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On the minimized decomposition theory of valuations

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APPENDIX: **On the nature of base fields**

by
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Abstract

In this note we discuss the behavior of minimized inertia/decomposition groups of valuations, and prove similar results to the ones for tame inertia. The results are technical tools for a host of questions in Bogomolov's birational anabelian program.

Key Words: Anabelian geometry, function fields, Riemann-Zariski space, (generalized) [quasi] prime divisors, decomposition graphs, Hilbert decomposition theory, pro-ℓ Galois theory, algebraic/étale fundamental group, (split) [semi-stable] families of curves, alteration/modification theory, ℓ-adic/Prontrjagin duality.
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1 Motivation / Introduction

We begin by recalling that Bogomolov's birational anabelian program originates form [Bo], and aims to reconstruct function fields $K|k$ over algebraically closed base fields k from the *pro-ℓ abelian-by-central* Galois group Π^c_K of K, provided $\ell \neq \mathrm{char}(K)$ and $\mathrm{td}(K|k) > 1$. When completed, this would go far beyond Grothendieck's birational anabelian program, see [G1], [G2], which was asking to recover the isomorphy type of finitely generated infinite fields K from their *full absolute Galois group* G_K. Bogomolov's program is far from complete, although there has been progress towards tackling it, see e.g., Bogomolov–Tschinkel [B-T1], [B-T2] and Pop [P4], in the case k is an algebraic closure of a finite field. Finally, there is progress on Bogomolov's program over more general base fields k, namely those of finite Kronecker dimension, e.g., algebraic closures of global fields, see Pop [P5], and Silberstein [Sb]. The content of this note is essential to that progress.

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In order to explain the difficulties of going beyond the case where k is an algebraic closure of a finite field, and to put the results of this note in the right perspective, let me first recall briefly the strategy to tackle Bogomolov's program, as presented in Pop [P3]. The strategy is similar to the one employed in Grothendieck's birational anabelian geometry: One first develops a *local theory* similar to that in Neukirch [Ne], and then *globalizes* the local information to reconstruct K from its Galois theory.

We will focus here on the local theory, and explain what difficulties one encounters. The local theory is concerned mainly with recovering the space of arithmetically significant valuations v of K, more precisely, their Galois theoretical invariants, i.e., the inertia/decomposition groups $T_v \subset Z_v$ above v, and the partial order $v \le w$. To proceed, let us introduce the precise terminology, which is as follows. First, given a function field $K|k$ (where the base field k will be usually algebraically closed) with $\mathrm{char}(k) \neq \ell$, a prime divisor of $K|k$ is any valuation v of K which is trivial on k and has value group $vK = \mathbb{Z}$ and residue field $Kv|k$ a function field with $\mathrm{td}(Kv|k) = \mathrm{td}(K|k) - 1$. Basics of valuation theory and algebraic geometry apply to show that for a valuation v of K the following are equivalent:

a) v is a prime divisor of $K|k$.

b) v is trivial on k and $\mathrm{td}(Kv|k) = \mathrm{td}(K|k) - 1$.

c) The center of v on some normal model of $K|k$ is a prime Weil divisor.

For reader's sake, recall that a (normal/regular) model of $K|k$ is any (normal/regular) integral k-variety X with $K = k(X)$ the function field of X. Further, the center of v on X is the unique point $x \in X$ –if such a point x exists– whose local ring $\mathcal{O}_{X,x}$ is dominated by the valuation ring $\mathcal{O}_v$, notation $\mathcal{O}_{X,x} \prec \mathcal{O}_v$. By the valuation criterion, the point $x \in X$ with $\mathcal{O}_{X,x} \prec \mathcal{O}_v$ is unique, if it exists; and it exists, if X is proper, e.g., projective.

An obvious generalization of the prime divisors are the (generalized) prime r-divisors of $K|k$, which are defined inductively as follows: The prime 1-divisors of $K|k$ are precisely the prime divisors v of $K|k$, and inductively, a valuation $\mathfrak{v}$ of K is called a prime r-divisor for some $r > 1$, if there exists a prime $(r-1)$ divisor $\mathfrak{w}$ of $K|k$ such that $\mathfrak{v} > \mathfrak{w}$ and the valuation theoretical quotient $v_r := \mathfrak{v}/\mathfrak{w}$ on the residue field $K\mathfrak{w}|k$ is a prime divisor. Notice that one gets inductively that $\mathfrak{v}$ is trivial on k, and $K\mathfrak{v}|k$ is a function field. Further, $K\mathfrak{v}|k$ is finite if and only if $r = \mathrm{td}(K|k)$, thus $K\mathfrak{v} = k$ because k is algebraically closed. We notice that a prime r-divisor has $vK = \mathbb{Z}^r$ ordered lexicographically. By abuse of language, we will say that the trivial valuation $\mathfrak{v}_0$ of $K|k$ is the generalized prime divisors of rank zero of $K|k$, and will speak about generalized prime divisors, if the rank r is not relevant for the context.

To complete the picture, the set of all the generalized prime divisors of $K|k$ gives rise to the total divisor graph $\mathcal{D}^{\mathrm{tot}}_K$ of $K|k$, whose vertices are indexed by the residue fields $K\mathfrak{v}$, and an edge from $K\mathfrak{w}$ to $K\mathfrak{v}$ exists if and only if $\mathfrak{w} \le \mathfrak{v}$ and $\mathfrak{v}/\mathfrak{w}$ has rank ≤ 1, and further $\mathfrak{v}/\mathfrak{w}$ is the only edge from $K\mathfrak{w}$ to $K\mathfrak{v}$, oriented if $\mathfrak{w} < \mathfrak{v}$, non-oriented if $\mathfrak{w} = \mathfrak{v}$.

Via the Galois correspondence and the Hilbert decomposition theory, one attaches to the total divisor graph $\mathcal{D}^{\mathrm{tot}}_K$ of $K|k$ its Galois theoretical counterpart, which is the total decomposition graph $\mathcal{G}^{\mathrm{tot}}_K$ for $K|k$ or for Π_K, which is a graph in bijection with $\mathcal{D}^{\mathrm{tot}}_K$, but whose vertices and edges are "decorated" with subquotients of Π^c_K as follows: Each vertex $K\mathfrak{v}$ is endowed with the corresponding $\Pi_{K\mathfrak{v}} = Z_\mathfrak{v}/T_\mathfrak{v}$, and each edge $\mathfrak{v}/\mathfrak{w}$ is endowed with the corresponding inertia/decomposition subgroups $T_{\mathfrak{v}/\mathfrak{w}} \subset Z_{\mathfrak{v}/\mathfrak{w}}$ of $\mathfrak{v}/\mathfrak{w}$ in $\Pi_{K\mathfrak{w}}$. To fix notations, if $\mathfrak{v}$ is a prime r-divisor of $K|k$, by abuse of language, we will say that $T_\mathfrak{v} \subset Z_\mathfrak{v}$ is an r-divisorial group in Π_K, respectively, that $T_v \subset Z_v$ is a divisorial group, if v is a prime divisor of $K|k$.

One of the main points of the local theory is that the total decomposition graph $\mathcal{G}^{\rm tot}_K$ can be recovered (using group theoretical recipes about pro-ℓ abelian-by-central groups) from $\Pi^c_K \to \Pi_K$ endowed with all the divisorial groups $T_v \subset Z_v$ of Π_K. The recipes to do so use in a central way the main results from Pop [P2] and [P3], as follows: First recall that by Theorems A, from [P2], the set $\mathfrak{In}_k(K)$ of all the inertia elements at valuations v of K which are trivial on k is closed in Π_K, and second, since $\mathrm{char}(k) \neq \ell$, all such inertia elements are actually tame inertia elements. Hence by Theorem B of [P2], it follows that $\mathfrak{In}_k(K)$ is nothing but the topological closure in Π_K of the set of divisorial inertia elements $\mathfrak{In.div}(K) := \cup_v T_v$, i.e., inertia elements at prime divisors v of $K|k$. Hence, one concludes that for every generalized prime divisor $\mathfrak{v}$ of $K|k$ one has: $T_\mathfrak{v} \subset \mathfrak{In}_k(K)$, and by [P3], [P4] it follows that the r-divisorial groups $T_\mathfrak{v} \subset Z_\mathfrak{v}$ are precisely the maximal pairs of subgroups $T \subset Z$ of Π_K satisfying the following: First, Z contains a subgroup $\Delta \cong \mathbb{Z}_\ell^{\mathrm{td}(K|k)}$ whose preimage under $\Pi^c_K \to \Pi_K$ is abelian. Second, $T \cong \mathbb{Z}_\ell^r$ and the preimage of T under $\Pi^c_K \to \Pi_K$ is the center of the preimage of Z under $\Pi^c_K \to \Pi_K$.

Unfortunately, at the moment, there is neither a strategy to recover the divisorial subgroups $T_v \subset Z_v$, nor one to recover $\mathfrak{In}_k(K)$, using the group theoretical information encoded in the pro-ℓ group Π^c_K. The best we can do thus far is to recover pieces of information about the (generalized) quasi prime divisors of $K|k$, defined as follows. First, the quasi prime divisors of $K|k$ are the valuations v of K, not necessarily trivial on k, but satisfying the following:

i) $vK/vk \cong \mathbb{Z}$ as groups, and $Kv|kv$ is a function field with $\mathrm{td}(Kv|kv) = \mathrm{td}(K|k) - 1$.

ii) v is minimal among the valuations of K satisfying condition i) above.

Recall that the condition ii) asserts that if w is any valuation of K satisfying i) and having $\mathcal{O}_v \subset \mathcal{O}_w$, then $w = v$; or equivalently, vK does not contain any convex divisible subgroup. We notice that the conditions at i) can be weakened, because the following are equivalent:

a) v is a quasi prime divisor of $K|k$.

b) v is minimal with the properties: $\mathrm{td}(Kv|kv) = \mathrm{td}(K|k) - 1$, $vK \neq vk$.

One should remark that if v is a quasi prime divisor of $K|k$, then v has no transcendence defect, i.e., it satisfies the Abhyankar equality. As in the case of the prime divisors, one defines the quasi prime r-divisors of $K|k$ inductively as follows: First, the quasi prime 1-divisors of $K|k$ are by definition the quasi prime divisors of $K|k$. Second, for $r > 1$, one defines the quasi prime r-divisors inductively, as being the valuations $\mathfrak{v}$ of $K|k$ such that there exists a quasi prime $(r-1)$-divisor $\mathfrak{w}$ of $K|k$ such that $\mathfrak{v} > \mathfrak{w}$ and the valuation theoretical quotient $v_r := \mathfrak{v}/\mathfrak{w}$ is a quasi prime divisor of the residue field $K\mathfrak{w}|k\mathfrak{w}$. Notice that, inductively, one has the following: If $\mathfrak{v}$ is quasi prime r-divisor of $K|k$, then $\mathfrak{v}K/\mathfrak{v}k \cong \mathbb{Z}^r$, and $K\mathfrak{v}|k\mathfrak{v}$ is a function field with $\mathrm{td}(K\mathfrak{v}|k\mathfrak{v}) = \mathrm{td}(K|k) - r$. In particular, for all quasi prime r-divisors one has $r \leq \mathrm{td}(K|k)$, and $r = \mathrm{td}(K|k)$ if and only if $K\mathfrak{v} = k\mathfrak{v}$. Finally, it makes sense to say that the trivial valuation is the (unique) quasi prime 0-divisor. Then the set of all the generalized quasi prime divisors is a tree-like partially ordered set, and as in the case of generalized prime divisors, it gives rise to the total quasi divisorial graph $\mathcal{Q}_K^{\mathrm{tot}}$.

Unfortunately, the situation with the Galois theoretical counterpart of the total quasi divisorial graph is much more involved than in the case of generalized prime divisors. To make a long story short, if one restricts to the subgraph of generalized quasi prime divisors $\mathfrak{v}$ with $\mathrm{char}(K\mathfrak{v}) \neq \ell$, then one can develop a perfectly satisfactory theory, completely parallel to the case of generalized prime divisors, by replacing $\mathfrak{In}_k(K)$ by the set of all the *tame inertia elements* $\mathfrak{In.tm}(K)$ in Π_K, see Pop [P2], [P4]. On the other hand, at the moment, we do not have a method to single out the quasi prime divisors $\mathfrak{v}$ having $\mathrm{char}(K\mathfrak{v}) \neq \ell$ using the information encoded in Π_K^c. The best one can do thus far is as follows: First, for an arbitrary valuation v of K, let $U_v^1 := 1+\mathfrak{m}_v \subset \mathcal{O}_v^\times =: U_v$ be the groups of principal v-units, respectively the group of v-units in $K^\times$, and set $K^{Z^1} := K[\sqrt[\ell^\infty]{U_v^1}]$, $K^{T^1} := K[\sqrt[\ell^\infty]{U_v}]$. We further denote $Z_v^1 := \mathrm{Gal}\big(K'\,|\,K^{Z^1}\big)$ and $T_v^1 := \mathrm{Gal}\big(K'\,|\,K^{T^1}\big)$, and call $T_v^1 \subset Z_v^1$ the minimized inertia/decomposition groups of v in Π_K, and further call $\Pi_{Kv}^1 := Z_v^1/T_v^1 = \mathrm{Gal}\big(K^{T^1}\,|\,K^{Z^1}\big)$ the minimized residue group at v. Thus one has canonical exact sequences

$$1 \to T_v^1 \to Z_v^1 \to \Pi_{Kv}^1 \to 1, \quad 1 \to U_v/U_v^1 = Kv^\times \to K^\times/U_v^1 \to K^\times/U_v \to 1$$

which are ℓ-adically dual to each other via Kummer theory. We notice that by general decomposition theory for valuations, it follows that $T_v^1 = T_v$ and $Z_v^1 = Z_v$ are the inertia/decomposition groups above v, and $\Pi_{Kv}^1 = \Pi_{Kv}$, provided $\mathrm{char}(Kv) \neq \ell$. Further, if $\mathrm{char}(Kv) = \ell$, then $T_v^1 \subseteq Z_v^1 \subseteq V_v$, where $V_v = T_v$ is the wild ramification group of v, and Π_{Kv}^1 has no interpretation as a Galois group over Kv. Hence T_v consists of tame inertia if and only if $\mathrm{char}(Kv) \neq \ell$, and $T_v = V_v$ consists of wild ramification elements if and only if $\mathrm{char}(Kv) = \ell$. Finally, we notice that one can write $\mathrm{Val}(K) = \mathrm{Val}_0(K) \cap \mathrm{Val}_\ell(K)$, where

$$\mathrm{Val}_0(K) := \{v\,|\,v(\ell)=0\} = \{v\,|\,\mathrm{char}(Kv) \neq \ell\},\ \mathrm{Val}_\ell(K) := \{v\,|\,v(\ell) > 0\} = \{v\,|\,\mathrm{char}(Kv) = \ell\}$$

are disjoint and closed subsets (in the patch topology) of $\mathrm{Val}(K)$. Thus letting $\mathcal{Q}_0(K|k)$ and $\mathcal{Q}_\ell(K|k)$ be the corresponding subset of quasi prime divisors of $K|k$, by Theorem A and Theorem B from Pop [P2] applied to the special case of Π_K, one gets:

a) The set of all the *tame inertia elements* $\mathfrak{In.tm}(K) = \cup_{v \in \mathrm{Val}_0(K)} T_v \subset \Pi_K$ and the set of all the *ramification elements* $\mathfrak{Ram}(K) = \cup_{v \in \mathrm{Val}_\ell(K)} T_v \subset \Pi_K$ are topologically closed, and have trivial intersection.

b) The set of all the *quasi divisorial tame inertia* $\mathfrak{In.tm.q.div}(K) = \cup_{v \in \mathcal{Q}_0(K)} T_v \subset \Pi_K$ is dense in $\mathfrak{In.tm}(K)$.

Our aim is to obtain similar results for the minimized inertia $\mathfrak{In}^1(K) = \cup_{v \in \mathrm{Val}(K)} T_v^1$. First we notice that $\mathrm{Val}_\ell(K)$ is non-empty if and only if $\mathrm{char}(K) = 0$. Thus if $\mathrm{char}(K) \neq 0$, it follows that $\mathrm{Val}_\ell(K)$ is empty, and $T_v^1 = T_v$ and $Z_v^1 = Z_v$ for all $v \in \mathrm{Val}(K)$, thus there is nothing new to prove. But if $\mathrm{char}(K) = 0$, then $\mathfrak{In}^1(K) = \mathfrak{In.tm}(K) \cup \mathfrak{In}^{T^1}(K)$, where

$$\mathfrak{In.tm}(K) = \cup_{v \in \mathrm{Val}_0(K)} T_v, \quad \mathfrak{In}^{T^1}(K) = \cup_{v \in \mathrm{Val}_\ell(K)} T_v^1$$

have trivial intersection. The facts a), b) above give a good description of the tame part $\mathfrak{In}.\mathfrak{tm}(K)$, but do not touch upon $\mathfrak{In}^{T^1}(K) = \cup_{v\in\mathrm{Val}_\ell(K)} T^1_v \subset \cup_{v\in\mathrm{Val}_\ell(K)} Z^1_v = \mathfrak{In}^{Z^1}(K)$ originating form the set of valuations $\mathrm{Val}_\ell(K)$ with residue characteristic equal to ℓ.

