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Cogalois Theory: an outline

by
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- Dedicated to Bazil Brînzănescu in honour of his 70th birthday -

Abstract

In this expository paper we present a short outline of *Cogalois Theory*.

Key Words: Field extension, radical extension, Kneser extension, strongly Kneser extension, Kneser Criterion, Galois extension, Cogalois extension, G-Cogalois extension, Galois Theory, Cogalois Theory, Galois group, Cogalois group, Kneser group, Kummer Theory, Abelian extension, profinite group, crossed homomorphism, algebraic number field, group-graded algebra, Hopf algebra, Abstract Cogalois Theory.
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Introduction

Cogalois Theory, a fairly new area in Field Theory, has been initiated in 1986 by Greither and Harrison [19], and then, systematically developed in the last 28 years. A basic source of this theory is the author's monograph [7]. Cogalois Theory deals with field extensions E/F which possess a Δ-*Cogalois correspondence*, i.e., a canonical *lattice isomorphism* between the lattice $\mathbb{I}(E/F)$ of all intermediate fields of E/F and the lattice $\mathbb{L}(\Delta)$ of all subgroups of a certain group Δ canonically associated with E/F. This situation is dual to that from *Galois Theory*: any finite Galois extension E/F possess a Γ-*Galois correspondence*, i.e., a canonical lattice *anti-isomorphism* between $\mathbb{I}(E/F)$ and $\mathbb{L}(\Gamma)$, where Γ is the Galois group of the extension E/F. An *Abstract Cogalois Theory* for arbitrary profinite groups, dual to the Abstract Galois Theory for such groups, is also discussed.

1. Basic notation

By $\mathbb{N}$ we denote the set $\{0, 1, 2, \dots\}$ of all natural numbers, by $\mathbb{N}^*$ the set $\mathbb{N} \setminus \{0\}$ of all strictly positive natural numbers, by $\mathbb{Z}$ the ring of all rational integers, and by $\mathbb{Q}$ (resp. $\mathbb{R}$, $\mathbb{C}$) the field of all rational (resp. real, complex) numbers. For any set M, not necessarily finite, $|M|$ will denote the cardinal number of M.

Throughout this paper F denotes a fixed field, $\mathrm{Char}(F)$ its characteristic, $e(F)$ its characteristic exponent (that is, $e(F) = 1$ if F has characteristic 0, and $e(F) = p$ if F has characteristic $p > 0$), and Ω a fixed algebraically closed field containing F as a subfield. Any extension of F is supposed to be a subfield of Ω.

For an arbitrary $\varnothing \neq S \subseteq \Omega$ and $n \in \mathbb{N}^*$ we denote throughout this paper:

$$S^* = S \setminus \{0\},\ S^n = \{\, x^n \mid x \in S\},\ \mu_n(S) = \{\, x \in S \mid x^n = 1\},$$

$$\mu(S) = \{\, x \in S \mid x^k = 1 \text{ for some } k \in \mathbb{N}^*\}.$$

For any $x \in \Omega^*$, $\widehat{x}$ will denote the coset xF^* of x in the quotient group Ω^*/F^*. By a primitive n-th root of unity we mean any generator of the cyclic group $\mu_n(\Omega)$; ζ_n will always denote such an element.

For a group G, the notation $H \leqslant G$ means that H is a subgroup of G, and the lattice of all subgroups of G is denoted by $\mathbb{L}(G)$. For any subset M of G, $\langle M \rangle$ denotes the subgroup of G generated by M.

For a field extension $F \subseteq E$, shortly, extension, we shall use the notation E/F. If E/F is an extension, then any subfield K of E with $F \subseteq K$ is called an *intermediate field* of the extension E/F, and $\mathbb{I}(E/F)$ will denote the set of all its intermediate fields. Note that $\mathbb{I}(E/F)$ is a complete lattice.

For all other undefined terms and notation concerning basic Field Theory the reader is referred to Bourbaki [18] and/or Karpilovsky [23].

