

## Three generated, squarefree, monomial ideals

by

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- Dedicated to Bazil Brînzănescu in honour of his 70th birthday -

### Abstract

Let  $I \supseteq J$  be two squarefree monomial ideals of a polynomial algebra over a field generated in degree  $\geq d$ , resp.  $\geq d + 1$ . Suppose that  $I$  is generated by three monomials of degrees  $d$ . If the Stanley depth of  $I/J$  is  $\leq d + 1$  then the usual depth of  $I/J$  is  $\leq d + 1$  too.

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### 1 Introduction

Let  $S = K[x_1, \dots, x_n]$ ,  $n \in \mathbf{N}$ , be a polynomial ring over a field  $K$ . Let  $I \supseteq J$  be two squarefree monomial ideals of  $S$  and  $u \in I \setminus J$  a monomial in  $I/J$ . For  $Z \subset \{x_1, \dots, x_n\}$  with  $(J : u) \cap K[Z] = 0$ , let  $uK[Z]$  be the linear  $K$ -subspace of  $I/J$  generated by the elements  $uf$ ,  $f \in K[Z]$ . A presentation of  $I/J$  as a finite direct sum of such spaces  $\mathcal{D} : I/J = \bigoplus_{i=1}^r u_i K[Z_i]$  is called a *Stanley decomposition* of  $I/J$ . Set  $\text{sdepth}(\mathcal{D}) := \min\{|Z_i| : i = 1, \dots, r\}$  and

$$\text{sdepth } I/J := \max\{\text{sdepth}(\mathcal{D}) : \mathcal{D} \text{ is a Stanley decomposition of } I/J\}.$$

Stanley's Conjecture says that the *Stanley depth*  $\text{sdepth}_S I/J \geq \text{depth}_S I/J$ . The Stanley depth of  $I/J$  is a combinatorial invariant and does not depend on the characteristic of the field  $K$ . If  $J = 0$  then this conjecture holds for  $n \leq 5$  by [12], or when  $I$  is an intersection of four monomial prime ideals by [11], [13], or an intersection of three monomial primary ideals by [23], or a monomial almost complete intersection by [3]. The Stanley depth and the Stanley's Conjecture are similarly given when  $I, J$  are not squarefree. In the non squarefree monomial ideals a useful inequality is  $\text{sdepth } I \leq \text{sdepth } \sqrt{I}$  (see [8, Theorem 2.1]).

Suppose that  $I$  is generated by squarefree monomials of degrees  $\geq d$  for some positive integer  $d$ . We may assume either that  $J = 0$ , or  $J$  is generated in degrees  $\geq d + 1$  after a multigraded isomorphism. We have  $\text{depth}_S I \geq d$  by [5, Proposition 3.1] and it follows  $\text{depth}_S I/J \geq d$  (see [15, Lemma 1.1]). Depth of  $I/J$  is a homological invariant and depends on the characteristic of the field  $K$ . The Stanley decompositions of  $S/J$  corresponds bijectively to partitions into intervals of the simplicial complex whose Stanley-Reisner ring is  $S/J$ . If Stanley's Conjecture holds then the simplicial complexes are partitionable (see [4]). Using this idea an equivalent definition of Stanley's depth of  $I/J$  was given in [5].

Let  $P_{I \setminus J}$  be the poset of all squarefree monomials of  $I \setminus J$  with the order given by the divisibility. Let  $\mathcal{P}$  be a partition of  $P_{I \setminus J}$  in intervals  $[u, v] = \{w \in P_{I \setminus J} : u|w, w|v\}$ , let us say  $P_{I \setminus J} = \cup_i [u_i, v_i]$ , the union being disjoint. Define  $\text{sdepth } \mathcal{P} = \min_i \deg v_i$ . Then  $\text{sdepth}_S I/J = \max_{\mathcal{P}} \text{sdepth } \mathcal{P}$ , where  $\mathcal{P}$  runs in the set of all partitions of  $P_{I \setminus J}$  (see [5], [21]).

For more than thirty years the Stanley Conjecture was a dream for many people working in combinatorics and commutative algebra. Many people believe that this conjecture holds and tried to prove directly some of its consequences. For example in this way a lower bound of depth given by Lyubeznik [10] was extended by Herzog et al. [6] for sdepth.

Some numerical upper bounds of sdepth give also upper bounds of depth, which are independent of char  $K$ . More precisely, write  $\rho_j(I \setminus J)$  for the number of all squarefree monomials of degrees  $j$  in  $I \setminus J$ .

**Theorem 1.1.** (Popescu [16, Theorem 1.3]) *Assume that  $\text{depth}_S(I/J) \geq t$ , where  $t$  is an integer such that  $d \leq t < n$ . If  $\rho_{t+1}(I \setminus J) < \alpha_t := \sum_{i=0}^{t-d} (-1)^{t-d+i} \rho_{d+i}(I \setminus J)$ , then  $\text{depth}_S(I/J) = t$  independently of the characteristic of  $K$ .*

The proof uses Koszul homology and is not very short. An extension is given below.

**Theorem 1.2.** (Shen [20, Theorem 2.4]) *Assume that  $\text{depth}_S(I/J) \geq t$ , where  $t$  is an integer such that  $d \leq t < n$ . If for some  $k$  with  $d+1 \leq k \leq t+1$  it holds  $\rho_k(I \setminus J) < \sum_{j=d}^{k-1} (-1)^{k-j+1} \binom{t+1-j}{k-j} \rho_j(I \setminus J)$ , then  $\text{depth}_S(I/J) = t$  independently of the characteristic of  $K$ .*

Shen's proof is very short, based on a strong tool, namely the Hilbert depth considered by Bruns-Krattenthaler-Uliczka [2] (see also [22], [7]). Thus it is important to have the right tool.

Let  $r$  be the number of the squarefree monomials of degrees  $d$  of  $I$  and  $B$  (resp.  $C$ ) be the set of the squarefree monomials of degrees  $d+1$  (resp.  $d+2$ ) of  $I \setminus J$ . Set  $s = |B|$ ,  $q = |C|$ . If  $r > s$  then Theorem 1.1 says that  $\text{depth}_S I/J = d$ , namely the minimum possible. This was done previously in [15] (the idea started in [14]). Moreover, Theorem 1.1 together with Hall's marriage theorem for bipartite graphs gives the following:

**Theorem 1.3.** (Popescu [15, Theorem 4.3]) *If  $\text{sdepth}_S I/J = d$  then  $\text{depth}_S I/J = d$ , that is Stanley's Conjecture holds in this case.*

The purpose of our paper is to study the next step in proving Stanley's Conjecture namely the following weaker conjecture.