The first result we announce is the following:

Theorem 1.1. *In the above notations, the following hold:*

1) *The subsets* $\mathfrak{In}^{T^1}(K) \subset \mathfrak{In}^{Z^1}(K) \subset \Pi_K$ *are closed in* Π_K.

2) *Actually more is true: Let* $\Delta \subseteq \Pi_K$ *be a closed subgroup such that for every open subgroup* $\Pi_i \subset \Pi_K$ *there exists* $v_i \in \mathrm{Val}(K)$ *such that* $\Delta \subseteq \Pi_i T^1_{v_i}$ *(resp.* $\Delta \subseteq \Pi_i Z^1_{v_i}$ *). Then there exists* $v \in \mathrm{Val}(K)$ *such that* $\Delta \subseteq T^1_v$ *(resp.* $\Delta \subseteq Z^1_v$ *).*

We remark that the result above has a kind of *general non-sense* type proof, being proved along the lines from Pop [P2], Theorem A, namely: Let $\mathrm{Val}(K)$ be the space of all the valuations of K endowed with the patch topology τ^{pa}, and $\mathrm{Sbg}(\Pi_K)$ the space of all the closed subgroups of Π_K endowed with the étale topology τ^{et}, see Section 2) for the definitions. Then one proves that the maps sending each $v \in \mathrm{Val}(K)$ to T^1_v, respectively Z^1_v, are continuous. One concludes by showing that Theorem 1.1 is a reinterpretation of this fact.

The next result is more technical, and does not follow by *general non-sense* type arguments. It reduces the detection of minimized inertia of a special class of generalized quasi prime divisors to identifying the minimized inertia of quasi prime divisors in a very special class of such, the so called c.r. quasi prime divisors. But first let us explain the terms. Let $K|k$ be a function field with k algebraically closed of characteristic $\neq \ell$. Recall that constant reductions (à la Deuring) of $K|k$ are valuations v of K which are not necessarily trivial on k and satisfy $\mathrm{td}(K|k) = \mathrm{td}(Kv|kv)$. In particular, constant reductions satisfy the Abhyankar equality, thus are defectless in the sense of Kuhlmann [Ku] and/or [K-K]. Then given any prime divisor v_0 of $Kv|kv$, it follows that the valuation theoretical composition $\mathfrak{v} := v_0 \circ v$ is a quasi prime divisor of $K|k$, which we will call a c.r. quasi prime divisor of $K|k$.

For a given valuation v_k of k, let $\mathcal{Q}_{v_k}(K|k)$ be the set of all the c.r. quasi prime divisors $\mathfrak{v}$ of $K|k$ with $\mathfrak{v}|_k = v_k$, and let $\mathcal{T}^1_{v_k}(K) \subset \Pi_K$ be the (topological) closure of the set $\cup_{\mathfrak{v}\in\mathcal{Q}_{v_k}(K|k)} T^1_{\mathfrak{v}}$. Notice that $\mathcal{T}^1_{v_k} \subset \mathfrak{In}^{T^1}(K)$, because the latter set is itself topologically closed in Π_K by Theorem 1.1 above, and therefore, $\mathcal{T}^1_{v_k}(K)$ consists of minimized inertia elements.

Next let w be a fixed valuation of $K|k$. We say that w is c.r. like, if there exists a k-subfield $k_1 \subset K$, depending on w, such that $k_1 w$ is algebraically closed and $Kw|k_1 w$ is a function field, i.e., a finitely generated field extension, which satisfies $\mathrm{td}(K|k_1) = \mathrm{td}(Kw|k_1 w)$. Further, a c.r. quasi prime w-divisor of $K|k$ is any valuation $\mathfrak{w}$ of K of the form $\mathfrak{w} := w_0 \circ w$, where w_0 is a c.r. quasi prime divisor of the function field $Kw|k_1 w$. The next result we want to announce is:

Theorem 1.2. *For every c.r. quasi prime* w*-divisor* $\mathfrak{w}$ *of* $K|k$*, one has* $T^1_{\mathfrak{w}} \subset T^1_w \cdot \mathcal{T}_{v_k}(K)$.

We should notice that it is generally believed that $T^1_v \subset \mathcal{T}_{v_k}(K)$ for every valuation v of K with $v_k = v|_k$, but as of now, there is no strategy to tackle this question. Nevertheless, if one restricts to the *tame inertia* $\mathfrak{In}.\mathfrak{tm}(K)$, i.e., if one has that $\mathrm{char}(kv) \neq \ell$, then the inclusion

$T_v \subset \mathcal{T}_{v_k}(K)$ is already known, and it follows (after some work) from Theorem B, Introduction, of Pop [P2].

Finally, Theorem 4.2 in Section 4 of this note, uses Theorems 1.1 and 1.2, to obtain an essential tool for determining the nature of k and $k\mathfrak{v}$, e.g., to characterize the generalized quasi prime divisors $\mathfrak{v}$ of $K|k$ with $K\mathfrak{v} \neq k\mathfrak{v}$ and $k\mathfrak{v}$ an algebraic closure of a finite field.

2 Proof of Theorem 1.1

For reader's sake, let us recall the basics concerning the Zariski topology and the patch topology on the set of (equivalence classes of) valuations. For an arbitrary field Ω, let $\mathrm{Val}(\Omega)$ be all the valuations rings, thus equivalence classes of valuations, or of places, of Ω. One defines the Zariski topology τ^{Zar} on $\mathrm{Val}(\Omega)$ as being the topology which has as a basis the sets of the form:

$$U_A := \{v \in \mathrm{Val}(\Omega) \mid v(a) \le 0,\ a \in A\}, \quad \forall \text{ finite } A \subset \Omega.$$

Since the trivial valuation lies in U_A for all finite sets $A \subset \Omega^*$, it follows that τ^{Zar} is not Hausdorff. Nevertheless, τ^{Zar} is quasi-compact. The constructible, thus Hausdorff, topology generated by τ^{Zar} is called the patch topology on $\mathrm{Val}(\Omega)$, denote τ^{pa}. A basis of this topology consists of all the sets of the form:

$$U_{A,B} := \{v \in \mathrm{Val}(\Omega) \mid v(a) \le 0, v(b) = 0,\ a \in A,\ b \in B\}, \quad \forall \text{ finite } A, B \subset \Omega.$$

By *general non-sense* about constructible topology, it follows that τ^{pa} is Hausdorff and compact, and the basic open subsets $U_{A,B}$ are actually open and closed. Thus $\mathrm{Val}(\Omega)$ endowed with the patch topology is a profinite topological space.

The Zariski topology and the patch topology behave nicely under field extensions as follows: Let $\tilde{\Omega}|\Omega$ be a field extension. Then the canonical restriction map

$$\mathrm{res} : \mathrm{Val}(\tilde{\Omega}) \to \mathrm{Val}(\Omega), \quad \tilde{v} \mapsto v := \tilde{v}|_{\Omega}$$

is surjective (by the Chavalley's theorem on the prolongation of places), and continuous in both the Zariski topology and the patch topology. Moreover, if $(\Omega_i)_i$ is an inductive family of fields, and $\Omega = \cup_i \Omega_i$ is the inductive limit, then $\big(\mathrm{Val}(\Omega_i)\big)_i$ endowed with the (surjective) restriction morphism $\mathrm{res}_{ji} : \mathrm{Val}(\Omega_j) \to \mathrm{Val}(\Omega_i)$ is a projective system, and $\mathrm{Val}(\Omega)$ is in a canonical way the projective limit of this projective system.

Second, let G be a profinite group. The set of all the closed subgroups $\mathrm{Sbg}\big(G\big)$ of G carries the étale topology τ^{et}, similar to τ^{Zar} on $\mathrm{Val}(\Omega)$, having a basis of open subsets given by:

$$U^{\mathrm{et}}_M := \{\Gamma \in \mathrm{Sbg}\big(G\big) \mid \Gamma \subseteq M\}, \quad \forall \text{ open subgroups } M \subseteq G.$$

Clearly, τ^{et} is quasi-compact and non-Hausdorff. The constructible topology on $\mathrm{Sbg}\big(G\big)$ generated by τ^{et} is called the strict topology τ^{st}. As above, τ^{st} is Hausdorff and compact, and has a basis of open (and closed) subsets given by

$$U^{\mathrm{st}}_{M,N} := \{\Gamma \in \mathrm{Sbg}\big(G\big) \mid \Gamma N = M\}, \quad \forall \text{ open subgroups } M, N \subseteq G,\ N \text{ normal}.$$

We next consider a special case of the situation above; see Pop [P2], Section 2, for a more general setting. Namely, for K an arbitrary field with $\mathrm{char}(K) \neq \ell$ and $\mu_{\ell^\infty} \subset K$, let $K'|K$ be the maximal pro-ℓ abelian extension, and Π_K be its Galois group. Let $(K_i|K)_i$ be an inductive family of finite subextensions of $K'|K$ with $\cup_i K_i = K'$, and for $K_i \subseteq K_j$, setting $\Pi_i := \mathrm{Gal}\big(K'\,|\,K_i\big)$ and $\overline{\Pi}_i = \mathrm{Gal}\big(K_i\,|\,K\big)$, let $\mathrm{pr}_i : \Pi_K \to \overline{\Pi}_i$, $\mathrm{pr}_{ji} : \overline{\Pi}_j \to \overline{\Pi}_i$ be the canonical projections (which are surjective). Thus recalling the minimized inertia/decompostion groups $T^1_v \subseteq Z^1_v$ of a valuation $v \in \mathrm{Val}(K)$ in Π_K, one gets canonical maps:

$$\psi^{T^1}\!, \psi^{Z^1}\colon \mathrm{Val}(K) \to \mathrm{Sbg}\big(\Pi_K\big),\ \psi^{T^1}_i\!, \psi^{Z^1}_i\colon \mathrm{Val}(K_i) \to \mathrm{Sbg}\big(\Pi_i\big),\ \psi^{\overline{T}^1}_i\!, \psi^{\overline{Z}^1}_i\colon \mathrm{Val}(K) \to \mathrm{Sbg}\big(\overline{\Pi}_i\big)$$

defined by $\psi^{T^1}(v) := T^1_v$, $\psi^{Z^1}(v) := Z^1_v$, $\psi^{T^1}_i(v) := T^1_v \cap \Pi_i$, $\psi^{Z^1}_i(v) := Z^1_v \cap \Pi_i$, and finally $\psi^{\overline{T}^1}_i(v) = \mathrm{pr}_i(T^1_v) =: \overline{T}^1_v$, $\psi^{\overline{Z}^1}_i(v) = \mathrm{pr}_i(Z^1_v) =: \overline{Z}^1_v$. Letting $\bullet$ be either T^1 or Z^1, by general decomposition theory, (although we cannot give a precise reference for this) one has that

$$\psi^\bullet = \varprojlim_{K_i} \psi^{\overline{\bullet}}_i, \quad \psi^\bullet_i = \ker(\psi^\bullet \to \psi^{\overline{\bullet}}_i) \text{ for every } K_i.$$

After this preparation we can announce the following:

Theorem 2.1. *In the above notations, the following hold:*

1) *The maps* $\psi^{T^1}\!, \psi^{Z^1} : \mathrm{Val}(K) \to \mathrm{Sbg}\big(\Pi_K\big)$ *are continuous, provided we endow* $\mathrm{Val}(K)$ *with the patch topology* τ^{pa} *and* $\mathrm{Sbg}\big(\Pi_K\big)$ *with the étale topology* τ^{et}.

2) *For* $\Sigma \subseteq \mathrm{Val}(K)$ *a* τ^{pa}*-closed subset, the sets* $\mathfrak{In.tm}_\Sigma(K)$, $\mathfrak{In}^{T^1}_\Sigma(K)$, $\mathfrak{In}^{Z^1}_\Sigma(K)$ *of all the corresponding elements at all the* $v \in \Sigma$ *are closed in* Π_K.

3) *Finally, in the situation above, let* $\Delta \subseteq \Pi_K$ *be a closed subgroup such that for every* $K_i|K$, *there exists* $v_i \in \Sigma$ *such that one of the conditions below holds:*

   i) $\mathrm{pr}_i(\Delta) \subseteq \mathrm{pr}_i(T^1_{v_i})$.
   ii) $\mathrm{pr}_i(\Delta) \subseteq \mathrm{pr}_i(Z^1_{v_i})$ *and* $\mathrm{char}(Kv_i) = \ell$.

   Then there exists $v \in \Sigma$ *such that* i) $\Delta \subseteq T^1_v$, *or* ii) $\Delta \subseteq Z^1_v$ *and* $\mathrm{char}(Kv) = \ell$.

Proof: To 1): By the discussion before the Theorem, it is sufficient to prove that the maps $\psi^{\overline{\bullet}}_i : \mathrm{Val}(K) \to \mathrm{Sbg}\big(\overline{\Pi}_i\big)$ are continuous, provided we endow $\mathrm{Val}(K)$ with the patch topology and $\mathrm{Sbg}\big(\overline{\Pi}_i\big)$ with the étale topology. Notice that since $\overline{\Pi}_i$ is finite, $\mathrm{Sbg}\big(\overline{\Pi}_i\big)$ consists of all the subgroups of $\overline{\Pi}_i$, and one checks easily that the sets $B_\Delta := \{\Gamma \in \mathrm{Sbg}\big(\overline{\Pi}_i\big)\,|\,\Delta \subseteq \Gamma\}$ with $\Delta \in \mathrm{Sbg}\big(\overline{\Pi}_i\big)$, represent a basis for the τ^{et}-closed subsets in $\mathrm{Sbg}\big(\overline{\Pi}_i\big)$. [Indeed: First, the complement of B_Δ is the union of all the basic open subsets U_{G_1} with $\Delta \not\subseteq G_1$, hence an τ^{et} open set. Second, the basic closed set which is the complement of U_{G_1} is exactly the union of all the subsets B_Δ with Δ all the subgroups $\Delta \not\subseteq G_1$.] We prove that $\psi^{\overline{\bullet}}_i$ are continuous by showing that the preimages of τ^{et}-closed subsets of the form B_Δ are τ^{pa}-closed.

By Kummer theory, it follows that every K_i is of the form $K_i = K[\sqrt[n_i]{A_i}\,]$, where $n_i = \ell^{e_i}$ is some power of ℓ, and $A_i \subset K^\times/n_i$ finite subgroup, and $\overline{\Pi}_i = \mathrm{Hom}\big(A_i, \mathbb{Z}/n_i(1)\big)$.

In particular, recalling the definition of minimized inertia/decomposition groups $T^1_v \subseteq Z^1_v$ of a valuation $v \in \mathrm{Val}(K)$ in Π_K, one gets:

a) $\overline{T}^1_v := \psi_i^{\overline{T}^1}(v) = \mathrm{Gal}\big(K_i\,|\,K_i^{T^1}\big)$, where $K_i^{T^1} = K[\sqrt[n_i]{U_{v,i}}\,]$ and $U_{v,i} := A_i/n_i \cap U_v/n_i$.

b) $\overline{Z}^1_v := \psi_i^{\overline{Z}^1}(v) = \mathrm{Gal}\big(K_i\,|\,K_i^{Z^1}\big)$, where $K_i^{Z^1} = K[\sqrt[n_i]{U^1_{v,i}}\,]$ and $U^1_{v,i} := A_i/n_i \cap U^1_v/n_i$.

Recalling the exact sequences $1 \to U_v \to K^\times \to vK \to 0$ and $1 \to U^1_v \to U_v \to Kv^\times \to 1$, one gets $1 \to U_v/n_i \to K^\times/n_i \to vK/n_i \to 0$ and $1 \to Kv^\times/n_i \to (K^\times/U^1_n)/n_i \to vK/n_i \to 0$. Hence $A_i/U_{v,i} \to vK/n_i$ is injective, and $A_i/U^1_{v,i} \subset (K^\times/U^1_n)/n_i$ fits in the last exact sequence.