2. Kneser and Cogalois extensions

For any field extension E/F we denote $T(E/F) := \{\, x \in E^* \mid x^n \in F^* \text{ for some } n \in \mathbb{N}^*\,\}$. Observe that for every $x \in T(E/F)$ there exists an $n \in \mathbb{N}^*$ such that $x^n = a \in F$, so x is an n-th *radical* of a, denoted by $\sqrt[n]{a}$. The quotient group $T(E/F)/F^*$ was called in [19] the *Cogalois group* of the extension E/F and denoted by $\mathrm{Cog}\,(E/F)$.

As in [11], a field extension E/F is said to be a *radical* extension if there exists a subset $A \subseteq T(E/F)$ such that $E = F(A)$, i.e., E is obtained by adjoining to the base field F an arbitrary set A of "radicals" over F. Clearly, one can replace A by the subgroup $G = F^*\langle\, A\,\rangle$ of the multiplicative group E^* of E generated by F^* and A. Thus, any radical extension E/F has the form $E = F(G)$, with $F^* \leqslant G \leqslant T(E/F)$. Such an extension is called *G-radical*.

A field extension E/F, which is not necessarily finite, has been called in [14] *G-Kneser* if it is a G-radical extension such that there exists a set of representatives for the factor group G/F^* which is linearly independent over F; in case the G-radical extension E/F is finite then the last condition can be expressed equivalently as $|G/F^*| = [\,E : F\,]$. The extension E/F is called *Kneser* if it is G-Kneser for some group G. As in [14], an extension E/F is said to be a *Cogalois extension* if it is $T(E/F)$-Kneser; for finite extensions these are exactly the Cogalois extensions introduced by Greither and Harrison in [19]. As in [19], a field extension E/F is said to be *pure* if $\mu_p(E) \subseteq F$ for all p, p odd prime or 4.

2.1. The Kneser Criterion ([24], [14]). *An arbitrary separable G-radical extension E/F is G-Kneser if and only if $\zeta_p \in G \Longrightarrow \zeta_p \in F$ for every odd prime p and $1 \pm \zeta_4 \in G \Longrightarrow \zeta_4 \in F$.* $\square$

2.2. The Greither-Harrison Criterion ([19], [14]). *An arbitrary extension E/F is Cogalois if and only if it is radical, separable, and pure.* □

2.3. Examples of Cogalois extensions. (1) Any finite G-radical extension E/F with E a subfield of $\mathbb{R}$ is clearly pure, hence it is Cogalois by the Greither-Harrison Criterion, and $\mathrm{Cog}\,(E/F) = G/F^*$.

For example, the extension $\mathbb{Q}\,(\sqrt[n_1]{a_1}\,, \ldots, \sqrt[n_r]{a_r}\,)/\mathbb{Q}$ with $r, n_1, \ldots, n_r, a_1, \ldots, a_r \in \mathbb{N}^*$, is a G-radical Cogalois extension, where $G = \mathbb{Q}^*\langle \sqrt[n_1]{a_1}, \ldots, \sqrt[n_r]{a_r}\,\rangle$, hence its Cogalois group is precisely $\mathbb{Q}^*\langle \sqrt[n_1]{a_1}\,, \ldots, \sqrt[n_r]{a_r}\,\rangle/\,\mathbb{Q}^*$. In particular, we have

$$[\,\mathbb{Q}\,(\sqrt[n_1]{a_1}\,, \ldots, \sqrt[n_r]{a_r}\,) : \mathbb{Q}\,] = |\,\mathbb{Q}^*\langle \sqrt[n_1]{a_1}\,, \ldots, \sqrt[n_r]{a_r}\,\rangle/\mathbb{Q}^*|.$$