**Conjecture 1.4.** *Suppose that  $I \subset S$  is minimally generated by some squarefree monomials  $f_1, \dots, f_k$  of degrees  $d$ , and a set  $H$  of squarefree monomials of degrees  $\geq d+1$ . Assume that  $\text{sdepth}_S I/J = d+1$ . Then  $\text{depth}_S I/J \leq d+1$ .*

The following theorem is a partial answer.

**Theorem 1.5.** *The above conjecture holds in each of the following two cases:*

1.  $k = 1$ ,
2.  $1 < k \leq 3$ ,  $H = \emptyset$ .

When  $k = 1$  and  $s \neq q+1$  the result was stated in [17] and [18]. The theorem follows from Proposition 3.1 and Theorems 2.3, 3.4.

We owe thanks to the Referee, who noticed some mistakes in a previous version of this paper, especially in the proof of Lemma 3.3.

## 2 Cases $r = 1$ and $d = 1$

Let  $I \supsetneq J$  be two squarefree monomial ideals of  $S$ . We assume that  $I$  is generated by squarefree monomials of degrees  $\geq d$  for some  $d \in \mathbf{N}$ . We may suppose that either  $J = 0$ , or is generated by some squarefree monomials of degrees  $\geq d+1$ . As above  $B$  (resp.  $C$ ) denotes the set of the squarefree monomials of degrees  $d+1$  (resp.  $d+2$ ) of  $I \setminus J$ .

**Lemma 2.1.** *Suppose that  $I \subset S$  is minimally generated by some square free monomials  $\{f_1, \dots, f_r\}$  of degrees  $d$ , and a set  $E$  of square free monomials of degrees  $\geq d+1$ . Assume that  $\text{sdepth}_S I/J \leq d+1$  and the above Conjecture 1.4 holds for  $k < r$  and for  $k = r$ ,  $|H| < |E|$  if  $E \neq \emptyset$ . If either  $C \not\subset (f_2, \dots, f_r, E)$ , or  $C \not\subset (f_1, \dots, f_r, E \setminus \{a\})$  for some  $a \in E$  then  $\text{depth}_S I/J \leq d+1$ .*

**Proof:** Let  $c \in (C \setminus (f_2, \dots, f_r, E))$ . Then  $c \in (f_1)$ , let us say  $c = f_1 x_t x_p$ . Set  $I' = (f_2, \dots, f_r, E, B \setminus \{f_1 x_t, f_1 x_p\})$ ,  $J' = I' \cap J$ . In the following exact sequence

$$0 \rightarrow I'/J' \rightarrow I/J \rightarrow I/(J + I') \rightarrow 0$$

the last term is isomorphic with  $(f_1)/(J + I') \cap (f_1)$  and has depth and  $\text{sdepth} \geq d+2$  because  $c \notin (J + I')$  (here it is enough that  $\text{depth} \geq d+1$ , which is easier to see). By [19, Lemma 2.2] we get  $\text{sdepth}_S I'/J' \leq d+1$ . It follows that  $\text{depth}_S I'/J' \leq d+1$  by hypothesis and so the Depth Lemma gives  $\text{depth}_S I/J \leq d+1$ .

Now, let  $I'' = (f_1, \dots, f_r, E \setminus \{a\})$  for some  $a \in E$  and  $c \in C \setminus I''$ . In the following exact sequence

$$0 \rightarrow I''/I'' \cap J \rightarrow I/J \rightarrow I/(J + I'') \rightarrow 0$$

the last term is isomorphic with  $(a)/(a) \cap (J + I'')$  and has depth and  $\text{sdepth} \geq d+2$  because  $c \notin J + I''$  and as above we get  $\text{depth}_S I/J \leq d+1$ . □

The following lemma could be seen somehow as a consequence of [17, Theorem 1.10], but we give here an easy direct proof.

**Lemma 2.2.** *Suppose that  $r = 1$ , let us say  $I = (f)$  and  $E = \emptyset$ . If  $\text{sdepth}_S I/J = d+1$ ,  $d = \text{deg } f$  then  $\text{depth}_S I/J \leq d+1$ .*

**Proof:** First assume that  $d > 0$ . Note that  $I/J \cong S/(J : f)$ . We have  $\text{sdepth}_S I/J = \text{sdepth}_S S/(J : f)$  and  $\text{depth}_S I/J = \text{depth}_S S/(J : f)$ . It is enough to treat the case  $d = 1$ . We may assume that  $x_1 | f$  and using [5, Lemma 3.6] after skipping the variables of  $f/x_1$  we may reduce our problem to the case  $d = 1$ .

Therefore we may assume that  $d = 1$ . If  $C = \emptyset$  then  $x_1 x_t x_k \in J$  for all  $1 < t < k \leq n$  and so  $(J : x_1)$  contains all squarefree monomials of degree two in  $x_t, t > 1$ , that is the annihilator of the element induced by  $x_1$  in  $I/J$  has dimension  $\leq 2$ . It follows that  $\text{depth}_S I/J \leq 2$ .

If let us say  $c = x_1 x_2 x_3 \in C$  then in the exact sequence

$$0 \rightarrow (B \setminus \{x_1 x_2, x_1 x_3\})/J \cap (B \setminus \{x_1 x_2, x_1 x_3\}) \rightarrow I/J \rightarrow I/J + (B \setminus \{x_1 x_2, x_1 x_3\}) \rightarrow 0$$

the last term is isomorphic with  $(x_1)/(J, (B \setminus \{x_1 x_2, x_1 x_3\}))$  and it has depth  $\geq 2$  and  $\text{sdepth} \geq 3$  because it has just the interval  $[x_1, c]$ . The first term is not zero since otherwise  $\text{sdepth}_S I/J = 3$ , which is false. Then the first term has  $\text{sdepth} \leq 2$  by [19, Lemma 2.2] and so it has depth  $\leq 2$  by [15, Theorem 4.3]. Now it is enough to apply the Depth Lemma.