We first show that $\psi_i^{\overline{T}^1}$ is continuous: Let $v \in \mathrm{Val}(K)$ be fixed. Since $U_{i,v}/n_i \subseteq A_i/n_i$ are finite abelian n_i-torsion groups, one can write A_i/n_i as a direct sum of cyclic subgroups, say generated by $(t_\alpha)_\alpha$ of orders $(d_\alpha)_\alpha$, and there exist $(e_\alpha)_\alpha$ with $e_\alpha | d_\alpha$ such that $U_{i,v}$ is the direct sum of the subgroups generated by $(t_\alpha^{e_\alpha})_\alpha$. Then the fact that $K^\times/n_i \to vK/n_i$ maps $A_i/U_{i,v}$ injectively into vK/n_i is equivalent to saying that for every (multiplicative) linear combination $t := \prod_\alpha t_\alpha^{m_\alpha}$ with $0 \le m_\alpha < d_\alpha$, one has: $v(t) = v(\theta^{n_i})$ for some $\theta \in K^\times$ if and only if $e_\alpha | m_\alpha$ for all α. Next, let $\Sigma_{i,v} \subset A_i$ be the (finite) set of all the multiplicative linear combinations $t := \prod_\alpha t_\alpha^{m_\alpha}$ as above, and for every valuation $w \in \mathrm{Val}(K)$ and $\theta \in K^\times$, consider the condition:

$$(*)_\theta \qquad\qquad w(t) \neq w(\theta^{n_i}) \quad \text{for all } t \in \Sigma_{i,v}.$$

Since $\Sigma_{i,v}$ is finite, the set $V_{\theta,v}$ of valuations w satisfying $(*)_\theta$ is obviously open and closed in the patch topology; hence $V_v := \cap_{\theta\in K^\times} V_{\theta,v}$ is closed in the patch topology. Moreover, if $w \in V_v$, then the map $K^\times/n_i \to wK/n_i$ is injective on $\Sigma_{i,v}$, and an easy argument shows that $U_{i,w} := A_i/n_i \cap U_w/n_i = \ker(A_i/n_i \to wK/n_i)$ must be contained in $U_{i,v}$. We thus conclude that there exists a τ^{pa} closed subset $V_v \subset \mathrm{Val}(K)$ such that for all $w \in V_v$ one has: $U_{i,w} \subseteq U_{i,v}$. Thus by the point a) of the discussion above, and in the notations from there, one has that $\overline{T}^1_v \subseteq \overline{T}^1_w$ for all $w \in V_v$. The converse implication is obvious, namely, if $\overline{T}^1_v \subseteq \overline{T}^1_w$, then by Kummer theory and point a) above, it follows that $U_{i,w} \subseteq U_{i,v}$.

Let $\Delta_i \subseteq \overline{\Pi}_i$ be a fixed subgroup, and $\mathcal{V}_{\Delta_i} := \{w \in \mathrm{Val}(K) \mid \Delta_i \subseteq \overline{T}^1_w\}$ be the preimage of the τ^{et}-closed set $B_{\Delta_i} \subseteq \mathrm{Sbg}\big(\overline{\Pi}_i\big)$. We claim that $\mathcal{V}_{\Delta_i}$ is τ^{pa}-closed in $\mathrm{Val}(K)$. Indeed, since $\overline{\Pi}_i$ is finite, thus $\mathrm{Sbg}\big(\overline{\Pi}_i\big)$ is finite, the set $\mathcal{G}_{\Delta_i} := \{\overline{T}^1_w \mid \Delta \subseteq \overline{T}^1_w\}$ is finite as well (maybe empty). Let $\mathcal{V}_0$ be a finite set of valuations $v \in \mathrm{Val}(K)$ such that $\mathcal{G}_{\Delta_i} = \{\overline{T}^1_v \mid v \in \mathcal{V}_0\}$. Then $\cup_{v\in\mathcal{V}_0} V_v = \mathcal{V}_{\Delta_i}$, and further $\cup_{v\in\mathcal{V}_0} V_v$ is τ^{pa}-closed, because each V_v is so by the discussion above. We conclude that $\psi^{\overline{T}^1}$ is continuous.

The continuity of $\psi^{\overline{Z}^1}$ is proved in a similar way, but starting with $A_i/U^1_{v,i} \subset (K^\times/U^1_n)/n_i$ which fits in the exact sequence $1 \to Kv^\times/n_i \to (K^\times/U^1_n)/n_i \to vK/n_i \to 0$, etc.

To 2): Since $\Sigma \subset \mathrm{Val}(K)$ is τ^{pa} closed, so are the sets $\Sigma_0 = \{v \in \Sigma \mid \mathrm{char}(Kv) \neq \ell\}$ and $\Sigma_\ell = \{v \in \Sigma \mid \mathrm{char}(Kv) = \ell\}$, and $\Sigma = \Sigma_0 \cup \Sigma_\ell$. Further, since $\psi^\bullet$ are continuous, it follows that $\psi^{T^1}(\Sigma_0)$, $\psi^{T^1}(\Sigma_\ell)$, $\psi^{Z^1}(\Sigma_\ell)$ are τ^{et} quasi compact subsets of $\mathrm{Sbg}\big(\Pi_K\big)$. But then it follows by *general non-sense* that $\mathfrak{In.tm}_\Sigma(K) := \cup_{v\in\Sigma_0} T_v$, $\mathfrak{In}^{T^1}_\Sigma(K) := \cup_{v\in\Sigma_\ell} T^1_v$, $\mathfrak{In}^{Z^1}_\Sigma(K) := \cup_{v\in\Sigma_\ell} Z^1_v$ are closed subsets of Π_K.

To 3): We prove i), because the proof of ii) is *mutatis mutandis* identical. Thus suppose that for every $K_i|K$ there exists $v_i \in \Sigma$ such that $\Delta_i := \mathrm{pr}_i(\Delta) \subseteq \psi_i^{\overline{T}^1}(v_i) = \mathrm{pr}_i(T^1_{v_i})$. Then in the notations from the proof of assertion 1), and taking into account the continuity of

$$\psi^{\overline{T}^1} : \Sigma \to \mathrm{Sbg}\big(\overline{\Pi}_i\big),$$

it follows that $\mathcal{V}_{\Delta_i} := \{v \in \Sigma \mid \Delta_i \subseteq \psi^{\overline{T}^1}(v)\}$ is closed in Σ, and is non-empty because $v_i \in \mathcal{V}_{\Delta_i}$. Thus $(\mathcal{V}_{\Delta_i})_i$ is a family of compact subsets of Σ, and we notice that $(\mathcal{V}_{\Delta_i})_i$ has the finite intersection property, because $\mathcal{V}_{\Delta_j} \subseteq \mathcal{V}_{\Delta_i}$ for $K_i \subseteq K_j$; hence $\cap_i \mathcal{V}_{\Delta_i}$ is non-empty. For every $v \in \cap_i \mathcal{V}_{\Delta_i}$ the following hold: Since ψ^{T^1} is the limit of the surjective projective system of maps $(\psi_i^{\overline{T}^1})_i$, we have that $T^1_v = \psi^{T^1}(v) \to \psi_i^{\overline{T}^1}(v) = \mathrm{pr}_i(T^1_v)$ is surjective, and $T^1_v = \psi^{T^1}(v)$ is the projective limit of the surjective projective system $\big(\psi_i^{\overline{T}^1}(v) = \mathrm{pr}_i(T^1_v)\big)_i$. Further, since $v \in \mathcal{V}_{\Delta_i}$, one has by the definition of $\mathcal{V}_{\Delta_i}$ that

$$\mathrm{pr}_i(\Delta) =: \Delta_i \subseteq \psi_i^{\overline{T}^1}(v) := \mathrm{pr}_i(T^1_v).$$

Since this holds for all $\mathrm{pr}_i : \Pi_K \to \overline{\Pi}_i$, we finally have $\Delta \subseteq T^1_v$, as claimed. □

3 Proof of Theorem 1.2

The proof of Theorem 1.2, although not too difficult, is quite involved, and we will end up by proving a more precise result, which is Theorem 3.2 below. Concerning the strategy of proof, one starts as in the proof of Theorem 1.1, namely: For every ℓ-power $n = \ell^e$ and every finite subgroup A of $K^\times/n$, we set $K_A := K[\sqrt[n]{A}\,]$ and consider the canonical projection $\Pi_K \to \overline{\Pi} := \mathrm{Gal}\big(K_A\,|\,K\big)$, $\sigma \mapsto \overline{\sigma}$. Further, for every valuation v of K, we let $T^1_v \to \overline{T}^1_v$ be the projection of T^1_v under $\Pi_K \to \overline{\Pi}$. Then the following assertions for $\sigma \in \Pi_K$ are equivalent:

i) $\sigma \in \Pi_K$ lies in $\mathcal{T}_{v_k}(K)$.

ii) $\forall\ n = \ell^e$, $A \subset K^\times/n$ finite subgroups, $\exists\ \mathfrak{v} \in \mathcal{Q}_{v_k}(K|k)$ such that $\overline{\sigma} \in \overline{T}^1_{\mathfrak{v}}$.

A) *An abstract result*

We next formulate and prove an abstract result, which will eventually imply Theorem 1.2. The situation is as follows: For $K|k$ as usual, consider the algebraically closed subfields $\lambda \subset \overline{K}$ with $\mathrm{td}(\lambda|k) = \mathrm{td}(K|k) - 1$, and for every such λ, set $\Lambda := K\lambda$ inside $\overline{K}$. Hence one has $\mathrm{td}(\Lambda|\lambda) = 1$, thus $\Lambda|\lambda$ is a function field in one variable, and there exists $t \in K$ such that Λ is finite separable over $\lambda(t)$. With $n = \ell^e$ and $A \subset K^\times/n$ as above, we consider the resulting finite abelian extension $\Lambda_A := \Lambda[\sqrt[n]{A}\,] = K_A\Lambda$ inside $\overline{K}$, the injective canonical projection $\mathrm{Gal}\big(\Lambda_A\,|\,\Lambda\big) =: \overline{\Pi}_\Lambda \to \overline{\Pi}$, and recall that for every prolongation $v_\Lambda|v$ of v to Λ, $\overline{\Pi}_\Lambda \to \overline{\Pi}$ gives rise by restriction to an embedding $\overline{T}^1_{v_\Lambda} \to \overline{T}^1_v$.

Next let v_Λ be a valuation of Λ and set $v_\lambda := v|_\lambda$. Then by the additivity of the residual transcendence degree, one has that $\mathrm{td}(\Lambda v_\Lambda|kv_\Lambda) = \mathrm{td}(\Lambda v_\Lambda|\lambda v_\Lambda) + \mathrm{td}(\lambda v_\lambda|kv_\lambda)$. Hence setting

$d := \mathrm{td}(\Lambda|k) = \mathrm{td}(\Lambda|\lambda) + \mathrm{td}(\lambda|k)$, and applying the Abhyankar (in)equality, it follows that $\mathrm{td}(\Lambda|\lambda) = 1 \geq \mathrm{td}(\Lambda v_\Lambda|\lambda v_\Lambda)$ and $\mathrm{td}(\lambda|k) = d-1 \geq \mathrm{td}(\lambda v_\lambda|k v_\lambda)$. We will say that a valuation v_Λ of Λ is a constant reduction of $\Lambda|k$ if it satisfies $\mathrm{td}(\Lambda|k) = \mathrm{td}(\Lambda v_\Lambda|k v_\Lambda)$, or equivalently, if $\mathrm{td}(\Lambda|\lambda) = 1 = \mathrm{td}(\Lambda v_\Lambda|\lambda v_\lambda)$ and $\mathrm{td}(\lambda|k) = d-1 = \mathrm{td}(\lambda v_\lambda|k v_\lambda)$. In particular, if v_Λ is a constant reduction of $\Lambda|k$, it follows that $\mathrm{td}(\Lambda|\lambda) = 1 = \mathrm{td}(\Lambda v_\Lambda|\lambda v_\Lambda)$, hence v_Λ defines a constant reduction of $\Lambda|\lambda$ in the usual way. Nevertheless, the converse of this assertion is not true, i.e., the set of all the constant reductions of $\Lambda|\lambda$, having a given restriction v_k to k, is much richer, provided $\lambda \neq k$.

Further, we will say that a valuation $\mathfrak{v}_\Lambda$ of Λ is a c.r. quasi prime divisor of $\Lambda|k$, if there exists a constant reduction v_Λ of $\Lambda|k$ and a prime divisor v_0 of $\Lambda v_\Lambda|\lambda v_\lambda$ such that $\mathfrak{v}_\Lambda = v_0 \circ v_\Lambda$. As in the case of constant reductions, a c.r. quasi prime divisor $\mathfrak{v}_\Lambda$ of $\Lambda|k$ is a c.r. quasi prime divisor of $\Lambda|\lambda$ as well. But if $\lambda \neq k$, then there exist c.r. quasi prime divisors of $\Lambda|\lambda$, which are not c.r. quasi prime divisors of $\Lambda|k$.

Definition 3.1. We say that $\overline{\sigma} \in \overline{\Pi}$ is a c.r. codimension one minimized inertia element, if there exists some $\Lambda|\lambda$ as above, and some c.r. quasi prime divisor $\mathfrak{w}_\Lambda$ of $\Lambda|\lambda$ such $\overline{\sigma}$ lies in the image of $\overline{T}^1_{\mathfrak{w}_\Lambda} \hookrightarrow \overline{\Pi}_\Lambda \to \overline{\Pi}$.

This being said, the abstract result we prove is the following:

Theorem 3.2. *In the above notations, let $\overline{\sigma} \in \overline{\Pi}$ be a c.r. codimension one minimized inertia element. Then there exist c.r. quasi prime divisors $\mathfrak{v}$ of $K|k$ with $\overline{\sigma} \in \overline{T}^1_{\mathfrak{v}}$. Moreover, if in the above notation, $\mathfrak{w}_\Lambda$ is a* c.r. *quasi prime divisor of $\Lambda|\lambda$ such that $\overline{\sigma}$ lies in the image of $\overline{T}^1_{\mathfrak{w}_\Lambda} \to \overline{\Pi}$, then one can choose the c.r. quasi prime divisor $\mathfrak{v}$ of $K|k$ such that $\mathfrak{w}_\Lambda|_k = \mathfrak{v}|_k$.*

Proof: First, in the notations from the definition above, let $\Lambda|\lambda$ and $\mathfrak{w}_\Lambda$ be a c.r. quasi prime divisor of $\Lambda|\lambda$ such that $\overline{\sigma}$ lies in the image of $\overline{T}^1_{\mathfrak{w}_\Lambda} \to \overline{\Pi}$. Then by definitions, $\Lambda|\lambda$ is a function field in one variable, thus $\Lambda = \lambda(C_\lambda)$ for some projective smooth λ-curve C_λ. Further, setting $w_\lambda := \mathfrak{w}_\Lambda|_\lambda$ and $v_k := w_\lambda|_k$, it follows that v_k is the restriction of w_λ to k as well. Equivalently, $\mathcal{O}_{\mathfrak{v}_\Lambda}$ dominates $\mathcal{O}_{w_\lambda}$, and both these valuation rings dominate $\mathcal{O}_{v_k}$.

Second, we briefly review a few generalities about the so called Riemann space $\mathrm{Val}(\Omega)$ of all the valuations of an arbitrary field Ω, see e.g. [Z-S] for details. Namely, let $\Omega_\nu \subset \Omega$ be a cofinal family of finitely generated subfields (over the prime field of Ω), i.e., $\Omega = \cup_\nu \Omega_\nu$. Then the set of the projective $\mathbb{Z}$-models of Ω_ν is set theoretically projectively ordered with respect to the domination relation, and if $(\mathcal{V}_{\mu_\nu})_{\mu_\nu}$ is a cofinal system of such models, then

$$(\dagger) \qquad \mathrm{Val}(\Omega) = \varprojlim_{\nu} \varprojlim_{\mu_\nu} \mathcal{V}_{\mu_\nu} \to \varprojlim_{\mu_\nu} \mathcal{V}_{\mu_\nu} = \mathrm{Val}(\Omega_\nu), \quad v_\Omega \mapsto v_{\Omega_\nu} := v_\Omega|_{\Omega_\nu}$$

where $\mathrm{Val}(\Omega) \to \mathrm{Val}(\Omega_\nu)$, $v_\Omega \mapsto v_{\Omega_\nu}$, is the restriction map from the Riemann space $\mathrm{Val}(\Omega)$ of Ω to that of Ω_ν. Recall that given $v_\Omega \in \mathrm{Val}(\Omega)$, and letting $x_{\mu_\nu} \in \mathcal{V}_{\mu_\nu}$ be the center of v_Ω on each $\mathcal{V}_{\mu_\nu}$, and $\mathcal{O}_{\mu_\nu}, \mathfrak{m}_{\mu_\nu}$ be its local ring, and $\kappa(x_{\mu_\nu}) := \mathcal{O}_{\mu_\nu}/\mathfrak{m}_{\mu_\nu}$, one has that:

$$(\ddagger) \qquad \mathcal{O}_{v_\Omega} = \cup_{\mu_\nu} \mathcal{O}_{\mu_\nu}, \quad \mathfrak{m}_{v_\Omega} = \cup_{\mu_\nu} \mathfrak{m}_{\mu_\nu}, \quad \Omega v_\Omega = \kappa(\mathfrak{m}_{v_\Omega}) = \cup_{\mu_\nu} \kappa(x_{\mu_\nu}).$$

Step 1. *Reviewing basics about families of curves*

Let $\Sigma_\lambda \subset C_\lambda(l)$ be any finite subset. By *general non-sense* about fields of definition, and more general, rings/schemes of definition, see e.g., Raynaud–Gruson [R-G], and/or de Jong [dJ1], [dJ2], the following hold:

1) There exists a subextension $k_1 \hookrightarrow l_1$ of $\lambda|k$ and a projective smooth geometrically integral l_1-curve C_1 satisfying the following:

   a) $l_1|k_1$ is a regular extension of finitely generated fields, i.e., l_1 is separable generated over k_1, and linearly disjoint from $\overline{k}_1$ over l_1.

   b) $C_\lambda = C_1 \times_{l_1} \lambda$ is the base change of C_1 under $l_1 \hookrightarrow \lambda$, and $\Sigma_\lambda \subset C_\lambda(\lambda)$ is the base change of a finite set $\Sigma_1 \subset C_1(l_1)$.

