangle contains vertices of $\mathcal{S}$ in its interior.*

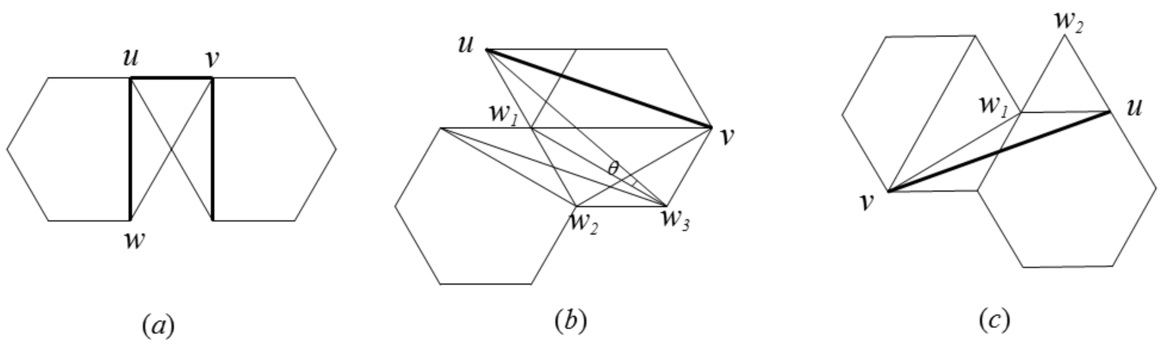

Figure 5: The non-obtuse basic triangle

Proof: We consider a non-obtuse triangulation of $\mathcal{S}$, and pay special attention to its basic triangles. The non-obtuse right triangle in Figure 5(a) contains a vertex of $\mathcal{S}$ in its interior. By Lemma 2, we only have to consider the cases when uv meets the interiors of a hexagonal face and a triangular face, or of two hexagonal faces.

If uv passes through the interiors of a triangle and a hexagon, and uvw is a basic triangle, then $\angle wuv \le \frac{\pi}{2}$ and $\angle wvu \le \frac{\pi}{2}$ imply $w \in \{w_1, w_2, w_3\}$, see Figure 5(b). However, $\angle uw_1v = \angle uw_2v = \frac{2\pi}{3}$ and $\angle uw_3v = \frac{\pi}{2} + \theta$ (where $0 < \theta < \frac{\pi}{6}$) if the segment uw_3 is chosen such that the triangle uvw_3 contains w_1 in its interior, or $\angle uw_3v = \frac{\pi}{2} - \theta$ if the choice of uw_3 is such that the triangle uvw_3 contains no vertex of $\mathcal{S}$ in its interior. Hence, there is no non-obtuse basic triangle having uv as an edge and containing a vertex of $\mathcal{S}$ in its interior.

If uv is an h-segment, then $\angle wuv \le \frac{\pi}{2}$ and $\angle wvu \le \frac{\pi}{2}$ imply for the third vertex $w \in \{w_1, w_2\}$, see Figure 5(c). However, $\angle uw_1v = \frac{5\pi}{6} > \frac{\pi}{2}$, $\angle uw_2v = \frac{2\pi}{3} > \frac{\pi}{2}$. Thus, there is no non-obtuse basic triangle with an h-segment as a side, the interior of which meets $V(\mathcal{S})$. □

We also formulate the following useful consequence.

Lemma 4. *If $\triangle uv$ and $\triangle vu$ are basic adjacent triangles, we have $n(u, v) \le 1$.*

Our first main result follows.

Theorem 1. *The surface of the regular truncated tetrahedron admits a non-obtuse triangulation with 10 triangles, and there is no non-obtuse triangulation with fewer triangles.*

Proof: Figure 6 describes the unfolded surface $\mathcal{S}$. Let a be a vertex of $\mathcal{S}$, and let b, c, d, e be vertices of $\mathcal{S}$ such that $d_{\mathscr{S}}(a, b) = d_{\mathscr{S}}(a, c) = d_{\mathscr{S}}(a, d) = d_{\mathscr{S}}(a, e) = 2$. Clearly, $|ab| = |ac| = |ad| = |ae| = \sqrt{3}$. Denote by f and g the centres of the two hexagons which are not adjacent to a. We obtain a geodesic triangulation $\mathscr{T}$ of $\mathcal{S}$ with the following 10 triangles: abc, acd, ade, aeg, agb, fbc, fcd, fde, feg, fgb.