Now assume that  $d = 0$ , that is  $I = S$ . Set  $S' = S[x_{n+1}]$ ,  $I' = (x_{n+1})$ ,  $J' = x_{n+1} J$ . We have  $\text{sdepth}_{S'} I'/J' = \text{sdepth}_S S'/JS' = 1 + \text{sdepth}_S S/J = 2$  using [5, Proposition 3.6]. From above we get  $\text{depth}_{S'} I'/J' \leq 2$  and it follows  $\text{depth}_S I/J \leq 1$ . □

The following theorem extends the above lemma and [18], its proof is given in the last section.

**Theorem 2.3.** *Suppose that  $I \subset S$  is minimally generated by a squarefree monomial  $\{f\}$ , of degree  $d$  and a set  $E \neq \emptyset$  of monomials of degrees  $d+1$ . Assume that  $\text{sdepth}_S I/J \leq d+1$ . Then  $\text{depth}_S I/J \leq d+1$ .*

**Lemma 2.4.** *Suppose that  $I = (x_1, x_2)$ ,  $E = \emptyset$ . If  $\text{sdepth}_S I/J = 2$  then  $\text{depth}_S I/J \leq 2$ .*

**Proof:** By [17, Proposition 1.3] we may suppose that  $C \not\subset (x_2)$ . Then apply Lemma 2.1, its hypothesis is given by Theorem 2.3.  $\square$

We need the following lemma, its proof is given in Section 4.

**Lemma 2.5.** *Suppose that  $I \subset S$  is minimally generated by some squarefree monomials  $\{f_1, f_2, f_3\}$  of degree  $d$  and that  $\text{sdepth } I/J = d + 1$ . If there exists  $c \in C \cap ((f_3) \setminus (f_1, f_2))$  then  $\text{depth}_S I/J \leq d + 1$ .*

**Proposition 2.6.** *Suppose that  $I = (x_1, x_2, x_3)$ ,  $E = \emptyset$ . If  $\text{sdepth}_S I/J = 2$  then  $\text{depth}_S I/J \leq 2$ .*

**Proof:** By [17, Proposition 1.3] we may suppose that  $C \not\subset (x_1, x_2)$ . Then we may apply Lemma 2.5.  $\square$

**Remark 2.7.** When  $J = 0$  the above proposition follows quickly from [1] (see also [5]).

### 3 Case $r, d > 1$

**Proposition 3.1.** *Suppose that  $I \subset S$  is generated by two squarefree monomials  $\{f_1, f_2\}$  of degrees  $d$ . Assume that  $\text{sdepth}_S I/J \leq d + 1$ . Then  $\text{depth}_S I/J \leq d + 1$ .*

**Proof:** We may suppose that  $I$  is minimally generated by  $f_1, f_2$  because otherwise apply the Theorem 2.3. Let  $w$  be the least common multiple of  $f_1, f_2$ . First suppose that  $C \not\subset (w)$ . This is the case when  $w \in J$ , or  $\deg w > d + 2$ , or  $w \in C$  and  $q > 1$ . Then it is enough to apply Lemma 2.1, the case  $r = 1$  being done in the Theorem 2.3. If  $q = 1$  then  $r > q$  and by [20, Corollary 2.6] (see also Theorem 1.2) we get  $\text{depth}_S I/J \leq d + 1$ . Assume that  $w \in B$ . After renumbering the variables  $x_i$  we may suppose that  $C = \{wx_i : 1 \leq i \leq q\}$  and so in  $B$  we have at least the elements of the form  $w, f_1x_i, f_2x_i, 1 \leq i \leq q$ . Thus  $s \geq 2q + 1 > q + 2$  when  $q > 1$  and by [16, Theorem 1.3] (see Theorem 1.1) we are done.  $\square$

**Lemma 3.2.** *Suppose that  $I \subset S$  is generated by three squarefree monomials  $\{f_1, f_2, f_3\}$  of degrees  $d$ ,  $\text{sdepth}_S I/J = d + 1$  and let  $w_{ij}$  be the least common multiple of  $f_i, f_j, 1 \leq i < j \leq 3$ . If  $w_{12}, w_{13}, w_{23} \in B$  and are different then  $\text{depth}_S I/J \leq d + 1$ .*

**Proof:** After renumbering the variables  $x_i$  we may assume that  $f_1 = x_1 \cdots x_d$  and  $f_2 = x_1 \cdots x_{d-1}x_{d+1}$ . We see that  $f_3$  must have  $d - 1$  variables in common with  $f_1$  and also with  $f_2$ . If  $f_3 \notin (x_1 \cdots x_{d-1})$  then we may suppose that  $f_3 = x_2 \cdots x_d x_{d+1}$  and  $w_{12} = w_{13}$ , which is false. It remains that  $f_3 \in (x_1 \cdots x_{d-1})$  so  $f_3 = x_1 \cdots x_{d-1}x_{d+2}$ . But this case may be reduced to  $d = 1$  which is done in Proposition 2.6.  $\square$

**Lemma 3.3.** *If  $C \subset (w_{12}, w_{13}, w_{23})$  and  $\text{sdepth}_S I/J \leq d + 1$  then  $\text{depth}_S I/J \leq d + 1$ .*

**Proof:** Note that if  $q < r = 3$  then  $\text{depth}_S I/J \leq d + 1$  by [20, Corollary 2.6] (see here Theorem 1.2). Suppose that  $q > 2$ .

Now assume that all  $w_{ij} \in B$ . Set  $C_{ij} = C \cap (w_{ij})$ ,  $q_{ij} = |C_{ij}|$  and  $B_{ij}$  the set of all  $b \in B$  which divide some  $c \in C_{ij}$ . If all  $w_{ij}$  are equal, let us say  $w_{ij} = w$ , then after renumbering the variables  $x_i$  the monomials of  $C$  have the form  $wx_t, 1 \leq t \leq q$ . Thus  $B$  contains  $w$  and  $f_jx_t$  for  $j \in [3]$  and  $t \in [q]$ . It follows that  $s \geq 3q + 1 > q + 3$  for  $q > 1$  and so  $\text{depth}_S I/J \leq d + 1$  by [16, Theorem 1.3]. Then we may suppose that all  $w_{ij}$  are different and we may apply Lemma 3.2.