2) Therefore, for any cofinal system of regular extensions of finitely generated fields $l_\nu|k_\nu$ of $\lambda|k$ with $k_1 \subset k_\nu \subset k$ and $l_1 \subset l_\nu \subset \lambda$, the base change $C_\nu := C_1 \times_{l_1} l_\nu$ is a projective smooth l_ν-curve with $\Sigma_\nu \subset C_\nu(l_\nu)$. Further, the base change of $\Sigma_\nu := \Sigma_1 \times_{l_1} l_\nu$ under $l_\nu \hookrightarrow \lambda$ equals the given Σ_λ.

For the above l_1-curve C_1 with $\Sigma_1 \subset C_1(l_1)$, and the resulting $C_\nu \to l_\nu \to k_\nu$, we will consider cofinal systems of models $\mathcal{X}_{\mu_\nu} \to \mathcal{S}_{\mu_\nu} \to \mathcal{V}_{\mu_\nu}$ with several extra properties, e.g., ones resulting from de Jong's theory of alteration, etc.

First, by *general non-sense* about schemes of definition, it follows that for any given proper $\mathbb{Z}$-models $\mathcal{V}'$, $\mathcal{S}'$, $\mathcal{X}'$ of k_ν, l_ν, C_ν, respectively, there exist projective birational morphisms of $\mathbb{Z}$-schemes $\mathcal{V}'' \to \mathcal{V}'$, $\mathcal{S}'' \to \mathcal{S}'$, $\mathcal{X}'' \to \mathcal{X}'$, and projective morphisms $\mathcal{X}'' \to \mathcal{S}'' \to \mathcal{V}''$ with generic fiber the given $C_\nu \to \operatorname{Spec} l_\nu \to \operatorname{Spec} k_\nu$. By de Jong [dJ2], Theorem 2.4, especially (vii), b), it follows that there exists a projective alteration $\mathcal{S}''' \to \mathcal{S}''$ such that the base change $\mathcal{X}''' \to \mathcal{S}'''$ is a projective semi-stable family of curves. Further, letting $\mathcal{Z}' \subset \mathcal{X}'$ be the Zariski closure of $\Sigma_\nu \subset C_\nu(l_\nu)$, it follows that the preimage of $\mathcal{Z}'$ under the canonical projection $\mathcal{X}''' \to \mathcal{X}'$ is of the form $\mathcal{Z}_1''' \cup \mathcal{D}_1'''$, where $\mathcal{Z}_1''' \subset \mathcal{X}'''(\mathcal{S}''')$ is a finite set of disjoint sections with values in the smooth locus of $\mathcal{X}''' \to \mathcal{S}'''$, and $\mathcal{D}_1'''$ is the preimage of some divisor $\mathcal{D}''' \subset \mathcal{S}'''$. In particular, the generic fiber $\mathcal{D}_1''' \times_{\mathcal{S}} l_\nu$ of $\mathcal{D}_1'''$ is empty, and therefore, the generic fiber $\mathcal{Z}_1''' \times_{\mathcal{S}'''} l_\nu$ of $\mathcal{Z}_1'''$ contains Σ_ν. Finally, replacing $\mathcal{Z}_1'''$ by the closure of $\mathcal{Z}'''$ of Σ_ν in $\mathcal{X}'''$, we can suppose that Σ_ν equals the generic fiber of $\mathcal{Z}'''$. Moreover, by Raynaud–Gruson [R-G], pp. 36-37, after replacing $\mathcal{V}'$ by its normalization $\mathcal{V}''' \to \mathcal{V}'$ under $\mathcal{S}''' \to \mathcal{V}'$, there exists a blowup $\mathcal{V}^{\text{IV}} \to \mathcal{V}'''$ such that the proper transform $\mathcal{S}^{\text{IV}} \to \mathcal{V}'''$ of $\mathcal{S}''' \to \mathcal{V}'''$ is a projective flat morphism. Thus replacing $\mathcal{X}''' \to \mathcal{S}'''$ by the base change under $\mathcal{S}^{\text{IV}} \to \mathcal{S}'''$, and $\mathcal{Z}'''$ by the corresponding base change, we get that $\mathcal{S}^{\text{IV}} \to \mathcal{V}^{\text{IV}}$ is a projective flat morphism with geometrically integral fibers, $\mathcal{X}^{\text{IV}} \to \mathcal{S}^{\text{IV}}$ is a projective semi-stable curve, and $\mathcal{Z}^{\text{IV}} \subset \mathcal{X}^{\text{IV}}(\mathcal{S}^{\text{IV}})$ is a finite set of disjoint sections with support in the smooth locus of $\mathcal{X}^{\text{IV}} \to \mathcal{S}^{\text{IV}}$, and having Σ_ν as generic fiber. Finally, [dJ1], Theorem 5.8, is applicable to the projective semi-stable family of curves $\mathcal{X}^{\text{IV}} \to \mathcal{S}^{\text{IV}}$, and therefore, there exists an projective alteration $\mathcal{S}^{\text{V}} \to \mathcal{S}^{\text{V}}$, a projective split semi-stable family of curves $\mathcal{X}^{\text{V}} \to \mathcal{S}^{\text{V}}$ and a dominant birational morphism $\mathcal{X}^{\text{V}} \to \mathcal{X}^{\text{IV}} \times_{\mathcal{S}^{\text{IV}}} \mathcal{S}^{\text{V}}$, and a finite set of disjoint sections $\mathcal{Z}^{\text{V}} \subset \mathcal{X}^{\text{V}}(\mathcal{S}^{\text{V}})$ whose generic fiber is the given Σ_ν. (This is not explicitly stated in Theorem 5.8 of loc.cit., but sorting through the proof, one can easily see that this assertion is proved in the course of the proof.) Therefore, after replacing $\mathcal{V}'^{v}$ by its normalization under the field extension $\kappa(\mathcal{V}'^{v}) \hookrightarrow \kappa(\mathcal{S}^{v})$, and performing what was done above (including the corresponding base changes, etc.), we conclude:

3) There exist **cofinal families of fields/models** as follows:

   a) Finitely generated subfields $k_\nu \hookrightarrow l_\nu$ of $k \hookrightarrow \lambda$, i.e., $k = \cup_\nu k_\nu$ and $\lambda = \cup_\nu l_\nu$.
   b) Projective flat morphism of projective normal $\mathbb{Z}$-models $\mathcal{S}_{\mu_\nu} \to \mathcal{V}_{\mu_\nu}$ for $l_\nu \hookleftarrow k_\nu$ having geometrically integral fibers.
   c) Projective split semi-stable families of curves $\mathcal{X}_{\mu_\nu} \to \mathcal{S}_{\mu_\nu}$.
   d) A finite set of disjoint sections $\mathcal{Z}_{\mu_\nu} \subset \mathcal{X}_{\mu_\nu}(\mathcal{S}_{\mu_\nu})$ with support in the smooth locus of $\mathcal{X}_{\mu_\nu} \to \mathcal{S}_{\mu_\nu}$ and having generic fiber $\Sigma_\nu \subset C_\nu(l_\nu)$.

As a corollary of the discussion above, one has the following: Let v_Λ be an arbitrary valuation of Λ with $v_\Lambda|_k = v_k$, and set $v_\lambda := v_\Lambda|_\lambda$ i.e., $\mathcal{O}_{v_\Lambda}$ dominates $\mathcal{O}_{v_\lambda}$, $\mathcal{O}_{v_k}$. For every

$$\mathcal{X}_{\mu_\nu} \to \mathcal{S}_{\mu_\nu} \to \mathcal{V}_{\mu_\nu}$$

consider the centers $x_{\mu_\nu} \mapsto s_{\mu_\nu} \mapsto z_{\mu_\nu}$ of v_Λ on the models above (which exist by the valuation criterion for properness), and the canonical embeddings of their local rings and residue fields:

$$\mathcal{O}_{x_{\mu_\nu}}, \mathfrak{m}_{x_{\mu_\nu}} \hookleftarrow \mathcal{O}_{s_{\mu_\nu}}, \mathfrak{m}_{s_{\mu_\nu}} \hookleftarrow \mathcal{O}_{z_{\mu_\nu}}, \mathfrak{m}_{z_{\mu_\nu}}, \quad \kappa(x_{\mu_\nu}) \hookleftarrow \kappa(s_{\mu_\nu}) \hookleftarrow \kappa(z_{\mu_\nu}).$$

By the discussion at the beginning of the proof, and taking into account that the families $(\,\cdot\,)_{\mu_\nu}$ above are co-final, it follows that one has the following:

$$(*) \qquad \mathcal{O}_{\mathfrak{w}_\Lambda} = \cup_{\mu_\nu} \mathcal{O}_{x_{\mu_\nu}}, \quad \mathfrak{m}_{\mathfrak{w}_\Lambda} = \cup_{\mu_\nu} \mathfrak{m}_{x_{\mu_\nu}}, \quad \Lambda\mathfrak{w}_\Lambda = \cup_{\mu_\nu} \kappa(x_{\mu_\nu}),$$

and similarly for $\mathcal{O}_{v_k} \hookrightarrow \mathcal{O}_{w_\lambda} \hookrightarrow \mathcal{O}_{\mathfrak{w}_\Lambda}$, $\mathfrak{m}_{v_k} \hookrightarrow \mathfrak{m}_{w_\lambda} \hookrightarrow \mathfrak{m}_{\mathfrak{w}_\Lambda}$, and $kv_k \hookrightarrow \lambda w_\lambda \hookrightarrow \Lambda\mathfrak{w}_\Lambda$.

Step 2. *A transfer principle*

Recall that $\mathfrak{w}_\Lambda$ is a c.r. quasi prime divisor of $\Lambda|\lambda$ such that $\mathfrak{w}_\Lambda|_k = v_k$. In particular, $\mathfrak{w}_\Lambda = w_0 \circ w_\Lambda$, where w_Λ is a constant reduction of $\Lambda|\lambda$, and w_0 is a prime divisor of the residue function field $\Lambda w_\Lambda|\lambda w_\lambda$. In particular, the following hold:

a) $\mathfrak{w}_\Lambda|_k = w_\Lambda|_k = v_k$, thus $\mathcal{O}_{\mathfrak{w}_\Lambda}$, $\mathcal{O}_{w_\Lambda}$ both dominate $\mathcal{O}_{v_k}$.

b) One has a canonial exact sequence: $0 \to w_0(\Lambda w_\Lambda) = \mathbb{Z} \to \mathfrak{w}_\Lambda\Lambda \to w_\Lambda\Lambda = w_\lambda\lambda \to 0$.

c) $\mathcal{O}_{\mathfrak{w}_\Lambda}$ contains elements of minimal positive value π, and $\mathfrak{m}_{\mathfrak{w}_\Lambda} = \pi\mathcal{O}_{\mathfrak{w}_\Lambda}$ for any such π.

Further, if $\mathfrak{v}_\Lambda$ is a c.r. quasi prime divisor of $\Lambda|k$, then $\mathfrak{v}_\Lambda$ is definitely a c.r. quasi prime divisor of $\Lambda|\lambda$. But the converse of this assertion is true only if $\lambda|k$ has no proper constant reductions, which holds if and only if $\lambda = k$.

Proposition 3.3. (Transfer Principle). *In the above notations, suppose that $\mathfrak{w}_\Lambda$ is a* c.r. *prime divisor of $\Lambda|\lambda$ with $\mathfrak{m}_{\mathfrak{w}_\Lambda} = \pi\mathcal{O}_{\mathfrak{w}_\Lambda}$, and $u_1, \dots, u_n \in \mathcal{O}_{\mathfrak{w}_\Lambda}^\times$ be given. Then there exist* c.r. *quasi prime divisors $\mathfrak{v}_\Lambda$ of $\Lambda|k$ with $\mathfrak{v}_\Lambda|_k = \mathfrak{w}_\Lambda|_k$, $\mathfrak{m}_{\mathfrak{v}_\Lambda} = \pi\mathcal{O}_{\mathfrak{v}_\Lambda}$, and $u_1, \dots, u_n \in \mathcal{O}_{\mathfrak{v}_\Lambda}^\times \cdot k^\times$.*

Proof: In the discussion at Step 2 above, let C_λ be the projective smooth model of $\Lambda|\lambda$. Let $k_1 \hookrightarrow l_1$ and the projective smooth curve C_1 in Step 1 above be chosen such that $\pi, u_1, \dots, u_n \in l_1(C_1)$, and $\Sigma_1 \subset C_1(l_1)$ be the support of the divisors of the rational functions $\pi, u_1, \dots, u_n$. We also let $D_0 = \sum_i m_i P_i$, be the zero divisor of $\pi \in l_1(C_1)$, hence in particular, one has that $\{P_i\}_i \subset \Sigma_1 \subset C_1(l_1)$.

For models $\mathcal{X}_{\mu_\nu} \to \mathcal{S}_{\mu_\nu} \to \mathcal{V}_{\mu_\nu}$ satisfying conditions from item 3) in Step 2 above, and $\pi \in l_\nu(C_\nu)$, we consider the base changes

$$\mathcal{X} := \mathcal{X}_{\mu_\nu} \times_{\mathcal{V}_{\mu_\nu}} \mathcal{O}_{v_k} \to \mathcal{S}_{\mu_\nu} \times_{\mathcal{V}_{\mu_\nu}} \mathcal{O}_{v_k} =: \mathcal{S}, \quad \mathcal{Z} := \mathcal{Z}_{\mu_\nu} \times_{\mathcal{V}_{\mu_\nu}} \mathcal{O}_{v_k} \subset \mathcal{X}_{\mu_\nu}(\mathcal{S}_{\mu_\nu}) \times_{\mathcal{V}_{\mu_\nu}} \mathcal{O}_{v_k} \subset \mathcal{X}(\mathcal{S}).$$

1) Letting $l := l_\nu k$ be the compositum of l_ν and k, and recalling the projective smooth l_ν-curve C_ν, the support $\Sigma_\nu \subset C_\nu(l_\nu)$ of the divisors of the given rational functions $\pi, u_1, \dots, u_n \in l_1(C_1) \subset l_\nu(C_\nu)$, and the zero divisor $D_0 = \sum_i m_i P_i$ of π, one has:

   a) The generic fiber of $\mathcal{X} \to \mathcal{S}$ is nothing but $C_l := C_\nu \times_{l_\nu} l = C_1 \times_{l_1} l$.

   b) $\mathcal{X} \to \mathcal{S}$ is a projective split semi-stable family of curves (as base change of such).

2) Let $\mathcal{D}_0 = \{\sigma_{P_i}(\mathcal{S})\}_i$ be the closure of $\{P_i\}_i$ in $\mathcal{X}$. Then $\mathcal{D}_0 \subset \mathcal{Z} \subset \mathcal{X}(\mathcal{S})$ are sets of disjoint sections with support in the smooth locus of $\mathcal{X} \to \mathcal{S}$, and the following hold:

   a) The rational map $\mathcal{X} \dashrightarrow \mathbb{P}^1_{\mathcal{S}}$ defined by $\pi \in \kappa(\mathcal{X}) = l(C_l)$ is everywhere defined, and $\sum_i m_i \sigma_{P_i}(\mathcal{S})$ are the zero sections of π.

   b) Let $\mathcal{X}_\xi \to \xi$ the fiber of $\mathcal{X} \to \mathcal{S}$ at some $\xi \in \mathcal{S}$. The restriction of π to the fiber $\mathcal{X}_\xi$ is a rational function having $\mathcal{D}_0(\xi) := \sum_i m_i \sigma_{P_i}(\xi)$ as its zero divisor.