(2) A quadratic extension $\mathbb{Q}(\sqrt{d}\,)/\mathbb{Q}$, where $d \neq 1$ is a square-free integer, is Cogalois if and only if $d \neq -1, -3$. Observe that the extension $\mathbb{Q}(\sqrt{-3}\,)/\mathbb{Q}$ is $\mathbb{Q}^*\langle\sqrt{-3}\,\rangle$-Kneser but it is not Cogalois. □

3. G-Cogalois extensions

For an arbitrary G-radical extension E/F, the maps

$$\varphi : \mathbb{I}(E/F) \longrightarrow \mathbb{L}(G/F^*),\ \varphi(K) = (K \cap G)/F^*,$$

and

$$\psi : \mathbb{L}(G/F^*) \longrightarrow \mathbb{I}(E/F),\ \psi(H/F^*) = F(H),$$

arise in a very natural way and establish a so called *Cogalois connection* between the lattices $\mathbb{I}(E/F)$ and $\mathbb{L}(G/F^*)$. We say that the G-radical extension E/F is an extension with G/F^*-*Cogalois correspondence* if the maps φ and ψ defined above are isomorphisms of lattices, inverse to one another. It turns out that the G-Kneser extensions with G/F^*-*Cogalois correspondence* are precisely the so called *strongly G-Kneser* extensions; these are the G-radical extensions E/F such that, for every intermediate field K of E/F, the extension E/K is K^*G-Kneser, or equivalently, the extension K/F is $K^* \cap G$-Kneser.

The most interesting strongly G-Kneser extensions are those extensions which additionally are separable, called *G-Cogalois extensions*; they play in Cogalois Theory the same role as Galois extensions play in Galois Theory. These extensions are completely characterized within the class of G-radical extensions by means of a crucial *General Purity Criterion* presented below. In order to state it we have to introduce some notation.

Denote by $\mathbb{P}$ the set of all positive prime numbers, by $\mathcal{P}$ the set $(\mathbb{P}\setminus\{2\})\cup\{4\}$, by $\mathbb{D}_n$ the set of all positive divisors of a given number $n \in \mathbb{N}^*$, and by $\mathcal{P}_n$ the set $\mathcal{P} \cap \mathbb{D}_n$. Recall that an extension E/F is called *pure* when $\mu_p(E) \subseteq F$ for all $p \in \mathcal{P}$. More generally, if $\varnothing \neq \mathcal{Q} \subseteq \mathcal{P}$, we say that an extension E/F is *$\mathcal{Q}$-pure* if $\mu_p(E) \subseteq F$ for all $p \in \mathcal{Q}$. If $n \in \mathbb{N}^*$, then an extension E/F is called n-pure if it is $\mathcal{P}_n$-pure. For any torsion multiplicative group T with identity element e we denote by $\mathcal{O}_T$ the set of all orders of elements of T. When the subset $\mathcal{O}_T$ of $\mathbb{N}^*$ is a finite set, then the least number $m \in \mathbb{N}^*$ such that $T^m = \{e\}$ is the *exponent* $\exp(T)$ of T and the group T is called $\exp(T)$-*bounded*. For any G-radical extension E/F, G/F^* is a torsion group, so $\mathcal{O}_{G/F^*} \subseteq \mathbb{N}^*$. A G-radical extension E/F is said to be *n-bounded extension* if $\exp(G/F^*) = n$. For any G-radical extension E/F we denote $\mathcal{P}_G := \mathcal{P} \cap \mathcal{O}_{G/F^*}$.