Next assume that  $w_{12}, w_{13} \in B$  and  $w_{23} \in C$ . As above we can assume that  $f_2 = x_1 \cdots x_d, f_3 = x_3 \cdots x_{d+2}$  and  $f_1 = x_2 \cdots x_{d+1}$ . We have  $C \subset C_{12} \cap C_{13}$ , and  $q = q_{12} + q_{13} - 1$  because  $w_{23} \in C_{12} \cap C_{13}$ . As in the case of  $r = 2$  we have  $|B_{12}| = 2q_{12} + 1$  and  $|B_{13} \setminus B_{12}| \geq 2q_{13} - \min\{q_{12}, q_{13}\}$ . It follows that  $s \geq 2q + 4 - \min\{q_{12}, q_{13}\} > q + 3$ , which implies  $\text{depth}_S I/J \leq d + 1$  by [16, Theorem 1.3]. Note that if  $w_{23} \in J$ , or  $\deg w_{23} > d + 2$  then  $q = q_{12} + q_{13}$  and we get in the same way that

$s \geq 2q + 2 - \min\{q_{12}, q_{13}\} \geq q + 3$ . Thus  $\text{depth}_S I/J \leq d + 1$  unless  $q_{12} = q_{13} = 1$ . The last case is false because  $q > 2$ .

Suppose that all  $w_{ij}$  are different,  $w_{12} \in B$  and  $w_{23}, w_{13} \in C$ . We may assume that  $f_2 = x_1 \cdots x_d$ ,  $f_3 = x_3 \cdots x_{d+2}$  and  $f_1 = x_2 \cdots x_d \cdot x_{d+3}$ . We have  $q = q_{12} + 2$ ,  $B_{12} \cap B_{13} \subset \{x_{d+1}f_1, x_{d+2}f_1\}$  and so  $|B_{13} \setminus B_{12}| \geq 2$ . Also note that  $B_{23} \cap (B_{12} \cup B_{13}) \subset \{x_{d+1}f_2, x_{d+2}f_2, x_2f_3\}$  and so  $|B_{23} \setminus (B_{12} \cup B_{13})| \geq 1$ . It follows that  $s \geq 2q_{12} + 1 + 2 + 1 = 2q$ . If  $q > 3$  we get  $s > q + 3$  and so  $\text{depth}_S I/J \leq d + 1$  by [16]. If  $q = 3$  then  $q_{12} = 1$  and so  $B_{12} = \{w_{12}, x_t f_1, x_t f_2\}$  for some  $x_t \nmid f_1, x_t \nmid f_2$ . If  $t = d + 1$  or  $t = d + 2$  then we see that  $|B_{13} \setminus B_{12}| \geq 3$  and so  $s > 6 = r + q$ , which is enough. If  $t > d + 3$  then  $s$  is even bigger than 7. If let us say  $w_{23} \in J$ , or  $\deg w_{23} > d + 2$  then  $q = q_{12} + 1$  and as above  $s \geq 2q_{12} + 1 + 2 = 2q + 1 > q + 3$  because  $q \geq 3$ , which is again enough. If also  $w_{13} \in J$ , or  $\deg w_{13} > d + 2$  then  $q = q_{12}$  and as above  $s \geq 2q_{12} + 1 = 2q + 1 > q + 3$  because  $q \geq 3$ .

Suppose that  $w_{12} \in B$  and  $w_{23} = w_{13} \in C$ . We may assume that  $f_2 = x_1 \cdots x_d$ ,  $f_3 = x_3 \cdots x_{d+2}$  and  $f_1 = x_1 \cdots x_{d-1} \cdot x_{d+2}$ . We have  $q = q_{12}$  and  $B_{12} \supset B_{13}$ . Thus  $s \geq 2q_{12} + 1 = 2q + 1 > q + 3$  and so again  $\text{depth}_S I/J \leq d + 1$ .

Finally if all  $w_{ij}$  are in  $C$  (they must be different, otherwise  $q \leq 2$  which is false) then  $q = 3$ ,  $q_{ij} = 1$  and we get  $s \geq 12 > q + 3$  which is again enough.  $\square$

**Theorem 3.4.** *Suppose that  $I \subset S$  is generated by three squarefree monomials  $\{f_1, f_2, f_3\}$  of degrees  $d$ , and  $\text{sdepth}_S I/J = d + 1$ . Then  $\text{depth}_S I/J \leq d + 1$ .*

**Proof:** We may suppose that  $I$  is minimally generated by  $f_1, f_2, f_3$  because otherwise apply Proposition 3.1. If  $C \not\subset (w_{12}, w_{13}, w_{23})$  then apply Lemma 2.5. Thus we may suppose that  $C \subset (w_{12}, w_{13}, w_{23})$  and we may apply Lemma 3.3.  $\square$

#### 4 Proof of Lemma 2.5

Let  $c = f_3 x_{i_3} x_{j_3}$  and set  $I' = (f_1, f_2, B \setminus \{f_3 x_{i_3}, f_3 x_{j_3}\})$ ,  $J' = I' \cap J$ . Consider the following exact sequence

$$0 \rightarrow I'/J' \rightarrow I/J \rightarrow I/(I' + J) \rightarrow 0.$$

The last term has  $\text{sdepth} = d + 2$  so by [19, Lemma 2.2] we get that the first term has  $\text{sdepth} \leq d + 1$ . If  $\text{depth} I'/J' \leq d + 1$  then by Depth Lemma we are done. It is **enough to show that**  $\text{sdepth}_S I'/J' = d + 1$  **implies**  $\text{depth}_S I'/J' \leq d + 1$ , or directly  $\text{depth}_S I/J \leq d + 1$ . Note that if  $\text{sdepth}_S I'/J' = d$  then  $\text{depth}_S I'/J' = d$  by Theorem 1.3. Let  $B', C', E'$  be similar to  $B, C, E$  in the case of  $I'/J'$ .

We see that  $E' \subset (f_3)$ . We may suppose that  $C' \subset ((f_1) \cap (f_2)) \cup (E')$  and  $E' \neq \emptyset$ , otherwise apply Lemma 2.1 with the help of Theorem 2.3.