3) Since $\mathfrak{w}_\Lambda|_k = w_\Lambda|_k = v_k$, and $\mathfrak{w}_\Lambda|_\lambda = w_\Lambda|_\lambda = w_\lambda$, setting $\kappa := kv_k$, and letting $\mathcal{X}_\kappa \to \mathcal{S}_\kappa$ be the closed fiber of $\mathcal{X} \to \mathcal{S}$, the following hold:

   a) $w_\Lambda, \mathfrak{w}_\Lambda$ have the same center s on $\mathcal{S}$, and $s \in \mathcal{S}_\kappa$. Further, the centers $y \mapsto s$ of w_Λ, and $x \mapsto s$ of $\mathfrak{w}_\Lambda$, on $\mathcal{X} \to \mathcal{S}$ lie in the closed fiber $\mathcal{X}_\kappa \to \mathcal{S}_\kappa$.

   b) Since $\mathfrak{m}_{w_\Lambda} \subset \mathfrak{m}_{\mathfrak{w}_\Lambda}$, it follows that x lies in the closure of y, thus $y \in \mathrm{Spec}(\mathcal{O}_x)$, and if $\mathfrak{p}_y := \mathfrak{m}_w \cap \mathcal{O}_x$ is the prime ideal defining $y \in \mathrm{Spec}(\mathcal{O}_x)$, one has:

      i) $\mathcal{O}_y = (\mathcal{O}_x)_{\mathfrak{p}_y}$.

      ii) $\kappa(y) = \mathrm{Quot}(\mathcal{O}_x/\mathfrak{p}_y)$.

4) Since $\mathcal{O}_{\mathfrak{w}_\Lambda} = \cup_{\mu_\nu} \mathcal{O}_{x_{\mu_\nu}}$, $\Lambda\mathfrak{w}_\Lambda = \cup_{\mu_\nu} \kappa(x_{\mu_\nu})$, and $\mathcal{O}_{w_\Lambda} = \cup_{\mu_\nu} \mathcal{O}_{y_{\mu_\nu}}$, $\Lambda w_\Lambda = \cup_{\mu_\nu} \kappa(y_{\mu_\nu})$, for all sufficiently large models $\mathcal{X}_{\mu_\nu} \to \mathcal{S}_{\mu_\nu} \to \mathcal{V}_{\mu_\nu}$, the following hold:

   a) $L := l(C_l) = l_\nu(C_\nu)\, l = l_1(C_1)\, l$.

   b) Since w_Λ is a constant reduction of $L|l$, hence Lw_Λ is finitely generated over lw_Λ, one eventually has $Lw_\Lambda = \kappa(y_{\mu_\nu})\, lw_\Lambda = \kappa(y)$, hence $\Lambda w_\Lambda = Lw_\Lambda\, \lambda w_\lambda = \kappa(y)\lambda w_\lambda$.

   c) Hence since $Lw_\Lambda = \kappa(y) = \mathrm{Quot}(\mathcal{O}_x/\mathfrak{p}_y)$, one has that $\mathcal{O}_x/\mathfrak{p}_y = \mathcal{O}_{w_0}$.

d) Since $\mathrm{td}(Lv_\Lambda|kv_\Lambda) \leq \mathrm{td}(K|k) < \infty$, eventually $\mathrm{td}(L\mathfrak{w}_\Lambda|kv_k) = \mathrm{td}(\kappa(x)|\kappa)$.

e) $u_1, \dots, u_n \in \mathcal{O}_x^\times$ and $\pi \in \mathcal{O}_x$. Therefore, $\mathfrak{m}_{w_0} = \pi\,\mathcal{O}_{w_0} = (\pi, \mathfrak{p}_y)/\mathfrak{p}_y$.

Next, set $\overline{\mathcal{O}}_x := \mathcal{O}_x \otimes_{\mathcal{O}_{v_k}} \kappa = \mathcal{O}_x/\mathfrak{m}_{v_k}\mathcal{O}_x$, $\overline{\mathcal{O}}_s := \mathcal{O}_s \otimes_{\mathcal{O}_{v_k}} \kappa = \mathcal{O}_s/\mathfrak{m}_{v_k}\mathcal{O}_s$, and consider the base change $\mathcal{X}_{\overline{\mathcal{O}}_s} := \mathcal{X} \otimes_{\mathcal{S}} \overline{\mathcal{O}}_s$ of $\mathcal{X} \to \mathcal{S}$ under $\mathrm{Spec}\,\overline{\mathcal{O}}_s \hookrightarrow \mathcal{S}$. Then the closed fiber of $\mathcal{X}_{\overline{\mathcal{O}}_s}$, i.e., the fiber at the closed point $s \in \mathrm{Spec}\,\overline{\mathcal{O}}_s$, equals the special fiber $\mathcal{X}_s \to s$ of $\mathcal{X} \to \mathcal{S}$ at $s \in \mathcal{S}$. Let $\overline{\eta} \in \mathrm{Spec}\,\overline{\mathcal{O}}_s$ be a generic point, and consider the resulting commutative diagrams:

$$\begin{array}{ccccc} \mathcal{X}_{\overline{\mathcal{O}}_s} & \hookrightarrow & \mathcal{X}_\kappa & \hookrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}\,\overline{\mathcal{O}}_s & \hookrightarrow & \mathcal{S}_\kappa & \hookrightarrow & \mathcal{S} \end{array} \qquad \begin{array}{ccccc} \mathcal{X}_{\overline{\eta}} & \hookrightarrow & \mathcal{X}_{\overline{\mathcal{O}}_s} & \hookleftarrow & \mathcal{X}_s \\ \downarrow & & \downarrow & & \downarrow \\ \overline{\eta} & \hookrightarrow & \mathrm{Spec}\,\overline{\mathcal{O}}_s & \hookleftarrow & s \end{array}$$

We notice that all the vertical morphisms in the diagrams above are base changes of the projective split semi-stable family of curves $\mathcal{X} \to \mathcal{S}$, and therefore, they define projective split semi-stable families of curves over the corresponding bases. In particular, $\mathcal{X}_\kappa \to \mathcal{S}_\kappa$ is a projective flat morphism of projective connected pure dimensional κ-varieties, of dimensions $\dim(\mathcal{X}_\kappa) = \mathrm{td}(L|k)$ and $\dim(\mathcal{S}_\kappa) = \mathrm{td}(l|k)$, where $\mathrm{td}(l|k) = \mathrm{td}(L|k) - 1$. Further, $\overline{\mathcal{O}}_s \hookrightarrow \overline{\mathcal{O}}_x$ is the canonical inclusion of the local rings of the morphism of κ-varieties $\mathcal{X}_\kappa \to \mathcal{S}_\kappa$ at $x \mapsto s$.

Therefore we conclude that $\overline{\mathcal{O}}_x$ is catenary and a flat $\overline{\mathcal{O}}_s$-algebra, of Krull dimension given by $\dim(\overline{\mathcal{O}}_x) = \mathrm{td}(L|k) - \mathrm{td}(\kappa(x)|\kappa)$. Further, $\overline{\mathcal{O}}_x$ satisfies: First, it contains (at least) one generic point η_1 of $\mathcal{X}_s$ such that y, thus x, lie in the closure $\mathcal{X}_1 := \overline{\{\eta_1\}}$ of η_1; and notice that $\mathcal{X}_1 \hookrightarrow \mathcal{X}_s$ is then an irreducible component of the fiber $\mathcal{X}_s$ at $s \in \mathcal{S}$. Second, every generic point $\eta_\alpha \in \overline{\mathcal{O}}_x$ is actually a generic point of $\mathcal{X}_s$, thus the closure $\mathcal{X}_\alpha := \overline{\{\eta_\alpha\}}$ of η_α is an irreducible component of $\mathcal{X}_s$. And since $\eta_\alpha \in \mathrm{Spec}\,\overline{\mathcal{O}}_x$, it follows that x lies in the closure $\mathcal{X}_\alpha$ of η_α. Hence finally, $x \in \cap_{\eta_\alpha}\mathcal{X}_\alpha$, where $(\eta_\alpha)_\alpha$ is the set of all the generic points $\eta_\alpha \in \mathrm{Spec}\,\overline{\mathcal{O}}_x$. Third, recall that by item 4), b) and d) above, one has that $\mathcal{O}_x/\mathfrak{p}_x = \mathcal{O}_{w_0}$ is a discrete valuation ring having valuation ideal equal to $\pi\mathcal{O}_{w_0}$, i.e., π is a uniformizing parameter of $\mathcal{O}_{w_0}$. Therefore, the local ring $\mathcal{O}_{\mathcal{X}_s,x}$ of $x \in \mathcal{X}_s$ satisfies:

$$\mathcal{O}_{\mathcal{X}_s,x} = \overline{\mathcal{O}}_x/\overline{\mathfrak{p}}_y = \mathcal{O}_x/\mathfrak{p}_y = \mathcal{O}_{w_0}, \quad \mathfrak{m}_{\mathcal{X}_s,x} = \overline{\mathfrak{m}}_x/\overline{\mathfrak{p}}_y = \mathfrak{m}_x/\mathfrak{p}_x = \pi\,\mathcal{O}_{w_0},$$

thus concluding that $x \in \mathcal{X}_s$ is a zero of the rational function defined by π on the fiber $\mathcal{X}_s$. Hence by the discussion at item 2) above, it follows that $x = \sigma_{P_{i_0}}(s) \in \mathcal{X}_s$ for some P_{i_0}, and after renumbering $(P_i)_i$, we can suppose that $i_0 = 1$, i.e., $x = \sigma_{P_1}(s)$. On the other hand, by the construction/definition of $\sigma_{P_i}(\mathcal{S}) \in \mathcal{Z} \subset \mathcal{X}(\mathcal{S})$, it follows that $x = \sigma_{P_1}(s)$ lies in the smooth locus of $\mathcal{X} \to \mathcal{S}$, thus in the smooth locus of $\mathcal{X}_s \to \mathcal{S}_s$. Hence recalling the fact proved above, namely that $x \in \cap_{\eta_\alpha}\mathcal{X}_\alpha$, we have: First, since any two distinct irreducible components of $\mathcal{X}_s$ meet is a double point, thus a singular point of $\mathcal{X}_s$, and second, since $x = \sigma_{P_1}(s)$ lies in the smooth locus of $\mathcal{X} \to \mathcal{S}$, thus in the smooth locus of $\mathcal{X}_s \to s$, it follows that $\mathcal{X}_1 \hookrightarrow \mathcal{X}_s$ is the unique irreducible component of $\mathcal{X}_s$ containing $x = \sigma_{P_1}(s)$. Hence by the discussion above, $\mathrm{Spec}\,\overline{\mathcal{O}}_x$ has a unique generic point, which is η_1.

Equivalently, $\overline{\mathcal{O}}_x$ has a unique minimal prime ideal, which equals the nilpotent radical of $\overline{\mathcal{O}}_x$. On the other hand, $\overline{\mathcal{O}}_x$ is reduced (as being a local ring of the reduced κ-variety $\mathcal{X}_\kappa$), and therefore, $\overline{\mathcal{O}}_x$ is actually an integral domain. Therefore, $\overline{\mathcal{O}}_s \hookrightarrow \overline{\mathcal{O}}_x$ is an integral domain as well. Letting $\overline{\eta}$ be the generic point of $\operatorname{Spec} \overline{\mathcal{O}}_s$, it follows that the generic fiber $\mathcal{X}_{\overline{\eta}} \to \overline{\eta}$ is integral as well. Further, $\sigma_{P_1}(\overline{\eta})$ is a simple zero of π viewed as a rational function on $\mathcal{X}_{\overline{\eta}}$.

Now recalling the given functions $u_1, \ldots, u_n \in \mathcal{O}^{\times}_{\mathfrak{w}_\Lambda}$, and that $u_1, \ldots, u_n \in \mathcal{O}^{\times}_x$, we claim that $\sigma_{P_1}(\overline{\eta})$ is neither a zero nor a pole of any of the rational functions on $\mathcal{X}$ defined by any of the $u_1, \ldots, u_n$. Indeed, by contradiction, suppose that $\sigma_{P_1}(\overline{\eta})$ is a zero (or a pole) of some u_i. Since x lies in the closure of $\overline{\eta}$, it follows that $\sigma_{P_1}(s)$ lies in the closure of $\sigma_{P_1}(\overline{\eta})$. And since the later is a zero (or a pole) of the rational function defined by u_i on the generic fiber $\mathcal{X}_{\overline{\eta}}$, it follows that $x = \sigma_{P_1}(s)$ is a zero (or a pole) of the rational function defined by u_i on $\mathcal{X}_s$, thus contradicting the fact $u_i \in \mathcal{O}^{\times}_x$.

Finally, since π is a uniformizing parameter of $\mathcal{O}_{\mathcal{X}_s,x} = \mathcal{O}_{w_0}$, it follows that $x = \sigma_{P_1}(s)$ is a simple zero of π on $\mathcal{X}_s$. Therefore, by the discussion at item 2) above, it follows that the multiplicity m_1 of P_1 in $D_0 = \sum_i m_i P_i$ is $m_1 = 1$. Hence by the discussion at item 2) above, $x_{\overline{\eta}} := \sigma_{P_1}(\overline{\eta}) \in \mathcal{X}_{\overline{\eta}}$ is a simple zero of π on the generic fiber $\mathcal{X}_{\overline{\eta}} \to \overline{\eta}$ of $\mathcal{X}_{\overline{\mathcal{O}}_s}$. Further, $x_{\overline{\eta}}$ is neither a zero, nor a pole of any of the $u_1, \ldots, u_n$ on the fiber $\mathcal{X}_{\overline{\eta}}$.

Since $\operatorname{td}(l|k) = \dim(\mathcal{S}_\kappa) = \operatorname{td}\big(\kappa(\overline{\eta})|\kappa\big)$, it follows that every valuation v_l of $l|k$ which dominates $\mathcal{O}_{\mathcal{S},\overline{\eta}}$ must satisfy $\operatorname{td}(l|k) = \operatorname{td}(lv_l|kv_l)$, and therefore, v_l is a constant reduction of $l|k$ with $v_l|_k = v_k$, and further, $\kappa(v_l)|\kappa(\overline{\eta})$ is finite.[1]

For such a constant reduction v_l of $l|k$, consider the base change $\mathcal{X}_{v_l} := \mathcal{X} \times_{\mathcal{S}} \mathcal{O}_{v_l}$ defined by the canonical embedding $\operatorname{Spec} \mathcal{O}_{v_l} \to \mathcal{S}$. Since being a projective split semi-stable curve is invariant under base change, and $\mathcal{X} \to \mathcal{S}$ is such a family of curves, one has:

5) $\mathcal{X}_{v_l}$ is a projective split semi-stable curve over $\mathcal{O}_{v_l}$, having l-generic fiber the given projective smooth l-curve C_l. Further the following hold:
   a) The special fiber $\mathcal{X}_{v_l,s} := \mathcal{X}_{v_l} \times_{\mathcal{O}_{v_l}} \kappa(v_l)$ is the base change $\mathcal{X}_{v_l,s} = \mathcal{X}_{\overline{\eta}} \times_{\kappa(\overline{\eta})} \kappa(v_l)$ of the projective geometrically integral smooth $\kappa(\overline{\eta})$-curve $\mathcal{X}_{\overline{\eta}}$ under $\kappa(\overline{\eta}) \hookrightarrow \kappa(v_l)$.
   - Thus $\mathcal{X}_{v_l,s}$ is a projective geometrically integral smooth $\kappa(v_l)$-curve.

   b) $x_{v_l,s} := \sigma_{P_1}(\mathfrak{m}_{v_l})$ is the base change of $x_{\overline{\eta}} = \sigma_{P_1}(\overline{\eta})$ under $\kappa(\overline{\eta}) \hookrightarrow \kappa(v_l)$, thus it is a simple $\kappa(v_l)$-rational zero of π on the special fiber $\mathcal{X}_{v_l,s}$ of $\mathcal{X}_{v_l}$.
   c) Let $\mathcal{X}_{1,v_l} \hookrightarrow \mathcal{X}_{v_l,s}$ be the unique irreducible component containing $x_{v_l,s}$, and η_{1,v_l} be its generic point. Then the local ring $\mathcal{O}_{\eta_{1,v_l}}$ is dominated by the local ring of a unique constant reduction v_L of the function field $L|l$ such that $v_L|_l = v_l$, thus one also has $v_L|_k = v_k$.
   d) Since v_l is a constant reduction of $l|k$, and k is algebraically closed, one has that $v_l l = v_k k$. Further, since v_L is a constant reduction of $L|l$, it follows by the Abhyankar (in)equality that $v_L L/v_l l$ is a torsion group. Hence since $v_l l = v_k k$ is divisible, it follows that $v_L L = v_l l = v_k k$. Hence for each i, there exist elements $a_i \in k^{\times}$ such that $u_i/a_i \in \mathcal{O}^{\times}_{v_L}$, and further, $x_{v_l,s}$ is neither a zero, nor a pole, of any of the functions u_i/a_i viewed as rational functions on the fiber $\mathcal{X}_{v_l,s}$.