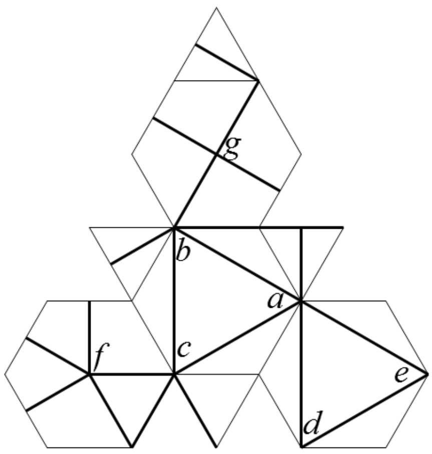

Figure 6: A non-obtuse triangulation of $\mathcal{S}$

The three triangles abc, ade and fcd are equilateral triangles. Each of the remaining seven triangles have two angles of $\frac{\pi}{2}$ and one angle of $\frac{\pi}{3}$. Thus, $\mathscr{T}$ is a non-obtuse triangulation.

Now we prove that ten is the smallest possible number of non-obtuse triangles.

Let $\mathscr{T}_0$ be a non-obtuse triangulation of $\mathcal{S}$. First, observe that $|\mathscr{T}_0|$ is even, because $3|\mathscr{T}_0|$ is the total number of edges, counted twice (fact true for any triangulation of a compact surface without boundary).

If $|\mathscr{T}_0| = 4$, then $\mathscr{T}_0$ is isomorphic to K_4; if $|\mathscr{T}_0| = 6$, then $\mathscr{T}_0$ is isomorphic to the 1-skeleton of a double pyramid over a triangle. In both cases, there are vertices with degree 3. However, at each vertex of $\mathcal{S}$ the total angle is $\frac{5\pi}{3}$. So, the degree of such a vertex in any non-obtuse triangulation of $\mathcal{S}$ using that vertex is at least 4. The degree of other vertices is also at least 4. So, a contradiction is obtained.

If $|\mathscr{T}_0| = 8$, then $\mathscr{T}_0$ is isomorphic to the 1-skeleton of a regular octahedron. Remember that a non-obtuse triangle of $\mathscr{T}_0$ contains at most one vertex in its interior. $\mathcal{S}$ has twelve vertices, hence at least four vertices of $\mathscr{T}_0$ coincide with vertices of $\mathcal{S}$.

If all six vertices of $\mathscr{T}_0$ coincide with vertices of $\mathcal{S}$, then the eight basic triangles must contain the remaining 6 vertices of $\mathcal{S}$ in their interiors. However, by Lemma 4, there are only at most four vertices of $\mathcal{S}$ in their interiors, a contradiction.

If there are exactly five vertices of $\mathscr{T}_0$ in $V(\mathcal{S})$, then they form four basic triangles with one common vertex. They contain at most two vertices of $\mathcal{S}$ in their interiors, by Lemma 4. Therefore, the remaining at least 5 vertices of $\mathcal{S}$ must be contained in the interiors of the other four triangles of $\mathscr{T}_0$. Hence, there is a triangle with at least two vertices of $\mathcal{S}$ in its interior, a contradiction.

If exactly four vertices of $\mathscr{T}_0$ coincide with vertices of $\mathcal{S}$, and the four vertices determine two adjacent triangles of $\mathscr{T}_0$, then, by Lemma 4, the two triangles only contain at most one vertex of $\mathcal{S}$ in their interiors. Hence, the remaining six triangles must contain at least 7 vertices of $\mathcal{S}$ in their interiors, which is impossible by the same argument as before. Therefore, the four vertices determine a 4-cycle C_4 in $\mathscr{T}_0$ which decomposes $\mathcal{S}$ into two regions $\mathcal{S}_1$ and $\mathcal{S}_2$, each of which contains a further vertex of $\mathscr{T}_0$.