Set  $I'_E = (f_1, f_2)$ ,  $J'_E = I'_E \cap J'$  and for all  $i \in [n] \setminus \text{supp } f_1$  such that  $f_1 x_i \in B' \setminus (f_2)$  set  $I'_i = (f_2, B' \setminus \{f_1 x_i\})$ ,  $J'_i = I'_i \cap J'$ . We may suppose that  $\text{sdepth}_S I'_E/J'_E \geq d + 2$  and  $\text{sdepth}_S I'_i/J'_i \geq d + 2$ . Indeed, otherwise one of the left terms from the following exact sequences

$$0 \rightarrow I'_E/J'_E \rightarrow I'/J' \rightarrow I'/I'_E + J' \rightarrow 0,$$

$$0 \rightarrow I'_i/J'_i \rightarrow I'/J' \rightarrow I'/I'_i + J' \rightarrow 0,$$

have  $\text{depth} \leq d + 1$  by Proposition 3.1 and Theorem 2.3. With the Depth Lemma we get  $\text{depth}_S I'/J' \leq d + 1$  since the right terms above have  $\text{depth} \geq d + 1$ . Let  $\mathcal{P}_E, \mathcal{P}_i$  be partitions of  $I'_E/J'_E, I'_i/J'_i$  with  $\text{sdepth} d + 2$ . We may choose  $\mathcal{P}_E$  and  $\mathcal{P}_i$  such that each interval starting with a squarefree monomial of degree  $\leq d + 1$  ends with a monomial from  $C'$ .

Our goal is mainly to **reduce our problem to the case when**  $w_{13}, w_{12} \in B' \cup C'$ .

**Case 1**  $C' \not\subset (f_1, f_3) \cap (f_2, f_3)$

Let for example  $c = f_1x_u x_v \in C' \setminus (f_2, f_3)$ , set  $I'' = (f_2, B' \setminus \{f_1x_u, f_1x_v\})$ ,  $J'' = I'' \cap J'$  and consider the exact sequence:

$$0 \rightarrow I''/J'' \rightarrow I'/J' \rightarrow I'/(I'' + J') \rightarrow 0.$$

The last term has sdepth  $d + 2$  so by [19, Lemma 2.2] we see that the first term has sdepth  $\leq d + 1$ . Using Theorem 2.3 we have  $\text{depth}_S I''/J'' \leq d + 1$  and then by the Depth lemma we get  $\text{depth}_S I'/J' \leq d + 1$  ending Case 1.

Let  $f_1 = x_1 \dots x_d$ , in  $\mathcal{P}_E$  we have the intervals  $[f_1, c_1], [f_2, c_2]$  and so at least one of  $c_1, c_2$ , let us say  $c_1 = f_1x_ix_j$ , is not a multiple of  $w_{12}$ . In  $\mathcal{P}_i$  we have the interval  $[b, c_1]$  for some  $b \in E'$ , otherwise replacing the interval  $[f_1x_j, c_1]$  or the interval  $[c_1, c_1]$  with the interval  $[f_1, c_1]$  we get a partition  $\mathcal{P}$  for  $I'/J'$  with sdepth  $= d + 2$ .

**Case 2** There exists  $t \in [n]$ ,  $t \notin \text{supp } f_1 \cup \{i\}$  such that  $\mathcal{P}_i$  contains the interval  $[f_1x_t, f_1x_tx_i]$ , or  $[f_1x_tx_i, f_1x_tx_i]$ .

In this case changing in  $\mathcal{P}_i$  the hinted interval with  $[f_1, f_1x_tx_i]$  we get a partition of  $I'/J'$  with sdepth  $\geq d + 2$  which is false.

As we have seen above we may suppose that in  $\mathcal{P}_i$  there exists an interval  $[b, c_1]$  with  $c_1 \in (f_1) \cap (E') \subset (w_{13})$ . **It follows that**  $w_{13} \in B' \cup C'$ . We may assume that if  $w_{13} \in B'$  then  $x_i \nmid w_{13}$ , otherwise change  $i$  by  $j$ . Thus  $c_1 = x_iw_{13}$  or  $c_1 = w_{13}$ . If  $C' \cap (f_1x_i) = \{c_1\}$  then in  $\mathcal{P}_j$  we have the interval  $[f_1x_i, c_1]$ , that is we are in Case 2. Then there exists another monomial  $c' \in C' \cap (f_1x_i)$ . We may suppose that  $[c', c']$  is not in  $\mathcal{P}_i$ , because otherwise we are in Case 2. If we have  $[u, c']$  in  $\mathcal{P}_i$  for some  $u \in E'$  then  $c' \in (w_{13})$  and so  $c' = c_1$  if  $w_{13} \in C'$ , otherwise  $c' = x_iw_{13} = c_1$  because  $x_i \nmid w_{13}$ . Contradiction! Then in  $\mathcal{P}_i$  we have the interval  $[f_2, c']$  or the interval  $[f_2x_k, c']$  for some  $k$ . Thus  $c' \in (w_{12})$  **and so**  $w_{12} \in B' \cup C'$ . Note that  $w_{12} \neq w_{13}$  because  $c_1 \in (w_{13}) \setminus (f_2)$ .

**Case 3**  $w_{12}, w_{13} \in C'$ .

In this case  $c_1 = w_{13}$ ,  $c' = w_{12}$  and so  $f_2, f_3 \in (x_i)$ . Then in  $\mathcal{P}_i$  we have the interval  $[f_1x_j, f_1x_jx_u]$ ,  $u \neq i$  and  $f_1x_jx_u \notin (f_2, f_3)$  because  $f_1x_jx_u \notin (x_i)$ , that is we are in Case 1.

**Case 4**  $w_{12} \in B', w_{13} \in C'$ .

Thus  $c_1 = w_{13}$ . We may assume that  $w_{12} = x_1 \dots x_{d+1}$ ,  $f_2 = x_2 \dots x_{d+1}$ ,  $i \neq d + 1 \neq j$  and  $c' = x_1 \dots x_{d+1}x_i$ . We also see that  $f_3 \in (x_ix_j)$  because  $c_1 = w_{13}$ . In  $\mathcal{P}_i$  we have the interval  $[f_1x_j, f_1x_jx_u]$ ,  $u \neq i$ . If  $u \neq d + 1$  then  $f_1x_jx_u \notin (f_2, f_3)$ , that is we are in Case 1. Otherwise  $u = d + 1$ , and so  $x_jw_{12} \in C'$ , in particular  $f_2x_j \in B'$ . We see that in  $\mathcal{P}_i$  we can have either  $w_{12} \in [f_2, c']$  or there exists an interval  $[w_{12}, w_{12}x_k]$ . If  $k = j$  then  $w_{12}x_k$  is the end of the interval starting with  $f_1x_j$ , which is false. If  $k = i$  then we are in Case 2. Thus  $i \neq k \neq j$ .