One concludes the poof of the Transfer Principle as follows: First, $lv_l = \kappa(v_l)$ is the residue field the constant reduction defined by v_l on l, and Lv_L is the residue function field of the constant

[1] Actually, $\kappa(\overline{\eta}) = \kappa(v_l)$, for $\mathcal{S}$ "sufficiently large," but we will not need this fact.

reduction v_L of $L|l$. Therefore, the relative algebraic closure $l_L \subset Lv_L$ of lv_l in Lv_L is finite over lv_l, and $Lv_L|l_L$ is the function field of the projective smooth curve l_L-curve $X_{l_L} := \mathcal{X}_{v_l,s} \times_{lv_l} l_L$. Second, the lv_l-rational point $x_{1,v_l} \in \mathcal{X}_{1,v_l}$ gives rise by base change to an l_L-rational point $x_{l_L} \in \mathcal{X}_{l_L}$.

6) Let v_0 be the prime divisor of $Lv_L|l_L$ defined by x_{l_L}, and set $\mathfrak{v}_L := v_0 \circ v_L$. Recalling the quasi prime $\mathfrak{w}_\Lambda$-divisor from Theorem 3.2, the following hold:

   a) One has $\mathfrak{v}_L|_l = v_L|_l = v_l$ and $\mathfrak{v}_L|_k = v_L|_k = v_k$. Hence since $\mathfrak{w}_\Lambda|_k = v_k$, one finally gets $\mathfrak{v}_L|_k = \mathfrak{w}_\Lambda|_k$.

   b) Since v_L is a constant reduction of $L|k$, and its restriction to l is the constant reduction v_l of $l|k$, it follows that v_L is a constant reduction of $K|k$. Hence $\mathfrak{v}_L$ is a c.r. quasi prime divisor of $K|k$.

   c) π is a uniformizing parameter of the discrete valuation ring $\mathcal{O}_{v_0}$. Hence $\mathfrak{v}_L(\pi)$ is the minimal positive element of $\mathfrak{v}_L L$, and $\mathfrak{m}_{\mathfrak{v}_L} = \pi\mathcal{O}_{\mathfrak{v}_L}$.

   d) The elements $u'_1 := u_1/a_1, \dots, u'_n := u_n/a_n$ are v_L-units, and their v_L-residues $\overline{u}'_1, \dots, \overline{u}'_n$ in Lv_L are v_0-units. Hence $u'_1, \dots, u'_n \in \mathcal{O}^\times_{\mathfrak{v}_L}$, thus $u_1, \dots, u_n \in \mathcal{O}^\times_{\mathfrak{v}_L} \cdot k^\times$.

We now conclude the proof of the Transfer Principle by letting $\mathfrak{v}_\Lambda$ be any prolongation of $\mathfrak{v}_L$ to the compositum $\Lambda = L\lambda$ defined by the canonical inclusions $l \hookrightarrow l_L \hookrightarrow \lambda$. □

Step 3. *Concluding the proof of Theorem 3.2*

In the notations from the beginning of the proof of Theorem 3.2, recall that $\overline{\sigma} \in \overline{\Pi}$ was the image of some $\overline{\sigma}_{\mathfrak{w}_\Lambda} \in \overline{T}^1_{\mathfrak{w}_\Lambda}$ under the canonical inclusion $\overline{T}^1_{\mathfrak{w}_\Lambda} \to \overline{\Pi}$, where $\mathfrak{w}_\Lambda$ is a c.r. quasi prime divisor of $\Lambda|\lambda$, etc. In particular, if $\pi \in \mathcal{O}_{\mathfrak{w}_\Lambda}$ is such that $\mathfrak{m}_{\mathfrak{w}_\Lambda} = \pi\mathcal{O}_{\mathfrak{w}_\Lambda}$, then one has that $\mathfrak{w}_\Lambda\Lambda/n = \mathfrak{w}_\Lambda(\pi)\mathbb{Z}/n$ is Pontrjagin dual to $\overline{T}^1_{\mathfrak{w}_\Lambda}$, thus $\overline{\sigma}_{\mathfrak{w}_\Lambda} : A \to \mathbb{Z}/n(1)$ factors through $A \to \mathfrak{w}_\Lambda\Lambda/n = \mathfrak{w}_\Lambda(\pi)\mathbb{Z}/n$, and we can write $A = u_0^{\mathbb{Z}} \cdot B$, with $B = \ker(\overline{\sigma}_{\mathfrak{w}_\Lambda})$ and $u_0 \in K^\times$ such that $\overline{\sigma}_{\mathfrak{w}_\Lambda}(u_0) = \overline{\sigma}_{\mathfrak{w}_\Lambda}(\pi)$. Moreover, without loss of generality, we can suppose that $\overline{\sigma}_{\mathfrak{w}_\Lambda}(\pi) \in \mathbb{Z}/n(1)$ is a generator of $\overline{T}^1_{\mathfrak{w}_\Lambda}$.

Let $u_1, \dots, u_n \in \mathcal{O}_{\mathfrak{w}_\Lambda} \cap K^\times$ be generators of B (when viewed as a subgroup of $K^\times/n$). By the transfer principle, there exists a c.r. quasi prime divisor $\mathfrak{v}_\Lambda$ such that $\mathfrak{m}_{\mathfrak{v}_\Lambda} = \pi\mathcal{O}_{\mathfrak{v}_\Lambda}$, hence $\mathfrak{v}_\Lambda\Lambda/n = \mathfrak{v}_\Lambda(\pi)\mathbb{Z}/n$, and $u_1, \dots, u_n \in \mathcal{O}^\times_{\mathfrak{v}_\Lambda} \cdot k^\times$. But then it follows that $\overline{\sigma}_{\mathfrak{w}_\Lambda} : A \to \mathbb{Z}/n(1)$ factors through $A \to \mathfrak{v}_\Lambda\Lambda/n$, i.e., there exists $\overline{\sigma}_{\mathfrak{v}_\Lambda} \in \overline{T}^1_{\mathfrak{v}_\Lambda}$ such that $\overline{\sigma}_{\mathfrak{w}_\Lambda} = \overline{\sigma}_{\mathfrak{v}_\Lambda}$ inside $\overline{\Pi}_\Lambda$. Finally, let $\mathfrak{v} := \mathfrak{v}_\Lambda|_K$ be the resulting c.r. quasi prime divisor of $K|k$, and recall that $\overline{\sigma}_{\mathfrak{v}_\Lambda} = \overline{\sigma}_{\mathfrak{w}_\Lambda}$ on the image $A_\Lambda \subset \Lambda^\times/n$. Hence letting $\imath : A \to A_\Lambda$ be the canonical map induced by $K^\times \hookrightarrow \Lambda^\times$, we have the following situation:

a) $\overline{\sigma}$ is the image of $\overline{\sigma}_{\mathfrak{w}_\Lambda} \in \overline{T}^1_{\mathfrak{w}_\Lambda}$ under $\overline{\Pi}_\Lambda \to \overline{\Pi}$, i.e., $\overline{\sigma} = \overline{\sigma}_{\mathfrak{w}_\Lambda} \circ \imath$ on $A \subset K^\times/n$.

b) $\overline{\sigma}_{\mathfrak{w}_\Lambda} = \overline{\sigma}_{\mathfrak{v}_\Lambda}$ as elements of $\overline{\Pi}_\Lambda$, i.e., as maps on $A_\Lambda = \imath(A)$.

c) Let $\overline{\sigma}_{\mathfrak{v}} \in \overline{T}^1_{\mathfrak{v}}$ be the image of $\overline{\sigma}_{\mathfrak{v}_\Lambda}$ under the canonical injective projection $\overline{T}^1_{\mathfrak{v}_\Lambda} \to \overline{T}^1_{\mathfrak{v}}$. Then $\overline{\sigma}_{\mathfrak{v}} = \overline{\sigma}_{\mathfrak{v}_\Lambda} \circ \imath$, and therefore: $\overline{\sigma}_{\mathfrak{v}} = \overline{\sigma}_{\mathfrak{v}_\Lambda} \circ \imath = \overline{\sigma}_{\mathfrak{w}_\Lambda} \circ \imath = \overline{\sigma}$ as maps on $A \subset K^\times/n$.

We therefore proved that $\overline{\sigma} \in \overline{\Pi}$ is the image of $\overline{\sigma}_{\mathfrak{v}} \in \overline{T}^1_{\mathfrak{v}}$ under the canonical injective projection $\overline{T}^1_{\mathfrak{v}} \hookrightarrow \overline{\Pi}_\Lambda \to \overline{\Pi}$. This concludes the proof of Theorem 3.2. □

B) *Proof of Theorem 1.2*

Recalling the discussion at the beginning of this section and using the ideas developed in the proof of Theorem 1.1, the proof of Theorem 1.2 is reduced to proving the following: Let $\mathfrak{w} = w_0 \circ w$ be as in Theorem 1.2. Then for every $n = \ell^e$, a finite subgroup $A \subset K^\times/n$, and the corresponding $\Pi_K \to \overline{\Pi} := \mathrm{Gal}\big(K_A\,|\,K\big)$, the following holds:

(!) *For every* $\overline{\sigma} \in \overline{T}^1_{\mathfrak{w}}$ *there exists some* $\mathfrak{v} \in \mathcal{Q}_{v_k}(K|k)$ *such that* $\overline{\sigma} \in \overline{T}^1_w \cdot \overline{T}^1_{\mathfrak{v}}$.

We notice that it is sufficient to prove assertion (!) for any "sufficiently large" finite subgroup $A \subset K^\times/n$, which means that if the assertion (!) holds for some finite subgroup $A' \subset K^\times/n$ with $A \subset A'$, then the assertion (!) holds for A as well. We call this the enlargement principle.

Recall that for an arbitrary valuation v of K, denoting by $U_v := \mathcal{O}_v^\times$ the group of v-units, one has a canonical exact sequence $1 \to U_v \to K^\times \to vK \to 0$. Hence for every $n = \ell^e$, one gets canonically the exact sequence $1 \to U_v/n \to K^\times/n \to vK/n \to 0$, because vK is torsion free. By Kummer theory, $T^1_v/n \to \Pi_K/n$ is Pontrjagin dual to $K^\times/n \to vK/n$, thus every element $\overline{\sigma} \in \overline{T}^1_v$, viewed as a character $\overline{\sigma} : A \to \mathbb{Z}/n(1)$, factors through $A \to vK/n$. We also notice that by the Abhyankar (in)equality, the rational rank of vK/vk if bounded by $\mathrm{td}(K|k)$. Hence, taking into account that $vk \subset vK$ is divisible (because k is algebraically closed), it follows that $vK/n = (vK/vk)/n$ is a free $\mathbb{Z}/n$-module of rank bounded by $\mathrm{td}(K|k)$. In particular, by the enlargement principle above, we can suppose that $A \to vK/n$ is surjective.

Recall that $\mathfrak{w} = w_0 \circ w$, where w_0 is a quasi prime divisor of the function field $Kw|k_1w$. One has canonical exact sequences of the form

$$0 \to w_0(Kw) \cong \mathbb{Z} \to \mathfrak{w}K \to wK \to 0, \quad 0 \to w_0(Kw)/n \cong \mathbb{Z}/n \to \mathfrak{w}K/n \to wK/n \to 0,$$

the latter exact sequence being Pontrjagin dual to $1 \to T^1_w/n \to T^1_{\mathfrak{w}}/n \to T^1_{w_0}/n \to 1$, where, by abuse of language/notation, $T^1_{w_0}/n$ is the dual of $w_0(Kw)/n$. Further, by the enlargement principle above, without loss of generality we can suppose that $A \to \mathfrak{w}K/n$ is surjective, thus $A \to wK/n$ is surjective as well, etc., and $\overline{T}^1_{\mathfrak{w}} = T^1_{\mathfrak{w}}/n$, $\overline{T}^1_w = T^1_w/n$, and $\overline{T}^1_{w_0} = T^1_{w_0}/n$.

Next, interpreting $\overline{\sigma}_{\mathfrak{w}} \in \overline{T}^1_{\mathfrak{w}}$ as a character $\overline{\sigma}_{\mathfrak{w}} : K/n \to \mathbb{Z}/n(1)$, let us consider the restriction $\overline{\sigma}_0$ of $\overline{\sigma}_{\mathfrak{w}}$ to the image of $w_0(Kw)/n \hookrightarrow \mathfrak{w}K/n$. Then $\overline{\sigma}_w := \overline{\sigma}_{\mathfrak{w}}\overline{\sigma}_0^{-1}$ is trivial on $w_0(Kw)/n$, thus it factors though wK/n. Hence $\overline{\sigma}_w$ lies in the image of $\overline{T}^1_w$ under the canonical inclusion $\overline{T}^1_w \hookrightarrow \overline{T}^1_{\mathfrak{w}}$. Finally, the given element $\overline{\sigma}_{\mathfrak{w}}$ satisfies $\overline{\sigma}_{\mathfrak{w}} = \overline{\sigma}_0\,\overline{\sigma}_w$.

Hence we conclude that in order to prove assertion (!) for $\overline{\sigma}_{\mathfrak{w}}$, it is sufficient to prove that there exists some $\mathfrak{v} \in \mathcal{Q}_{v_k}(K|k)$ such that $\overline{\sigma}_0 \in \overline{T}^1_{\mathfrak{v}}$. For this we employ Theorem 3.2.

Let $\pi \in K$ be a fixed w-unit such that its residue $\overline{\pi} \in Kw$ is a uniformizing parameter of $\mathcal{O}_{w_0}$ (which is a DVR of Kw). Since k_1w is algebraically closed, $Kw|k_1w$ has (separable) transcendence bases of the form $t_1 = \pi, \ldots, t_r$. Let $l_1 \subset K$ be the relative algebraic closure of $k_1\big((t_i)_{1<i\leq r}\big)$ in K. Then $l_1(\pi) \hookrightarrow K$ is a finite separable extension, and further one has:

a) $\mathrm{td}(K|l_1) = 1 = \mathrm{td}(Kw|l_1w)$, hence w is a constant reduction of $K|l_1$.

b) w_0 is a quasi prime divisor of the function field $Kw|l_1w$ such that $\mathfrak{w} = w_0 \circ w$ and $K\mathfrak{w}$ is separable over l_1w.

c) π is a w-unit, and its residue in Kw is a uniformizing parameter of $\mathcal{O}_{w_0}$.

d) $\overline{\sigma}_0 \in \overline{T}^1_{\mathfrak{w}}$ viewed as a character $\overline{\sigma}_0 : \mathfrak{w}K/n \to \mathbb{Z}/n(1)$ factors through the canonical embeddding $\mathfrak{w}(\pi)\,\mathbb{Z}/n \hookrightarrow \mathfrak{w}K/n$.

Let λ be the algebraic closure and $\Lambda := K\lambda$ in some fixed algebraic closure $\overline{K}$ of K, and $\mathfrak{w}_\Lambda$ be some prolongation of $\mathfrak{w}$ to Λ. Then we are in context of Theorem 3.2. Hence there exists a c.r. quasi prime divisor $\mathfrak{v}$ of $K|k$ such that $\overline{\sigma}_0 \in \overline{T}^1_{\mathfrak{v}}$.

This concludes the proof of Theorem 1.2.

4 Application: The nature of the residue field

In this section, we keep notations as introduced in the Introduction. We consider a fixed valuation w of K having the property:

The value group wK has no ℓ-divisible convex subgroups, and K has a subfield $\mathsf{k}_1 \subset K$ satisfying: $\mathsf{k}_1 w = kw$ and $\mathrm{td}(K|\mathsf{k}_1) = 1 = \mathrm{td}(Kw|\mathsf{k}_1 w)$.

We notice that all the generalized quasi prime r divisors w of $K|k$ with $r = \mathrm{td}(K|k) - 1$ have the properties asked for above.

To simplify notations, we set $K_0 := Kw$, $kw =: k_0 := \mathsf{k}_1 w$, and notice that w is a constant reduction of the function field in one variable $K|\mathsf{k}_1$, thus $K_0|k_0$ is a function field in one

The problem we address now is about giving a recipe for deciding whether k_0 is an algebraic closure of a finite field, and further, given an algebraically closed subfield $l \subseteq k$, to decide whether the residue fields $l_0 \hookrightarrow k_0$ are actually equal. Moreover, that recipe should involve solely Π_K endowed with the given $T^1_w \subset Z^1_w$ and the family of minimized quasi divisorial subgroups $T^1_{\mathfrak{v}} \subset Z^1_{\mathfrak{v}}$ (which is provided to us by the local theory).