Thus, the further two vertices of $\mathscr{T}_0$ are not in $V(\mathcal{S})$, and the remaining 8 vertices of $\mathcal{S}$ lie in the interiors of the eight triangles of $\mathscr{T}_0$. Therefore, each triangle must contain one vertex of $\mathcal{S}$ in its interior. Then, by Lemma 1, all edges uv of C_4 are h-segments. However, on one hand, the total angle at each vertex of $\mathcal{S}$ is $\frac{5\pi}{3}$, so the two angles formed by any two consecutive edges of C_4, say uv and vw, are between $\frac{2\pi}{3}$ and π. On the other hand, it is immediately checked that $\angle uvw < 2\pi/3$, and a contradiction is obtained. □

3 Acute triangulations

Here we prove our second main result.

Theorem 2. *The surface of the regular truncated tetrahedron admits an acute triangulation with* 12 *triangles, and there is no acute triangulation with fewer triangles.*

Proof: Let a, b, c, d be the centres of the four hexagons, let ef be the common edge of the two hexagons with centres b and d, and gh the common edge of the two hexagons with centres a and c, as shown in Figure 7. Clearly, we get a non-obtuse geodesic triangulation $\mathscr{T}_0$ of $\mathcal{S}$ with the following 12 triangles: abe, ade, abg, adh, agh, bcf, bcg, bef, cdf, cdh, cgh, def.

Notice that there are two right angles at each vertex of $\mathscr{T}_0$; to be precise, all the sixteen angles $\angle bag$, $\angle dah$, $\angle abe$, $\angle cbf$, $\angle bcg$, $\angle dch$, $\angle ade$, $\angle cdf$, $\angle aeb$, $\angle aed$, $\angle cfb$, $\angle cfd$, $\angle bga$, $\angle bgc$, $\angle dha$ and $\angle dhc$ are equal to $\frac{\pi}{2}$.

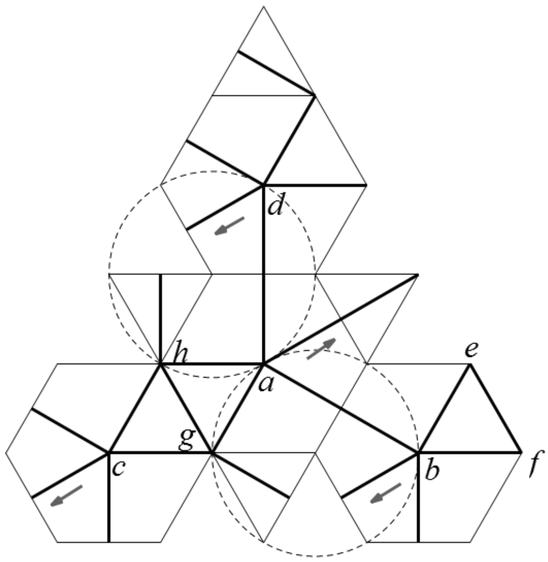

Figure 7: An acute triangulation of $\mathcal{S}$

Now, we will slightly change the position of four vertices of $\mathscr{T}_0$ to get an acute triangulation of $\mathcal{S}$.

Step 1: Slide a slightly to a' in direction $\overrightarrow{ae}$. Then the six right angles $\angle bag$, $\angle dah$, $\angle bga$, $\angle dha$, $\angle abe$ and $\angle ade$ become less than $\frac{\pi}{2}$.

Step 2: Slide b slightly to b' in direction $\overrightarrow{bg}$. Thus, the four right angles $\angle cbf$, $\angle cfb$, $\angle aeb$ and $\angle bcg$ become less than $\frac{\pi}{2}$.

Step 3: Slide c slightly to c' in direction $\overrightarrow{cf}$. Clearly, the four right angles $\angle dch$, $\angle bgc$, $\angle dhc$ and $\angle cdf$ become less than $\frac{\pi}{2}$.

Step 4: Slide d slightly to d' in direction $\overrightarrow{dh}$. Thus, the two right angles $\angle aed$ and $\angle cfd$ become less than $\frac{\pi}{2}$.

At every step, all angles which were previously acute should remain acute. Thus, we obtain an acute geodesic triangulation of $\mathcal{S}$ with the 12 triangles $a'b'e$, $a'd'e$, $a'b'g$, $a'd'h$, $a'gh$, $b'c'f$, $b'c'g$, $b'ef$, $c'd'f$, $c'd'h$, $c'gh$, $d'ef$.