When in  $\mathcal{P}_i$  there exists the interval  $[w_{12}, w_{12}x_k]$  then there exists also the interval  $[f_1x_k, f_1x_kx_t]$ . If  $f_1x_kx_t \in (f_2)$  then  $t = d + 1$  and so  $f_1x_kx_t = x_kw_{12}$  which is not possible because  $x_kw_{12}$  is in  $[w_{12}, x_kw_{12}]$ . If  $f_1x_kx_t \in (f_3)$  then  $\{k, t\} = \{i, j\}$  which is not possible since  $k \notin \{i, j\}$ . Then  $f_1x_kx_t \notin (f_2, f_3)$ , that is we are in Case 1. It remains the case when  $w_{12}$  is in the interval  $[f_2, c']$ . In  $\mathcal{P}_i$  we have an interval  $[f_2x_j, f_2x_lx_j]$  for some  $l$ . If  $f_2x_jx_l \in (f_1)$  then  $l = 1$  and so  $f_2x_jx_l = x_jw_{12}$  which is already the end of the interval starting with  $f_1x_j$ . Contradiction! Thus  $f_2x_lx_j \in (f_3)$ , otherwise we are in Case 1. We get  $l = i$  and changing  $[f_2, c'], [f_2x_j, f_2x_ix_j]$  with  $[f_2, f_2x_ix_j], [w_{12}, c']$  we arrive in Case 2.

**Case 5**  $w_{12} \in C', w_{13} \in B'$ .

Thus we may assume that  $w_{12} = c' = x_1 \dots x_{d+1}x_i$ ,  $f_2 = x_3 \dots x_{d+1}x_i$ . As  $c_1 \in (w_{13})$  we have  $w_{13} \in \{f_1x_i, f_1x_j\}$ . If  $w_{13} = f_1x_i$  then in  $\mathcal{P}_i$  we have an interval  $[f_1x_j, f_1x_jx_u]$ . If  $f_1x_jx_u \in (f_2)$  then  $u = i$ . Also if  $f_1x_jx_u \in (f_3)$  we get  $f_1x_jx_u \in (w_{13})$  and we get again  $u = i$ , that is we are in Case 2. Thus  $f_1x_jx_u \notin (f_2, f_3)$  and we arrive in Case 1.

Then, we may suppose that  $w_{13} = f_1x_j$ . Since  $f_1x_{d+1}|c'$  we see that  $f_1x_{d+1} \in B'$ . In  $\mathcal{P}_i$  we can have the interval  $[f_1x_{d+1}, f_1x_{d+1}x_m]$ . If  $f_2x_{d+1}x_m \in (f_2)$  then  $m = i$ , that is we are in Case 2. Then  $f_2x_{d+1}x_m \in (f_3)$  because otherwise we are in Case 1. It follows that  $m = j$  and we have  $[f_1x_{d+1}, f_1x_{d+1}x_j]$  in  $\mathcal{P}_i$ . Then the interval  $[f_1x_j, f_1x_jx_p]$  existing in  $\mathcal{P}_i$  has  $p \neq j$  and also  $p \neq i$

because otherwise we are in Case 2. Thus we must also have an interval  $[f_1x_p, f_1x_px_k]$  with  $k \neq j$  and also  $k \neq i$ , otherwise we are in Case 2. Then  $f_1x_px_k \notin (f_2, f_3)$ , that is we are in Case 1.

**Case 6**  $w_{12}, w_{13} \in B'$ .

We may assume that  $w_{12} = x_1 \dots x_{d+1}$ ,  $f_2 = x_2 \dots x_{d+1}$  and  $c' = x_1 \dots x_{d+1}x_i$ . If  $w_{23} \in B'$  then all  $w_{ij}$  are different and by Lemma 3.2 we get  $\text{depth}_S I/J \leq d + 1$ . Thus we may suppose that  $w_{23} \in C'$ . We may choose  $f_3 = x_1x_3 \dots x_dx_i$  or  $f_3 = x_1x_3 \dots x_dx_j$ . If  $f_3 = x_1x_3 \dots x_dx_i$  then in  $\mathcal{P}_i$  we have as above the interval  $[f_1x_j, f_1x_{d+1}x_j]$ . Indeed, if we have  $[f_1x_j, f_1x_mx_j]$  then  $f_1x_mx_j \notin (f_3)$  and so  $f_1x_mx_j \in (f_2)$ , otherwise we are in Case 1. It follows  $m = d + 1$ . As  $f_2x_j|x_jw_{12} = f_1x_{d+1}x_j$  we get  $f_2x_j \in B'$ . Let  $[f_2, f_2x_jx_k]$  or  $[f_2x_j, f_2x_jx_k]$  be the existing interval of  $\mathcal{P}_i$  containing  $f_2x_j$ . Note that  $f_2x_jx_k \notin (f_3)$  and if  $f_2x_jx_k \in (f_1)$  then  $f_2x_jx_k = x_jw_{12}$  which appeared already in the previous interval. Thus  $f_2x_jx_k \notin (f_1, f_3)$ , that is we are in Case 1.

It remains that  $f_3 = x_1x_3 \dots x_dx_j$  and, as before, we have in  $\mathcal{P}_i$  the interval  $[f_1x_j, f_1x_{d+1}x_j]$ . We see then  $f_2x_j \in B'$  and we must have also an interval  $[f_2, f_2x_jx_k]$  or  $[f_2x_j, f_2x_jx_k]$ . If  $f_2x_jx_k \in (f_1) \cup (f_3)$  then we get  $k = 1$  and so  $f_2x_jx_k = x_jw_{12}$  which appeared in the previous interval. It follows that  $f_2x_jx_k \notin (f_1, f_3)$ , that is we are in Case 1.