3.1. The General Purity Criterion [3]. *A separable G-radical extension is G-Cogalois if and only if it is $\mathcal{P}_G$-pure.* □

From 3.1 it follows immediately that any Cogalois extension E/F is $T(E/F)$-Cogalois. Note that when a G-radical extension E/F is n-bounded, then $\mathcal{O}_{G/F^*} = \mathbb{D}_n$, hence $\mathcal{P}_G = \mathcal{P} \cap \mathbb{D}_n = \mathcal{P}_n$, and so, we obtain:

3.2. The n-Purity Criterion ([11], [14]). *A separable n-bounded G-radical extension E/F is G-Cogalois if and only if it is n-pure. In particular, a finite separable G-radical extension E/F with* $\exp(G/F^*) = n$ *is G-Cogalois if and only if it is n-pure.* □

Using 3.1 and 3.2 one deduces that the class of finite or infinite G-Cogalois extensions is fairly large, including besides the Cogalois extensions, the *classical Kummer extensions* [23], the *neat presentations* [19], as well as various generalizations of these two types of extensions: the *generalized Kummer extensions* [11], [14], the *Kummer extensions with few roots of unity* [1], [6], [14], and the *quasi-Kummer extensions* [7] (see 5.1 for definitions).

3.3. The Kneser group of a G-Cogalois extension ([11], [14]). *Let E/F be an extension which is simultaneously G-Cogalois and H-Cogalois. Then $G = H$, and the uniquely determined group G/F^* is called the* Kneser group *of the extension E/F and is denoted by* $\mathrm{Kne}(E/F)$. □

3.4. Primitive elements [12]. *Let E/F be a finite G-Cogalois extension and let $(x_i)_{1\leqslant i\leqslant n}$ be a finite family of elements of G. If $\widehat{x_i} \neq \widehat{x_j}$ for every $i, j \in \{1, \ldots, n\}$, $i \neq j$, then $x_1 + \ldots + x_n$ is a primitive element of E/F if and only if $G = F^*\langle x_1, \ldots, x_n\rangle$; in particular, if $\{u_1, \ldots, u_r\}$ is any set of representatives of G/F^*, then $u_1 + \ldots + u_r$ is a primitive element of E/F.*

If $G = F^*\langle x_1, \ldots, x_n\rangle$ and $[\, F(x_1, \ldots, x_n) : F\,] = \prod_{i=1}^{n} [\, F(x_i) : F\,]$, then

$$F(x_1, \ldots, x_n) = F(x_1 + \ldots + x_n).$$ □

4. Galois G-Cogalois extensions

Let E/F be an arbitrary Galois extension. Then, the Galois group Γ of E/F is a profinite group, or equivalently, a Hausdorff, compact, and totally disconnected topological group with respect to its Krull topology.

Let $M \leqslant E^*$ be such that $\sigma(M) \subseteq M$ for every $\sigma \in \Gamma$. A *crossed homomorphism* (or an 1-*cocycle*) of Γ with coefficients in M is a map $f : \Gamma \to M$ satisfying the condition $f(\sigma\tau) = f(\sigma) \cdot \sigma(f(\tau))$ for every $\sigma, \tau \in \Gamma$. The set $Z^1_c(\Gamma, M)$ of all continuous 1-cocycles of Γ with coefficients in the discrete group M is an Abelian group. For any $\alpha \in M$, the 1-*coboundary* $f_\alpha : \Gamma \to M$, defined by $f_\alpha(\sigma) = \sigma(\alpha)\cdot\alpha^{-1}$, $\sigma \in \Gamma$, is a continuous map and the set $B^1(\Gamma, M) = \{\, f_\alpha \mid \alpha \in M\,\}$ of all 1-coboundaries of Γ with coefficients in M is a subgroup of $Z^1_c(\Gamma, M)$. The quotient group $Z^1_c(\Gamma, M)/B^1(\Gamma, M)$ is denoted by $H^1_c(\Gamma, M)$.

For an arbitrary extension E/F, the map

$$\psi : \mathrm{Cog}\,(E/F) \longrightarrow Z^1_c(\mathrm{Gal}\,(E/F), \mu(E)),\ \psi(\widehat{\alpha}) = f_\alpha,\ \alpha \in T(E/F),$$

is a well-defined group morphism. The *Hilbert's Theorem 90*, saying that if E/F is an arbitrary Galois extension then $H^1_c(\Gamma, E^*) = \mathbf{1}$, has the following reformulation in terms of Cogalois groups.