Now we prove that there exists no triangulation with 10 acute triangles. Let $\mathscr{T}$ be an acute triangulation of $\mathcal{S}$ and assume $|\mathscr{T}| = 10$. Then $\mathscr{T}$ is isomorphic to the 1-skeleton of a double pyramid over a pentagon. Besides two vertices of degree 5, $\mathscr{T}$ has five vertices of degree 4 forming a 5-cycle C_5; these five vertices must be vertices of $\mathcal{S}$. The cycle C_5 decomposes $\mathcal{S}$ into two regions, which will be denoted by $\mathcal{S}_u$ and $\mathcal{S}_v$. Let $u \in \mathcal{S}_u$ and $v \in \mathcal{S}_v$ be the two remaining vertices of $\mathscr{T}$.

<u>*Claim.*</u> No edge of C_5 is an h-segment.

Indeed, denote by u_i $(i = 1, 2, 3, 4, 5)$ the vertices of C_5. Suppose one of the edges of C_5, say u_1u_2, is an h-segment.

Let $u_1 = v_1, u_2 = v_2$, see Figure 8. Since $\mathscr{T}$ is acute, the angles formed by any two consecutive edges of C_5 are greater than $\frac{2\pi}{3}$ and less than π, hence $u_3 \in \{v_3, v_4\}$.

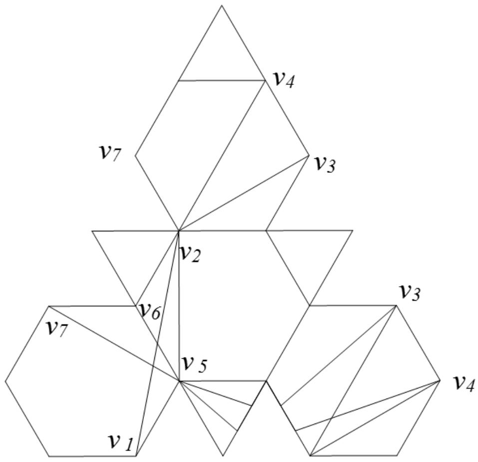

Figure 8:

If $u_3 = v_3$, then, similarly, $u_4 \in \{v_5, v_1\}$. If $u_3 = v_3$ and $u_4 = v_5$, then $u_5 = v_6$ or $u_5 = v_7$. But both v_5v_6 and v_5v_7 intersect v_1v_2, and a contradiction is obtained. (If $u_3 = v_3$ and $u_4 = v_1$, then we get a 3-cycle $v_1v_2v_3v_1$.)

If $u_3 = v_4$, then $u_4 \in \{v_5, v_1\}$. If $u_3 = v_4$ and $u_4 = v_5$, then $u_5 = v_6$ or $u_5 = v_2$. However, v_5v_6 and v_5v_2 intersect v_1v_2, contradiction.

The claim is true.

If $u, v \in V(\mathcal{S})$, then the remaining 5 vertices of $\mathcal{S}$ must be inside the interiors of the ten acute basic triangles of $\mathscr{T}$. By Lemma 2, this is impossible.

Otherwise, the ten triangles of $\mathscr{T}$ must contain at least 6 vertices of $\mathcal{S}$ in their interiors. By our Claim, no edge of C_5 is an h-segment. Then, by Lemma 1, the ten triangles of $\mathscr{T}$ together contain at most five vertices of $\mathcal{S}$ in their interiors, and we get again a contradiction. Hence, there is no acute triangulation of $\mathcal{S}$ with 10 triangles. □

Acknowledgement. Thanks are due to the referee, who made very valuable suggestions for improvement.

The first author gratefully acknowledges financial support by the State Foundation of China for Studying Abroad (CSC No. 201408130096).

The second author is grateful for financial support by NSF of China (11071055, 11471095), NSF of Hebei Province (A2012205080, A2013205189), Program for New Century Excellent Talents in University – Ministry of Education of China (NCET-10-0129), the project of Outstanding Experts' Overseas Training of Hebei Province and CSC (201508130089).

The third author thankfully acknowledges the financial support by the High-end Foreign Experts Recruitment Program of People's Republic of China. His work was also partly supported by a grant of the Roumanian National Authority for Scientific Research, CNCS – UEFISCDI, project number PN-II-ID-PCE-2011-3-0533.

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Received: 11.10.2014
Accepted: 26.11.2014

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