In order to announce the result answering the above question, we need a short preparation as follows. First, let $\mathcal{Q}(K|k)$ be the set of all the quasi prime divisors of $K|k$, and denote by $\mathcal{T}^1(K) \subset \Pi_K$ the topological closure of $\cup_{\mathfrak{v}\in\mathcal{Q}(K|k)} T^1_{\mathfrak{v}}$ in Π_K. Second, for $l \subseteq k$ and $l_0 \subseteq k_0$ as above, let $\mathcal{Q}_l(K|k)$ be the set of all the quasi prime divisors $\mathfrak{v}$ with $\mathfrak{v}|_l = w|_l$, and $\mathcal{T}^1_l(K) \subset \Pi_K$ be the topological closure of $\cup_{\mathfrak{v}\in\mathcal{Q}_l(K|k)} T^1_{\mathfrak{v}}$. Then $\mathcal{T}^1_l(K) \subseteq \mathcal{T}^1(K)$, and these sets consist of minimized inertia elements by Theorem 1.1, and $\mathcal{T}^1_l(K)$ consists of minimized inertia elements at valuations v with $v|_l = w|_l$. Recalling the canonical exact sequence

$$1 \to T^1_w \to Z^1_w \to Z^1_w/T^1_w =: \Pi^1_{K_0} \to 1,$$

let $\mathcal{T}_l(K_0) \subseteq \mathcal{T}^1(K_0)$ be the images of $\mathcal{T}^1_l(K) \cap Z^1_w \subseteq \mathcal{T}^1(K) \cap Z^1_w$ under $Z^1_w \to \Pi^1_{K_0}$.

By abuse of language, we call $\Pi^1_{K_0}$ the minimized residual group at w. Further, for any $v \geq w$, one has $T^1_w \hookrightarrow T^1_v \hookrightarrow Z^1_v \hookrightarrow Z^1_w$ canonically. Considering $v_0 := v/w$ on K_0, we set

$T^1_{v_0} := T^1_v/T^1_w \hookrightarrow Z^1_v/T^1_w =: Z^1_{v_0}$, and by abuse of language, we say that $T^1_{v_0} \hookrightarrow Z^1_{v_0} \hookrightarrow \Pi^1_{K_0}$ are the minimized residual inertia/decomposition groups at v_0.

Remark 4.1. As mentioned already in the Introduction, if $\mathrm{char}(K_0) = \ell$, then $T^1_v \subseteq T^1_v$ are contained in the wild ramification group of v, thus in the usual inertia group T_v. Therefore, $\mathrm{char}(K_0) = \ell$ implies that the residue group $\Pi^1_{K_0}$ is <u>not</u> a Galois group over K_0, and in particular, $T^1_{v_0} \hookrightarrow Z^1_{v_0}$ are not a true inertia and/or decomposition groups. In order to highlight this disparity, Topaz prefers to denote the minimized inertia/decomposition groups by $\mathrm{I}_v \subset \mathrm{D}_v$, see [To1], and Appendix below. In particular, this distinction becomes as imperative for $T^1_{v_0} \subseteq Z^1_{v_0}$, which would be denoted $\mathrm{I}_{v_0} \subset \mathrm{D}_{v_0}$, etc. The groups $T^1_{v_0} \subset Z^1_{v_0} \subset \Pi^1_{K_0}$ have, nevertheless, the following interpretation via Kummer theory: First, $T^1_w \hookrightarrow Z^1_w \to \Pi^1_{K_0}$ and $K_0^\times = U_w/U^1_w \hookrightarrow K^\times/U^1_w \to K/U_w$ are ℓ-adically dual to each other, and so are: $T^1_w \hookrightarrow T^1_v$ and $vK = K^\times/U_v \to K^\times/U_w = wK$, respectively, $T^1_w \hookrightarrow Z^1_v$ and $K^\times/U^1_v \to K^\times/U_w$. Hence the following are in ℓ-adic duality:

a) $T^1_{v_0} = T^1_v/T^1_w \hookrightarrow \Pi^1_{K_0}$ and $U_w/U^1_w \to U_w/U_v = K_0^\times/U_{v_0}$.

b) $Z^1_{v_0} = Z^1_v/T^1_w \hookrightarrow \Pi^1_{K_0}$ and $U_w/U^1_w \to U_w/U^1_v = K_0^\times/U^1_{v_0}$.

Finally, recall that a pro-ℓ abelian group G endowed with a system of procyclic subgroups $(T_\alpha)_\alpha$ is called complete curve like, if there exists a system of generators $(\tau_\alpha)_\alpha$ with $\tau_\alpha \in T_\alpha$ such that letting $T \subseteq G$ be the closed subgroup of G generated by $(\tau_\alpha)_\alpha$, the following hold:

i) $\prod_\alpha \tau_\alpha = 1$ and this is the only profinite relation satisfied by $(\tau_\alpha)_\alpha$.[2]

ii) The quotient G/T is a finite $\mathbb{Z}_\ell$-module.

The following fact was mentioned in Pop [P4] in the tame case, i.e., if $\mathrm{char}(K_0) \neq \ell$, and aspects of the question were revisited by Topaz [To2] in general, see the Appendix.

Theorem 4.2. *In the above notations, let $(T_\alpha)_\alpha$ be a maximal system of distinct maximal cyclic subgroups of $\Pi^1_{K_0}$ satisfying one of the following conditions:*

i) $T_\alpha \subset \mathcal{T}^1(K_0)$ for each α.

ii) $T_\alpha \subset \mathcal{T}^1_l(K_0)$ for each α.

Then $\Pi^1_{K_0}$ endowed with $(T_\alpha)_\alpha$ is complete curve like if and only if

a) k_0 is an algebraic closure of a finite field, provided i) *is satisfied.*

b) $l_0 = k_0$, provided ii) *is satisfied.*

Moreover, if so, then $T_\alpha = T^1_{v_\alpha}$ is the set of the minimized inertia groups in $\Pi^1_{K_0}$ at all the prime divisors $(v_\alpha)_\alpha$ of the function field in one variable $K_0|k_0$.

[2] This implies by definition that $\tau_\alpha \to 1$ in G, thus every open subgroup of G contains almost all T_α.

Proof: The proof is based on a few lemmas as follows:

Lemma 4.3. *In the above notations, let $\tau \in Z^1_w$ be any minimized inertia element having a non-trivial image $\tau_0 \in \Pi^1_{K_0}$ under $Z^1_w \to \Pi^1_{K_0}$. Then there exists a unique quasi prime divisor v_0 of $K_0|k_0$ such that setting $v_\tau = v_0 \circ w$ one has $\tau_0 \in T^1_{v_0} = T^1_{v_\tau}/T^1_w$.*

Proof: First, since τ is a minimized inertia element, there exist valuations v of K such that $\tau \in T^1_v$. For such a valuation v, consider the minimal coarsening $v_\tau \leq v$ such that τ is in the image of the canonical embedding $T^1_{v_\tau} \to T^1_v$. In other words, after identifying $T^1_{v_\tau} \to T^1_v$ with the ℓ-adic dual of $vK \to v_\tau K$, and viewing $\tau : vK \to \mathbb{Z}(1)$ as a character, the valuation v_τ is the minimal one such that $\ker(vK \to v_\tau K) \subset \ker(\tau)$. Then by the general theory of (minimized) core of valuations, see e.g., [P1], and [To1], it follows that $\tau \in T^1_{v_\tau}$, and v_τ depends on τ only, and not on v. Further, since $\tau \in Z^1_w$, and wK admits no divisible convex subgroups, it follows that $w \leq v_\tau$. Finally, $w < v_\tau$, because one has, first, $T^1_w \subseteq T^1_{v_\tau}$, second, $\tau_0 \in T^1_{v_\tau}/T^1_w$ is non-trivial. Further, by the same strategy, it follows that v_τ depends on τ only, and not on the valuation v we started with. Finally, setting $v_0 := v_\tau/w$, it follows that v_0 is a valuation of K_0 such that $v_0K_0 = \ker(v_\tau K \to wK)$ is not ℓ-divisible. On the other hand, since k_0 is algebraically closed, thus v_0k_0 is divisible, and $\mathrm{td}(K_0|k_0) = 1$, the Abhyankar (in)equality implies that v_0K_0/v_0k_0 is either trivial, or isomorphic to $\mathbb{Z}$. Thus since v_0K_0 is not ℓ-divisible, we conclude that $v_0K_0 \cong \mathbb{Z}$. Finally, the minimality of v_α with the property that $v_\alpha K$ has no ℓ-divisible convex subgroups is equivalent to the minimality of v_0 with v_0K_0/ℓ being non-trivial, thus v_0K_0 satisfying $v_0K_0 \cong \mathbb{Z}$. We thus conclude that v_0 is a quasi prime divisor of the function in one variable $K_0|k_0$. □

Lemma 4.4. *In the notations from the previous Lemma, $T^1_{v_0} \subset \Pi^1_{K_0}$ is a maximal procyclic subgroup of $\Pi^1_{K_0}$. Further, if $\tau, \sigma \in Z^1_w$ are minimized inertia elements having non-trivial images τ_0, σ_0 in $\Pi^1_{K_0}$, and $v_\tau = v_{0\tau} \circ w$, $v_\sigma = v_{0\sigma} \circ w$ are the corresponding valuations, then $T^1_{v_{0\tau}} \cap T^1_{v_{0\sigma}}$ is non-trivial if and only if $v_{0\tau} = v_{0\sigma}$, thus $T^1_{v_\tau} = T^1_{v_\sigma}$.*

Proof: By the Remark above, $T^1_{v_0} \hookrightarrow \Pi^1_{K_0}$ and $U_w/U^1_w \xrightarrow{f} K_0^\times/U_{v_{0\tau}} = U_w/U_{v_\tau}$ are in ℓ-adic duality. Hence the fact that $T^1_{v_{0\tau}}$ is a maximal procyclic subgroup of $\Pi^1_{K_0}$ is equivalent to the fact that $(U_w/U^1_w)/\ker(f)$ has no ℓ-torsion. On the other hand, $\ker(f) = U_{v_\tau}/U^1_w$, hence $(U_w/U^1_w)/\ker(f) = U_w/U_{v_\tau} \hookrightarrow K^\times/U_{v_\tau} = v_\tau K$ has no torsion.

For the second assertion of the Lemma, suppose that $v_{0\tau} \neq v_{0\sigma}$ of K_0. Then since quasi prime divisors are not comparable as valuations, or equivalently, the valuation rings $\mathcal{O}_{v_{0\sigma}}$, $\mathcal{O}_{v_{0\tau}}$ are not comparable w.r.t. inclusion, it follows that $\mathcal{O} := \mathcal{O}_{v_{0\sigma}} \cdot \mathcal{O}_{v_{0\tau}}$ stricly contains both rings. Further, $\mathcal{O}$ is the valuation ring $\mathcal{O} = \mathcal{O}_{v_0}$ of the maximal valuation v_0 with $v_0 \leq v_{0\tau}, v_{0\sigma}$, and one also has $U_{v_0} = U_{v_{0\tau}} \cdot U_{v_{0\sigma}}$. Hence, if $\theta \in T^1_{v_{0\tau}} \cap T^1_{v_{0\sigma}}$, then θ is trivial on $U_{v_{0\tau}}$ (because so are all elements of $T^1_{v_{0\tau}}$), and trivial on $U_{v_{0\sigma}}$ (because so are all elements of $T^1_{v_{0\sigma}}$). Hence θ is trivial on $U_{v_0} = U_{v_{0\tau}} \cdot U_{v_{0\sigma}}$, thus factors through $K_0^\times/U_{v_0}$. On the other hand, since $v_0 < v_{0\tau}, v_{0\sigma}$, the value group of v_0 is a divisible group. Thus θ is trivial. □

We conclude that the set of minimized inertia $\mathfrak{In}^1(K) = \cup_{v \in \mathrm{Val}(K)} T^1_v \subset \Pi_K$, which is closed in Π_K by Theorem 1.1, satisfies: The image of $\mathfrak{In}^1(K) \cap Z^1_w$ under $Z^1_w \to \Pi^1_{K_0}$ is actually the set $\cup_{v_0\in\mathcal{Q}(K_0|k_0)}T^1_{v_0}$ of all the quasi divisorial minimized inertia elements in $\Pi^1_{K_0}$. Moreover, for distinct quasi prime divisors $v_\alpha, v_\beta \in \mathcal{Q}(K_0|k_0)$, it follows that $T^1_{v_\alpha} \cap T^1_{v_\beta} = 1$.

1) Therefore we have: Let $(T_\alpha)_\alpha$ be a system of distinct maximal procyclic subgroups of $\Pi^1_{K_0}$ with $T_\alpha \subset \mathfrak{In}^1(K_0)$. Then there exist quasi prime divisors v_α of $K_0|k_0$ such that $T_\alpha = T^1_{v_\alpha}$. In particular, $T_\alpha \cap T_\beta = 1$ for $\alpha \neq \beta$.

Next let $\mathfrak{w}$ be a c.r. quasi prime w-divisor of $K|k$, i.e., $\mathfrak{w}$ is of the form $\mathfrak{w} = w_0 \circ w$, where w_0 is a c.r. quasi prime divisor of $K_0|k_0$. Then by Theorem 1.2, it follows that $T^1_{\mathfrak{w}} \subset T^1_w \cdot \mathcal{T}^1(K)$. Therefore, $T^1_{w_0} \subset \mathcal{T}^1(K_0)$ by the definition of $\mathcal{T}^1(K_0)$. Further, if $\mathfrak{w}|_l = w|_l$, then $T^1_{w_0} \subset \mathcal{T}^1_l(K)$. Hence using the assertion 1) above, we conclude:

2) The system of minimized inertia groups $(T_{v_\alpha})_\alpha$ of all the c.r. quasi prime divisors v_α of $K_0|k_0$ is contained in $\mathcal{T}^1(K_0)$.

3) The system of minimized inertia groups $(T_{v_\alpha})_\alpha$ of all the c.r. quasi prime divisors v_α of $K_0|k_0$ with v_α trivial on l_0 is contained in $\mathcal{T}^1_l(K_0)$.

Hence this reduces the assertion of Theorem 4.2 to the following: Let $K_0|k_0$ be a function field in one variable over an algebraically closed field k_0, and $l_0 \subset k_0$ be an algebraically closed subfield. Let $\Pi^1_{K_0}$ be the ℓ-adic dual of $K_0^\times$, and for every quasi prime divisor v_α of K_0, let $T^1_{v_\alpha} \hookrightarrow \Pi^1_{K_0}$ be the ℓ-adic dual of $K_0^\times \to v_\alpha K_0 = K_0/U_{v_\alpha}$. Then one has:

Lemma 4.5. (Complete curve like). *In the above notations, the following hold:*

1) *Let $(v_\alpha)_\alpha$ be all the* c.r. *quasi prime divisors of $K_0|k_0$. Then k_0 is an algebraic closure of a finite field if and only if $\Pi^1_{K_0}$ endowed $(T^1_{v_\alpha})_\alpha$ is complete curve like.*

2) *Let $(v_\beta)_\beta$ be all the* c.r. *quasi prime divisors of $K_0|k_0$ which are trivial on l_0. Then $k_0 = l_0$ if and only if $\Pi^1_{K_0}$ endowed $(T^1_{v_\beta})_\beta$ is complete curve like.*

For a proof, see Pop [P4] in the case $\mathrm{char}(k_0) \neq \ell$, and Topaz [To2] for the general case, which is the Appendix below. □

APPENDIX: **ON THE NATURE OF BASE FIELDS**

Adam Topaz[(2)**]

The purpose of this Appendix is to prove a technical part of Bogomolov's program in anabelian geometry, concerning recovering the *nature of base fields*, given enough information from the local theory. In broad terms, if v is a quasi-divisorial valuation, and thus the residue field of v is a function field over an algebraically closed field k, the question of determining the *nature*

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of the base fields asks –among other things– whether k and/or kv is an algebraic closure of a finite field.

The problem of recovering the nature of the base fields was solved by Pop [P4] in the *tame case*, i.e., in the case where the residue characteristic is prime to ℓ. In this appendix, we generalize the argument from loc.cit. so that it works even in characteristic ℓ, by working exclusively with the *minimized inertia / decomposition groups* of valuations. We recall the basic necessary facts about minimized decomposition theory below, see [To1] for details.

For the notions *divisorial valuation*, *quasi-divisorial valuation*, *constant-reduction (c.r.) quasi-divisorial valuation,* we refer to Pop [P0], [P1], and/or the present note.

A) *Notation and Main Theorem*

Throughout, $K|k$ is a function field in one variable over the algebraically closed field k, while $\mathrm{char}(k) = \ell$ is allowed. The group

$$\Pi(K) := \mathrm{Hom}(K^\times, \mathbb{Z}_\ell)$$

is called the **minimized pro-ℓ group** of K, and notice that $\Pi(K)$ is a pro-ℓ abelian free group with respect to the *point-wise convergence topology.* While $\Pi(K)$ is not a Galois group in the traditional sense, in the case where $\mathrm{char}\, K \neq \ell$, the group $\Pi(K)$ is (non-canonically) isomorphic to the maximal pro-ℓ abelian Galois group of K.

