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On a class of Henselian fields

Dedicated to the memory of Nicolae Popescu,
a great researcher and teacher in Algebra

by
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Abstract

In this note we introduce a new class of Henselian fields (called strongly Henselian fields), which generalize the case of a complete rank 1 Krull valued field. We characterize the class of closed subfields of a discrete complete rank 1 and of equal characteristic valued field. We also make some considerations on the automorphisms group of a complet nonalgebraically closed rank 1 valued field.

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1 Some introductory remarks

Let K be a commutative field of characteristic zero and let $F = K((X))$ be the field of Laurent power series $\sum_{n>-\infty}^{\infty} a_n X^n$ in the variable X, with coefficients a_n in K. In [6] the authors were interested in the study of the group $G = Aut(F/K)$ of all K-automorphisms of F, i.e. in those field automorphism of F which fix the elements of K. They proved there that there is a one-to-one and onto Galois type correspondence between the set of all closed (with respect to the usual X-order topology) subfields $L \subset F$, $K \subsetneq L$, and the set of all cyclic subgroups of G. Moreover, each finite subgroup of G is cyclic [6].

For $f = \sum_{n>-\infty}^{\infty} a_n X^n$, let us denote $v(f) = \min\{n : a_n \neq 0\}$ the common X-adic discrete (Krull) valuation of F. For any subfield L of F, let v_L be the restriction of v to L. In [6], Theorem 1, the authors proved that if $K \neq L$, v is the unique extension of v_L to F, i.e. if v_L does not split in F, or if $(L, v_L) \subseteq (F, v)$ is a Henselian extension, then L is closed in F and $[F : L] < \infty$.

Starting from this last result and by using our methods introduced in [9] and [1] for studying Henselian extensions of valued fields (even in a more general frame), we consider in this note a new class of Henselian fields, namely the strongly Henselian fields (Definition 1 bellow).

This class of strongly Henselian fields seems to be very appropriate for studying a more complicated problem: "Find the structure of all finite codimensional subfields of a given non-algebraically closed Henselian field". This problem was stated in 2007 by the late Professor Nicolae Popescu during our joint research work, partially finalized in the paper [6]. This problem is an analogue of the famous Artin Schreier theory for algebraically closed fields. Here, a valued field (F, v) is said to be a Henselian field if v can be uniquely extended to any algebraic extension T of F, i.e. if v does not split in any algebraic extension of F.

A motivation for introducing this new class of henselian fields is given in the following remarks.

Remark 1. *Let $F = K((X))$ be the Laurent power series with coefficients in a field K of characteristic zero and let v be the usual X-order valuation on F. Let $L_0 = K(X)$ be the rational function subfield of F. An important problem (with applications in other branches of mathematics!) is to study the algebraic elements of F, i.e. the series of f which are algebraic over L_0, or the algebraic function subfields of L_0. If one fixes a discrete valuation of rank 1 on L_0, trivial on K, say v_0, the X-order valuation on L_0 induced by v, could we make some remarks on all discrete rank 1 valuations w on F which extends v_0? For the moment, surely not, because there are "many" transcendental elements in F (over L_0). For instance, the power series which formally define $\exp X$, $\ln(1+X)$, $\sin X$, $\cos X$, etc., are in general transcendental elements over L_0. But the Henselian valued field (F, v) (it is complete!) has the following amazing property: if $K \nsubseteq L \subset F$, with $[F : L] < \infty$, and if v_L is the restriction of v to L, then v is the unique extension of v_L to F (see Theorem 1 bellow). In particular, if we study the above problem, it is sufficient "to diminish" F down to a finite codimension subfield L of F which contains L_0 and try to classify the discrete rank 1 valuations on L (instead of F) which extend v_0. This was my main example to motivate the following study of this new class of strongly Henselian fields.*

Definition 1. *A nontrivial Krull valued field (F, v) of rank 1 is called a strongly Henselian field if for any subfield $L \subseteq F$ with $[F : L] < \infty$ and with v_L, the restriction of v to L, nontrivial, (L, v_L) is a Henselian field.*

In Theorem 1 and in Example 1 we give important cases of strongly Henselian valued fields. In Example 2 we also give a very known situation of a Henselian field which is not strongly Henselian. This last particular example can be easily extended to some more general valued fields.