### 5 Proof of Theorem 2.3

Suppose that  $E \neq \emptyset$  and  $s \leq q + 1$ . We may assume that  $|B \setminus E| \geq 2$  because otherwise  $\text{depth}_S I/J \leq d + 1$  since the element induced by  $f$  in  $I/J$  is annihilated by all variables but one and those from  $\text{supp } f$ . For  $b = fx_i \in B$  set  $I_b = (B \setminus \{b\})$ ,  $J_b = J \cap I_b$ . If  $\text{sdepth}_S I_b/J_b \geq d + 2$  then let  $\mathcal{P}_b$  be a partition on  $I_b/J_b$  with  $\text{sdepth } d + 2$ . We may choose  $\mathcal{P}_b$  such that each interval starting with a squarefree monomial of degree  $d, d + 1$  ends with a monomial of  $C$ . In  $\mathcal{P}_b$  we have some intervals for all  $b' \in B \setminus \{b\}$  an interval  $[b', c_{b'}]$ . We define  $h : B \setminus \{b\} \rightarrow C$  by  $b' \rightarrow c_{b'}$ . Then  $h$  is an injection and  $|\text{Im } h| = s - 1 \leq q$  (if  $s = 1 + q$  then  $h$  is a bijection). We may suppose that all intervals of  $\mathcal{P}_b$  starting with a monomial  $v$  of degree  $\geq d + 2$  have the form  $[v, v]$ .

**Lemma 5.1.** *Suppose that the following conditions hold:*

1.  $s \leq q + 1$ ,
2.  $\text{sdepth}_S I_b/J_b \geq d + 2$ , for a  $b \in B \cap (f)$ ,
3.  $C \subset ((f) \cap (E)) \cup (\cup_{a, a' \in E, a \neq a'} (a) \cap (a'))$ .

*Then either  $\text{sdepth}_S I/J \geq d + 2$ , or there exists a nonzero ideal  $I' \subsetneq I$  generated by a subset of  $\{f\} \cup B$  such that  $\text{sdepth}_S I'/J' \leq d + 1$  for  $J' = J \cap I'$  and  $\text{depth}_S I/(J, I') \geq d + 1$ .*

**Proof:** Consider  $h$  as above for a partition  $\mathcal{P}_b$  with  $\text{sdepth } d + 2$  of  $I_b/J_b$  which exists by (2). A sequence  $a_1, \dots, a_k$  is called a *path* from  $a_1$  to  $a_k$  if  $a_i \in B \setminus \{b\}$ ,  $i \in [k]$ ,  $a_i \neq a_j$  for  $1 \leq i < j \leq k$ ,  $a_{i+1}|h(a_i)$  for  $1 \leq i < k$ , and  $h(a_i) \notin (b)$  for  $1 \leq i < k$ . This path is *bad* if  $h(a_k) \in (b)$  and it is *maximal* if all divisors from  $B$  of  $h(a_k)$  are in  $\{b, a_1, \dots, a_k\}$ . If  $a = a_1$  we say that the above path *starts with a*. Since  $|B \setminus E| \geq 2$  there exists  $a_1 \in B \setminus \{b\}$ . Set  $c_1 = h(a_1)$ . If  $c_1 \in (b)$  then the path  $\{a_1\}$  is maximal and bad. By recurrence choose if possible  $a_{p+1}$  to be a divisor from  $B$  of  $c_p$  which is not in  $\{b, a_1, \dots, a_p\}$  and set  $c_p = h(a_p)$ ,  $p \geq 1$ . This construction ends at step  $p = e$  if all divisors from  $B$  of  $c_{e-1}$  are in  $\{b, a_1, \dots, a_{e-1}\}$ . If  $c_i \notin (b)$  for  $1 \leq i < e - 1$  then  $\{a_1, \dots, a_{e-1}\}$  is a maximal path. If  $c_{e-1} \in (b)$  then this path is also bad. We have two cases:

- 1) there exist no maximal bad path starting with  $a_1$ ,
- 2) there exists a maximal bad path starting with  $a_1$ .

In the first case, set  $T_1 = \{b' \in B : \text{there exists a path } a_1, \dots, a_k \text{ with } a_k = b'\}$ ,  $G_1 = B \setminus T_1$  and  $I'_1 = (f, G_1)$ ,  $I''_1 = (G_1)$ ,  $J'_1 = I'_1 \cap J$ ,  $J''_1 = I''_1 \cap J$ . Note that  $I''_1 \neq 0$  because  $b \in I''_1$ . Consider the following exact sequence

$$0 \rightarrow I'_1/J'_1 \rightarrow I/J \rightarrow I/(J, I'_1) \rightarrow 0.$$

If  $T_1 \cap (f) = \emptyset$  then the last term has depth  $\geq d + 1$  and sdepth  $\geq d + 2$  using the restriction of  $\mathcal{P}_b$  since  $h(b') \notin I'_1$ , for all  $b' \in T_1$ . If the first term has sdepth  $\geq d + 2$  then by [19, Lemma 2.2] the middle term has sdepth  $\geq d + 2$ . Otherwise, the first term has sdepth  $\leq d + 1$  and we may take  $I' = I'_1$ .

If let us say  $a \in (f)$  for some  $a \in T_1$  then in the following exact sequence

$$0 \rightarrow I''_1/J''_1 \rightarrow I/J \rightarrow I/(J, I''_1) \rightarrow 0$$

the last term has sdepth  $\geq d + 2$  and depth  $\geq d + 1$  since  $h(a) \notin I''_1$  and we may substitute the interval  $[a, h(a)]$  from the restriction of  $\mathcal{P}_b$  to  $(T_1)$  by  $[f, h(a)]$ , the second monomial from  $[f, h(a)] \cap B$  being also in  $T_1$ . As above we get either sdepth<sub>S</sub>  $I/J = d + 2$ , or sdepth<sub>S</sub>  $I''_1/J''_1 \leq d + 1$ , depth<sub>S</sub>  $I/(J, I''_1) \geq d + 1$ .