For a valuation v of K, we denote by vK the value group, Kv the residue field, and $\mathcal{O}_v$ the valuation ring of K with valuation ideal $\mathfrak{m}_v$. Furthermore, we denote by $U_v = \mathcal{O}_v^\times$ the group of v-units, and $U_v^1 = (1 + \mathfrak{m}_v)$ the group of principal v-units. With this notation in mind, we introduce the **minimized inertia/decomposition** groups of v:

$$\mathrm{I}_v = \mathrm{Hom}(K^\times/U_v, \mathbb{Z}_\ell) \hookrightarrow \mathrm{Hom}(K^\times/U_v^1, \mathbb{Z}_\ell) =: \mathrm{D}_v \hookrightarrow \Pi(K),$$

and notice that these are closed subgroups of $\Pi(K)$.

Next let $l \subset k$ be a fixed algebraically closed subfield, and $\mathcal{V} \supset \mathcal{V}_l$ be the collection of all the c.r. quasi prime divisors of $K|k$, respectively of the c.r. quasi prime divisors of $K|k$ which are trivial on l. Then one has the following:

Fact 1. The minimized inertia groups of $K|k$ have the properties:

1. For every distinct $v, w \in \mathcal{V}$, one has $\mathrm{I}_v \cap \mathrm{I}_w = 1$.
2. I_v is a maximal procyclic subgroup of $\Pi(K)$ for every $v \in \mathcal{V}$.

Fact 2. Let G be a profinite abelian group, $(I_i)_i$ be a system of procyclic subgroups, and $\tau_i \in I_i$ a be generator of I_i for each i. The following assertions are equivalent:

i) Every open subgroup of G contains I_i for all but finitely many i.

ii) The pro-word $\tau_0 := \prod_i \tau_i$ is defined in G.

iii) There exists a continuous map $\prod_i I_i \to G$ which is the identity on each I_i.

The main result concerning detecting *the nature of the base field* is as follows.

Main Theorem. *In the above notations, the following hold:*

I. The nature of k**.** *The following are equivalent:*

i) *k is the algebraic closure of a finite field.*

ii) *There is a system of generators $(\tau_v)_{v\in\mathcal{V}}$ of the groups $(\mathrm{I}_v)_{v\in\mathcal{V}}$ satisfying the pro-relation $\prod_v \tau_v = 1$, and this is the only profinite relation satisfied by $(\tau_v)_{v\in\mathcal{V}}$.*

II. The equality $k = l$**.** *The following are equivalent:*

i) *One has $k = l$.*

ii) *There is a system of generators $(\tau_v)_{v\in\mathcal{V}_l}$ of the groups $(\mathrm{I}_v)_{v\in\mathcal{V}_l}$ satisfying the pro-relation $\prod_v \tau_v = 1$, and this is the only profinite relation satisfied by $(\tau_v)_{v\in\mathcal{V}_l}$.*

Moreover, let X be the unique projective smooth curve with $K = k(X)$, and $\pi_1^{\ell,\mathrm{ab}}(X)$ be its pro-ℓ abelian fundamental group.[3] *If $\mathcal{V}_*$ denotes either $\mathcal{V}$ or $\mathcal{V}_l$, and the above equivalent conditions are satisfied for $\mathcal{V}_*$, one has a canonical exact sequence:*

$$0 \to \mathbb{Z}_\ell \to \prod_{v\in\mathcal{V}_*} \mathrm{I}_v \to \Pi(K) \to \pi_1^{\ell,\mathrm{ab}}(X) \to 1.$$

B) *Basic facts about minimized inertia / decomposition*

For $f \in \mathrm{D}_v = \mathrm{Hom}(K^\times/U_v^1, \mathbb{Z}_\ell)$, let $f_v : Kv^\times = U_v/U_v^1 \subset K^\times/U_v^1 \to \mathbb{Z}_\ell$ be its restriction to $Kv^\times$. Then we get a canonical homomorphism $\mathrm{D}_v \to \Pi(Kv)$, $f \mapsto f_v$.

Fact 3. In the above notations, let w be a valuation of Kv. Then the following hold:

1. The canonical map $\mathrm{D}_v \to \Pi(Kv)$ induces an isomorphism $\mathrm{D}_v/\mathrm{I}_v \cong \Pi(Kv)$.
2. One has the following inequalities of subgroups of $\Pi(K)$:
$$\mathrm{I}_v \subset \mathrm{I}_{w\circ v} \subset \mathrm{D}_{w\circ v} \subset \mathrm{D}_v.$$
3. Identifying $\mathrm{D}_v/\mathrm{I}_v$ with $\Pi(Kv)$ as above, one has $\mathrm{D}_{w\circ v}/\mathrm{I}_v = \mathrm{D}_w$, $\mathrm{I}_{w\circ v}/\mathrm{I}_v = \mathrm{I}_w$.

Next recall the following basic properties of the quasi prime divisors v of $K|k$:

a) One has $\mathrm{td}(K|k) - 1 = \mathrm{td}(Kv|kv)$.

b) The value group vK contains no non-trivial ℓ-divisible convex subgroups.

c) One has an isomorphism $vK/vk \cong \mathbb{Z}$ as abstract groups.

[3] Recall that if g is the genus of X, then $\pi_1^{\ell,\mathrm{ab}}(X) \cong \mathbb{Z}_\ell^{2g}$ if $\mathrm{char}(k) \neq \ell$, and $\pi_1^{\ell,\mathrm{ab}}(X) \cong \mathbb{Z}_\ell^{\gamma}$ for some $0 \le \gamma \le g$ if $\mathrm{char}(k) = \ell$, called is the Hasse–Witt invariant of the Jacobian variety $\mathrm{Jac}(X)$ of X.

d) Any two distinct quasi prime divisors v and w are incomparable, i.e., $v \neq w$ implies that $\mathcal{O}_v$ and $\mathcal{O}_w$ are not comparable w.r.t. inclusion.

These properties have the following consequences for minimized decomposition theory.

Fact 4. In the above notations and context, the following hold:

1. Let v be a quasi prime divisor of $K|k$. Then $\mathrm{I}_v = \mathrm{D}_v$ and $\mathrm{I}_v \cong \mathbb{Z}_\ell$.
2. Let v, w be two distinct quasi prime divisors. Then one has $\mathrm{I}_v \cap \mathrm{I}_w = 1$.

The following lemma is the key point in the proof of our Main Theorem:

Lemma 5. *In the notations from the Main Theorem, let $\mathcal{V}_0$ denote the set of prime divisors of $K|k$. Then the following hold:*

1. *Every open subgroup of $\Pi(K)$ contains I_v for all but finitely many $v \in \mathcal{V}_0$.*
2. *The kernel of the canonical map $\iota_{\mathcal{V}_0} : \prod_{v\in\mathcal{V}_0} \mathrm{I}_v \to \Pi(K)$ is isomorphic to $\mathbb{Z}_\ell$.*
3. *One has a canonical exact sequence: $0 \to \mathbb{Z}_\ell \to \prod_{v\in\mathcal{V}_0} \mathrm{I}_v \to \Pi(K) \to \pi_1^{\ell,\mathrm{ab}}(X) \to 1$.*

Proof: Proof of Assertion (1): Assume first that U is an open subgroup of $\Pi(K)$ such that $\Pi(K)/U \cong \mathbb{Z}/\ell^n$, and let $f : \Pi(K) \to \mathbb{Z}/\ell^n$ be a surjective homomorphism with kernel U. Since $\mu_{\ell^\infty} \subset K$, and thus $\mathrm{Tor}^1_{\mathbb{Z}_\ell}(K^\times, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is ℓ-divisible, we see that the canonical map

$$\Pi(K) = \mathrm{Hom}(K^\times, \mathbb{Z}_\ell) \to \mathrm{Hom}(K^\times, \mathbb{Z}/\ell^n)$$

is surjective, and the kernel of this map is $\ell^n \cdot \Pi(K)$. Thus f factors through some homomorphism $g : \mathrm{Hom}(K^\times, \mathbb{Z}/\ell^n) \to \mathbb{Z}/\ell^n$. By Pontryagin duality, there exists some $x \in K^\times$ such that $g(h) = h(x)$ for all $h \in \mathrm{Hom}(K^\times, \mathbb{Z}/\ell^n)$. Hence our original map f is given by

$$f(\phi) = \phi(x) \pmod{\ell^n}$$

for all $\phi \in \Pi(K) = \mathrm{Hom}(K^\times, \mathbb{Z}_\ell)$. Now since $\mathcal{V}_0$ is the collection of all divisorial valuations of $K|k$, and $\mathrm{td}(K|k) = 1$, we see that $v(x) = 0$ for all but finitely many $v \in \mathcal{V}_0$. From this it follows that $U = \ker(f)$ contains I_v for all but finitely many $v \in \mathcal{V}_0$. Next, every open subgroup $U \subset \Pi(K)$ is of the form $U = U_1 \cap \cdots \cap U_r$ with $U_i \subset \Pi(K)$ open and $\Pi(K)/U_i \cong \mathbb{Z}/\ell^{n_i}$ for each i. Thus, assertion (1) follows.

Proof of Assertions (2) & (3): Let X be the unique complete normal model of $K|k$, and consider the canonical exact sequence:

$$0 \to K^\times/k^\times \xrightarrow{\mathrm{div}} \mathrm{Div}(X) \to \mathrm{Pic}(X) \to 0. \tag{1.2}$$

Since $\mathrm{I}_v = \mathrm{Hom}(K^\times/U_v, \mathbb{Z}_\ell) = \mathrm{Hom}(vK, \mathbb{Z}_\ell)$, we obtain a canonical isomorphism:

$$\mathrm{Hom}(\mathrm{Div}(X), \mathbb{Z}_\ell) \cong \textstyle\prod_{v\in\mathcal{V}_0} \mathrm{I}_v.$$

Moreover, $\iota_{\mathcal{V}_0} : \prod_{v\in\mathcal{V}_0} \mathrm{I}_v \to \Pi(K)$ is obtained by applying the functor $\mathrm{Hom}(\bullet, \mathbb{Z}_\ell)$ to the divisor map $\mathrm{div} : K^\times \to \mathrm{Div}(X)$, and since $k^\times$ is divisible, one has $\Pi(K) = \mathrm{Hom}(K^\times/k^\times, \mathbb{Z}_\ell)$ canonically. Thus, by applying $\mathrm{Hom}(\bullet, \mathbb{Z}_\ell)$ to (1.2), we obtain the exact sequence:

$$0 \to \mathrm{Hom}(\mathrm{Pic}(X), \mathbb{Z}_\ell) \to \prod_{v\in\mathcal{V}_0} \mathrm{I}_v \xrightarrow{\iota_{\mathcal{V}_0}} \Pi(K) \to \pi_1^{\ell,\mathrm{ab}}(X) \to 1. \tag{1.3}$$

To conclude the proof of assertion (2), we consider the following exact sequence:

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \xrightarrow{\deg} \mathbb{Z} \to 0. \tag{1.4}$$

Applying $\mathrm{Hom}(\bullet, \mathbb{Z}_\ell)$ to (1.4), we obtain the following short exact sequence:

$$0 \to \mathbb{Z}_\ell \to \mathrm{Hom}(\mathrm{Pic}(X), \mathbb{Z}_\ell) \to \mathrm{Hom}(\mathrm{Pic}^0(X), \mathbb{Z}_\ell).$$

But $\mathrm{Pic}^0(X)$ is divisible since k is algebraically closed, and thus $\mathrm{Hom}(\mathrm{Pic}^0(X), \mathbb{Z}_\ell) = 0$. Therefore, one has an isomorphism $\mathbb{Z}_\ell \cong \mathrm{Hom}(\mathrm{Pic}(X), \mathbb{Z}_\ell)$, and thus (1.3) turns into the following short exact sequence $0 \to \mathbb{Z}_\ell \to \prod_{v\in\mathcal{V}_0} \mathrm{I}_v \to \Pi(K) \to \pi_1^{\ell,\mathrm{ab}}(X) \to 1$. This concludes the proof of Lemma 5. □

C) *Proof of Main Theorem*

First, the implication i) ⇒ ii) follows directly from the Lemma 5 above: Namely, if k is an algebraic closure of a finite field, then k has no non-trivial valuations. Therefore, the quasi prime divisors and the prime divisors of $K|k$ are the same, i.e., $\mathcal{V} = \mathcal{V}_0$. Second, if $l = k$, then $\mathcal{V}_0 = \mathcal{V}_k = \mathcal{V}_l$. Thus the implication i) ⇒ ii) is a direct consequence of Lemma 5.

For the converse implication, suppose that condition ii) is satisfied. By contradiction, suppose that k is not algebraic over a finite field in case I, respectively that $k \neq l$ in case II. Then k has non-trivial valuations w_k, which in case II, are trivial on l. For such a valuation w_k, we consider a constant reduction w of K with $w|_k = w_k$. (Notice that such constant reductions w always exist: If $t \in K$ is a non-constant function, then any prolongation w of the Gauss valuation w_t of $k(t)$ to K will do the job.) In particular, $wK = wk$, because k is algebraically closed, and by Fact 3, it follows that I_w is trivial, and $\mathrm{D}_w \to \Pi(Kw)$ is an isomorphism. Thus we can identify $\Pi(Kw)$ canonically with a subgroup of $\Pi(K)$.

Let $\mathcal{V}_0$ denote the collection of the prime divisors of $K|k$, and let $\mathcal{W}_0$ denote the collection of prime divisors of $Kw|kw$. Furthermore, put

$$\mathcal{V}_w := \{w_0 \circ w \ : \ w_0 \in \mathcal{W}_0\}.$$

We put $\mathcal{V}_* := \mathcal{V}$ in case I, respectively $\mathcal{V}_* := \mathcal{V}_l$ in case II. Each $v \in \mathcal{V}_0$ is, in particular, a c.r. quasi prime divisor of $K|k$ (namely, with respect to the trivial valuation of K), so that $\mathcal{V}_0 \subset \mathcal{V}_*$. Also, for every valuation $w_0 \in \mathcal{W}_0$, the composition $w_0 \circ w \in \mathcal{V}_w$ is a c.r. quasi prime divisor of $K|k$, hence $\mathcal{V}_w \subset \mathcal{V}_*$.

Applying Lemma 5 to the set $\mathcal{V}_0$ of the prime divisors of $K|k$, we have an exact sequence

$$1 \to \mathbb{Z}_\ell \longrightarrow \prod_{v\in\mathcal{V}_0} \mathrm{I}_v \xrightarrow{\iota_{\mathcal{V}_0}} \Pi(K). \tag{1.5}$$

On the other hand, we can apply Lemma 5 to the set of prime divisors $\mathcal{W}_0$ of $Kw|kw$, to obtain another short exact sequence:

$$1 \to \mathbb{Z}_\ell \longrightarrow \prod_{w_0 \in \mathcal{W}_0} \mathrm{I}_{w_0} \xrightarrow{\iota_{\mathcal{W}_0}} \Pi(Kw).$$

Next recall that we identified canonically $\Pi(Kw) = \mathrm{D}_w/\mathrm{I}_w = \mathrm{D}_w$ as a subgroup of $\Pi(K)$. In light of this identification, Fact 3 implies that:

$$\mathrm{I}_{w_0} = \mathrm{I}_{w_0 \circ w}/\mathrm{I}_w = \mathrm{I}_{w_0 \circ w}$$

as subgroups of $\Pi(K)$. In particular, we obtain yet another short exact sequence:

$$1 \to \mathbb{Z}_\ell \to \prod_{v \in \mathcal{V}_w} \mathrm{I}_v \xrightarrow{\iota_{\mathcal{V}_w}} \Pi(K). \tag{1.6}$$

Combining (1.5) and (1.6), we see that the kernel of

$$\iota_{\mathcal{V}_0 \cup \mathcal{V}_w} : \prod_{v \in \mathcal{V}_0 \cup \mathcal{V}_w} \mathrm{I}_v = \left(\prod_{v \in \mathcal{V}_0} \mathrm{I}_v\right) \times \left(\prod_{v \in \mathcal{V}_w} \mathrm{I}_v\right) \to \Pi(K)$$

has $\mathbb{Z}_\ell$-rank ≥ 2 since $\mathcal{V}_0 \cap \mathcal{V}_w = \varnothing$. Equivalently, if $(\tau_v)_{v \in \mathcal{V}_0 \cup \mathcal{V}_w}$ is any system of generators of the system of procyclic groups $(\mathrm{I}_v)_{v \in \mathcal{V}_0 \cup \mathcal{V}_w}$, then there are at least two pro-relations between the generators $(\tau_v)_{v \in \mathcal{V}_0 \cup \mathcal{V}_w}$.

To conclude the proof of the theorem, we note that $\mathcal{V}_0 \cup \mathcal{V}_w \subset \mathcal{V}_*$. Thus if $(\tau_v)_{v \in \mathcal{V}_*}$ is any system of generators of the system of procyclic groups $(\mathrm{I}_v)_{v \in \mathcal{V}_*}$, then there are at least two pro-relations between these generators. This contradicts the assumptions of ii).

Applying Lemma 5, (3), one concludes the proof of the Main Theorem.

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