4.1. The Cogalois group via continuous 1-cocycles ([25], [15]). *For any Galois extension E/F, the map $\widehat{\alpha} \mapsto f_\alpha$ establishes a group isomorphism*

$$\mathrm{Cog}\,(E/F) \xrightarrow{\sim} Z^1_c(\mathrm{Gal}\,(E/F), \mu(E)). \qquad \square$$

As an immediate consequence of 4.1, it follows that if E/F is a finite Galois extension with $\mu(E)$ finite, then $\mathrm{Cog}(E/F)$ is a finite group (see [15]); in particular, for any extension K/L of algebraic number fields, which is not necessarily Galois, the group $\mathrm{Cog}(K/L)$ is finite (see [19]).

4.2. The Kneser group via continuous 1-cocycles [4]. *For any Galois G-Cogalois extension E/F, the map $\widehat{\alpha} \mapsto f_\alpha$ yields a group isomorphism*

$$\mathrm{Kne}\,(E/F) \xrightarrow{\sim} Z^1_c(\mathrm{Gal}\,(E/F), \mu_G(E)),$$

where $\mu_G(E) := \bigcup_{m \in \mathcal{O}_{G/F^*}} \mu_m(E)$. $\square$

From 4.2 we deduce that if E/F is a Galois n-bounded G-Cogalois extension, in particular a finite Galois G-Cogalois extension with $n = \exp(G/F^*)$, then there exists a group isomorphism

$$\mathrm{Kne}\,(E/F) \xrightarrow{\sim} Z^1_c(\mathrm{Gal}\,(E/F), \mu_n(E)).$$

4.3. Radical extensions via continuous 1-cocycles [4]. *Let E/F be a Galois extension with Galois group Γ, and let $K \in \mathbb{I}(E/F)$. Then K/F is a radical extension if and only if there exists $U \leqslant Z^1_c(\Gamma, \mu(E))$ such that* $\mathrm{Gal}\,(E/K) = \{\, \sigma \in \Gamma \mid h(\sigma) = 1,\ \forall\, h \in U\}$. $\square$

Using 4.3, one can obtain characterizations of G-Kneser and G-Cogalois subextensions of a given Galois extension E/F with Galois group Γ via subgroups of $Z^1_c(\Gamma, \mu(E))$ (see [4]).

For any topological group T we denote by $\widehat{T}$ the *character group* of T, that is, the group $\mathrm{Hom}_c(T, \mathbb{U})$ of all continuous group morphisms of T into the unit circle $\mathbb{U}$.

4.4. Abelian G-Cogalois extensions [9]. *If E/F is an Abelian G-Cogalois extension, then the discrete torsion Abelian groups* $\mathrm{Kne}\,(E/F)$ *and* $\widehat{\mathrm{Gal}\,(E/F)}$ *are isomorphic. In particular, the discrete torsion Abelian groups* $\mathrm{Cog}\,(E/F)$ *and* $\widehat{\mathrm{Gal}\,(E/F)}$ *are isomorphic for any Abelian Cogalois extension E/F.* $\square$

From 4.4 it follows immediately that for any finite Abelian G-Cogalois extension E/F, the finite Abelian groups $\mathrm{Kne}\,(E/F)$ and $\mathrm{Gal}\,(E/F))$ are isomorphic; in particular, the groups $\mathrm{Cog}\,(E/F)$ and $\mathrm{Gal}\,(E/F))$ are isomorphic for any finite Abelian Cogalois extension E/F.