In the second case, let  $a_1, \dots, a_{t_1}$  be a maximal bad path starting with  $a_1$ . Set  $c_j = h(a_j)$ ,  $j \in [t_1]$ . Then  $c_{t_1} = bx_{u_1}$  for some  $u_1$  and let us say  $b = fx_i$ . If  $a_{t_1} \in (f)$  then changing in  $\mathcal{P}_b$  the interval  $[a_{t_1}, c_{t_1}]$  by  $[f, c_{t_1}]$  we get a partition on  $I/J$  with sdepth  $d + 2$ . Thus we may assume that  $a_{t_1} \in E$ . If  $fx_{u_1} \in \{a_1, \dots, a_{t_1-1}\}$ , let us say  $fx_{u_1} = a_v$ ,  $1 \leq v < t_1$  then we may replace in  $\mathcal{P}_b$  the intervals  $[a_p, c_p]$ ,  $v \leq p \leq t_1$  with the intervals  $[a_v, c_{t_1}]$ ,  $[a_{p+1}, c_p]$ ,  $v \leq p < t_1$ . Now we see that we have in  $\mathcal{P}_b$  the interval  $[fx_{u_1}, fx_ix_{u_1}]$  and switching it with the interval  $[f, fx_ix_{u_1}]$  we get a partition with sdepth  $\geq d + 2$  for  $I/J$ . Thus we may assume that  $fx_{u_1} \notin \{a_1, \dots, a_{t_1}\}$ . Now set  $a_{t_1+1} = fx_{u_1}$ . Let  $a_{t_1+1}, \dots, a_k$  be a path starting with  $a_{t_1+1}$  and set  $c_j = h(a_j)$ ,  $t_1 < j \leq k$ . If  $a_p = a_v$  for  $v < t_1$ ,  $p > t_1$  then change in  $\mathcal{P}_b$  the intervals  $[a_j, c_j]$ ,  $v \leq j \leq p$  with the intervals  $[a_v, c_p]$ ,  $[a_{j+1}, c_j]$ ,  $v \leq j < p$ . We have in  $\mathcal{P}_b$  an interval  $[fx_{u_1}, fx_ix_{u_1}]$  and switching it to  $[f, fx_ix_{u_1}]$  we get a partition with sdepth  $\geq d + 2$  for  $I/J$ . Thus we may suppose that in fact  $a_p \notin \{b, a_1, \dots, a_{p-1}\}$  for any  $p > t_1$  (with respect to any path starting with  $a_{t_1+1}$ ). We have again two subcases:

- 1') there exist no maximal bad path starting with  $a_{t_1+1}$ ,
- 2') there exists a maximal bad path starting with  $a_{t_1+1}$ .

In 1') set  $T_2 = \{b' \in B : \text{there exists a path } a_{t_1+1}, \dots, a_k \text{ with } a_k = b'\}$ ,  $G_2 = B \setminus T_2$  and  $I'_2 = (f, G_2)$ ,  $I''_2 = (G_2)$ ,  $J'_2 = I'_2 \cap J$ ,  $J''_2 = I''_2 \cap J$ . As above, we see that if  $T_2 \cap (f) = \emptyset$  then we may take  $I' = I'_2$  and if  $T_2 \cap (f) \neq \emptyset$  then  $I' = I''_2$  works.

In the second case, let  $a_{t_1+1}, \dots, a_{t_2}$  be a maximal bad path starting with  $a_{t_1+1}$  and set  $c_j = h(a_j)$  for  $j > t_1$ . As we saw the whole set  $\{a_1, \dots, a_{t_2}\}$  has different monomials. As above  $c_{t_2} = bx_{u_2}$  and we may reduce to the case when  $fx_{u_2} \notin \{a_1, \dots, a_{t_1}\}$ . Set  $a_{t_2+1} = fx_{u_2}$  and again we consider two subcases, which we treat as above. Anyway after several such steps we must arrive in the case  $p = t_m$  when  $b|c_{t_m}$  and again a certain  $fx_{u_m}$  is not among  $\{a_1, \dots, a_{t_m}\}$  and taking  $a_{t_m+1} = fx_{u_m}$  there exist no maximal bad path starting with  $a_{t_m+1}$ . This follows since we may reduce to the case when the set  $\{a_1, \dots, a_{t_m}\}$  has different monomials and so the procedures should stop for some  $m$ . Finally, using  $T_m = \{b' \in B : \text{there exists a path } a_{t_m+1}, \dots, a_k \text{ with } a_k = b'\}$  as  $T_1$  above we are done.  $\square$

Now Theorem 2.3 follows from the next proposition, the case  $s > q + 1$  being a consequence of [16] (see here Theorem 1.1).

**Proposition 5.2.** *Suppose that  $I \subset S$  is minimally generated by a squarefree monomial  $f$  of degree  $d$ , and a set  $E$  of squarefree monomials of degrees  $\geq d + 1$ . Assume that sdepth<sub>S</sub>  $I/J = d + 1$  and  $s \leq q + 1$ . Then depth<sub>S</sub>  $I/J \leq d + 1$ .*

**Proof:** Apply induction on  $|E|$ , the case  $E = \emptyset$  follows from Lemma 2.2. Suppose that  $|E| > 0$ . We may assume that  $E$  contains just monomials of degrees  $d + 1$  by [17, Lemma 1.6]. Using Theorem 1.3 and induction on  $|E|$  apply Lemma 2.1. Thus we may suppose that  $C \subset ((f) \cap (E)) \cup (\cup_{a, a' \in E, a \neq a'} (a) \cap (a'))$ .

Let  $b \in (B \cap (f))$  and  $I'_b = (B \setminus \{b\})$ . Set  $J'_b = I'_b \cap J$ . Clearly  $b \notin I'_b$ . As in Case 1 from the previous section we see that if sdepth<sub>S</sub>  $I'_b/J'_b \leq d + 1$  then depth<sub>S</sub>  $I'_b/J'_b \leq d + 1$  by Theorem 1.3 and



so  $\text{depth}_S I/J \leq d+1$  by the Depth Lemma. Thus we may suppose that  $\text{sdepth}_S I'_b/J'_b \geq d+2$ . Applying Lemma 5.1 we get either  $\text{sdepth}_S I/J \geq d+2$  contradicting our assumption, or there exists a nonzero ideal  $I' \subsetneq I$  generated by a subset of  $\{f\} \cup B$  such that  $\text{sdepth}_S I'/J' \leq d+1$  for  $J' = J \cap I'$  and  $\text{depth}_S I/(J, I') \geq d+1$ . In the last case we see that  $\text{depth}_S I'/J' \leq d+1$  by induction hypothesis on  $|E|$  and so  $\text{depth}_S I/J \leq d+1$  by the Depth Lemma applied to the following exact sequence

$$0 \rightarrow I'/J' \rightarrow I/J \rightarrow I/(J, I') \rightarrow 0.$$

□

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