5. Applications of Cogalois Theory

5.1. Applications to Kummer Theory. As in [7], [14], we say that an extension E/F is a *classical n-Kummer extension* (resp. a *generalized n-Kummer extension, n-Kummer extension with few roots of unity, n-quasi Kummer extension*), where $n \in \mathbb{N}^*$, if $E = F(B)$ for some $\varnothing \neq B \subseteq E^*$, with $\gcd(n, e(F)) = 1$, $B^n \subseteq F$, and $\mu_n(\Omega) \subseteq F$ (resp. $\mu_n(E) \subseteq F$, $\mu_n(E) \subseteq \{-1, 1\}$, $\zeta_p \in F$ for every $p \in \mathcal{P}_n$). If E/F is any of these four types of Kummer extensions, then E/F is an $F^*\langle B\rangle$-Cogalois extension (see [7], [14]). This implies not only that the whole classical *Kummer Theory* can be easily deduced from this fact, but also that the various generalizations of classical Kummer extensions enjoy very similar properties to them.

5.2. Applications to Elementary Field Arithmetic. (1) A real number $\alpha > 0$ can be written as a finite sum of real numbers of type $\pm \sqrt[n_i]{a_i}\,,\ 1 \leqslant i \leqslant r,\ r,\, n_i,\, a_i \in \mathbb{N}^*$, if and only if the extension $\mathbb{Q}(\alpha)/\mathbb{Q}$ is radical (or Kneser, or Cogalois). We deduce that if a square-free integer $d \geqslant 2$ and $r \in \mathbb{Z}^*$ are such that $r > -\sqrt{d}$ and $\sqrt{r^2 - d} \notin \mathbb{Q}(\sqrt{d}\,)$, then $\sqrt{r+\sqrt{d}}$ cannot be written as a finite sum of real numbers of type $\pm \sqrt[n_i]{a_i}\,,\ 1 \leqslant i \leqslant r$, with $r,\, n_i,\, a_i \in \mathbb{N}^*$, since the extension $\mathbb{Q}\big(\sqrt{r+\sqrt{d}}\,\big)/\mathbb{Q}$ is not Cogalois (see [2]); in particular, this holds, e.g., for the number $\sqrt{1+\sqrt{2}}$.

(2) If F is any subfield of $\mathbb{R},\ r, n_1, \ldots, n_r \in \mathbb{N}^*$, and $a_1, \ldots, a_r \in F^*$ are positive, then $F(\sqrt[n_1]{a_1}\,, \ldots, \sqrt[n_r]{a_r}\,) = F\,(\sqrt[n_1]{a_1} + \cdots + \sqrt[n_r]{a_r}\,)$, by 3.4; this implies that

$$\sqrt[n_1]{a_1} + \cdots + \sqrt[n_r]{a_r} \in F \iff \sqrt[n_i]{a_i} \in F \ \text{ for all } i,\ 1 \leqslant i \leqslant r.$$

(3) If $r, n_0, n_1, \ldots, n_r \in \mathbb{N}^*$ and $a_0, a_1, \ldots, a_r \in \mathbb{Q}$ are positive, then $\sqrt[n_0]{a_0}$ can be written as a finite sum of monomials of form $c \cdot \sqrt[n_1]{a_1}^{\,j_1} \cdot \ldots \cdot \sqrt[n_r]{a_r}^{\,j_r}$, with $j_1, \ldots, j_r \in \mathbb{N}$ and $c \in \mathbb{Q}^*$, if and only if $\sqrt[n_0]{a_0}$ is itself such a monomial (see [2]).

(4) For further applications of Cogalois Theory to Elementary Field Arithmetic, see [2].

5.3. Applications to Algebraic Number Theory. The Kneser Criterion has nice applications not only in investigating field extensions with Cogalois correspondence, but also in proving some results in Algebraic Number Theory. Thus, a series of classical results due to Hasse [20], Besicovitch [17], Mordell [26], and Siegel [27] concerning the computation of degrees of particular radical extensions of algebraic number fields, can be very easily proved using the Kneser Criterion (see [7]).

A classical construction from 1920 in the Algebraic Number Theory, originating with Hecke [21], is the following one: to every algebraic number field K one can associate a so-called *system of ideal numbers* S, which is a certain subgroup of the multiplicative group $\mathbb{C}^*$ of complex numbers such that $K^* \leqslant S$ and the quotient group S/K^* is canonically isomorphic to the ideal class group $\mathcal{C}\ell_K$ of K. The equality $[\,K(S) : K\,] = |\mathcal{C}\ell_K|$ was claimed by Hecke [22, p.122] but never proved by him. To the best of our knowledge, no proof of this assertion is available in the literature, excepting the very short one in [13] based on Cogalois Theory.

5.4. Applications to Gröbner bases. Unexpected applications of the Kneser Criterion to *Gröbner bases* can be found in [16].

6. Connections with graded algebras and Hopf algebras

6.1. Cogalois Theory via graded algebras. The basic concepts of Cogalois Theory like G-radical, G-Kneser, and G-Cogalois field extension can be also described in terms of *Clifford extensions* and *strongly group-graded algebras* invented by Dade 1970 and 1980, respectively (see [5]). A similar approach in investigating Cogalois extensions E/F, finite or not, is due to Masuoka [25] using the concepts of group-graded field extension and coring.

6.2. Kneser and Cogalois extensions via Hopf algebras. The Kneser and Cogalois field extensions can be described in terms of *Galois H-objects* appearing in Hopf algebras as follows (see [5]). A G-radical field extension E/F is G-Kneser if and only if E is a *Galois* $F[G/F^*]$*-object* via the comodule structure given by the map $E \longrightarrow E \otimes F[G/F^*],\ x \mapsto x \otimes \widehat{g}$, for all $x \in Fg$ and $g \in G$. In particular, a field extension E/F is Cogalois if and only if E is a *Galois* $F[\,\mathrm{Cog}(E/F)\,]$-*object* with respect to the comodule structure given by the linear map $E \longrightarrow E \otimes F[\,\mathrm{Cog}(E/F)\,],\ x \mapsto x \otimes \widehat{g}$, for all $x \in Fg$ and $g \in T(E/F)$.

7. Abstract Cogalois Theory

An *Abstract Cogalois Theory* for arbitrary profinite groups, which is dual to the *Abstract Galois Theory*, has been developed in [10]. The basic concepts of the field theoretic Cogalois Theory, namely that of G-Kneser and G-Cogalois field extension, as well as their main properties are generalized to arbitrary profinite groups.

The main idea in doing so is to use the description presented in 4.1, via the Hilbert's Theorem 90, of the Cogalois group $\mathrm{Cog}(E/F)$ of an arbitrary Galois extension E/F as the group $Z^1_c(\mathrm{Gal}(E/F), \mu(E))$. Note that the multiplicative group $\mu(E)$ is isomorphic (in a non canonical way) to a subgroup of the additive group $\mathbb{Q}/\mathbb{Z}$, and that the basic groups appearing in the investigation of E/F from the Cogalois Theory perspective are subgroups of $\mathrm{Cog}(E/F)$. In this way, the above description of $\mathrm{Cog}(E/F)$ in terms of continuous 1-cocycles naturally suggests to study the abstract setting of subgroups of groups of type $Z^1_c(\Gamma, A)$, with Γ an arbitrary profinite group and A any subgroup of $\mathbb{Q}/\mathbb{Z}$ such that Γ acts continuously on the discrete group A. Thus, one can define the concepts of *Kneser subgroup* and *Cogalois subgroup* of the group $Z^1_c(\Gamma, A)$ and one can establish their main properties, including an *Abstract Kneser Criterion* for Kneser groups of cocycles and an *Abstract Quasi-Purity Criterion* for Cogalois groups of cocycles. Their proofs, involving cohomological as well as topological tools, are completely different from that of their field theoretic correspondents. A natural dictionary relates the basic notions of the (field theoretic) Cogalois Theory to their correspondents in the Abstract Cogalois Theory, which permit to retrieve easily most of the basic results of the former one from the corresponding results from the latter one (see [8]).

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