The next theorem gives an important class of strongly Henselian fields. We shall see a proof for this theorem and some particular examples in Section 2.

Theorem 1. *Let (F, v) be a nontrivial complete nonalgebraically closed valued field of rank 1. Then (F, v) is a strongly Henselian field.*

Corollary 1 says that if (F, v) is a strongly Henselian field and if $L \subset F$ is a finite separable extension, then L is closed in F. This result is a generalization of Theorem 1 from [6].

In Proposition 1 we prove that for any subfield K of (F, v), a complete nonalgebraically closed valued field of rank 1, any element $\sigma \in G = Aut(F/K)$ is continuous with respect to v. This is a generalization of an old result of O.F.G. Schilling [10] (see also [6] for a similar discussion in a particular situation).

Another basic result of this note is the following theorem.

Theorem 2. *Let (F, v) be a discrete complete rank 1 valued field of equal characteristic and let (L, v_L) be a nontrivial valued subfield of (F, v) such that the extension of fields F/L is algebraically separable and, if $\widehat{L}$, $\widehat{F}$ are the residue fields of L, F respectively, then $[\widehat{F} : \widehat{L}] < \infty$. Under these conditions the following statements are equivalent:*

i) $[F : L] < \infty$

ii) (L, v_L) *is a Henselian field*

iii) (L, v_L) *is topologically closed in* (F, v).

All proofs, examples and other remarks are given in section 2.

2 Proofs and other results

Proof of Theorem 1. Let L be a finite codimension subfield of F, i.e. $[F : L] = n < \infty$ such that v_L is not a trivial one. In order to prove that (L, v_L) is Henselian, it is sufficient to consider the particular case in which the extension F/L is separable. Indeed, let $L^* \subseteq F$ be the inseparable closure of L in F. Since v_L can be uniquely extended to L^*, it will be sufficient to prove that (L^*, v_{L^*}) is Henselian. But F/L^* is a finite separable extension. Thus we can assume that F/L is a finite separable extension.

Let now $\overline{L}$ be the least (finite) normal and separable extension of L which contains F and let $\overline{v}$ be the unique extension (as a valuation) of v to $\overline{L}$ (see [4], [7], or [2]). Let $G = Gal(\overline{L}/L)$ be the Galois group of the Galois extension $\overline{L}/L$. Since (F, v) is complete, it is a Henselian field ([4], [5], or [2]), it will be sufficient to prove that the unique valuation on $\overline{L}$ which extends v_L is $\overline{v}$.

Let us assume that w is another extension of v_L to $\overline{L}$. Since $\overline{L}/L$ is a Galois extension, there is an automorphism $\sigma \in G$ such that $w = \overline{v} \circ \sigma$ (see [5], or [2]). Let us prove that $(\overline{L}, w)$ is also a complete valued field (with respect to w). Indeed, if $\{x_n\}_n$ is a Cauchy sequence in $(\overline{L}, w)$, then $\overline{v}(\sigma(x_{n+1}) - \sigma(x_n)) \to \infty$, when $n \to \infty$ [4]. Thus $\{\sigma(x_n)\}_n$ is a Cauchy sequence in $(\overline{L}, \overline{v})$. But this last valued field is complete, as a finite extension of the complete field (F, v) (see [4], [7], [5], or [2]). Let $y \in L$, $y = \lim_{n\to\infty} \sigma(x_n)$, i.e. $\overline{v}(\sigma(x_n) - y) \to \infty$, when $n \to \infty$. Now, it is easy to see that $x = \sigma^{-1}(y) \in \overline{L}$ is the limit of $\{x_n\}_n$ in $(\overline{L}, w)$, i.e. $(\overline{L}, w)$ is a complete valued field.

The next step is to see that $\overline{L}$ cannot be algebraically closed. If $\overline{L}$ were algebraically closed, the Artin-Schreier theory of real closed fields (see [3]) would say that $[\overline{L} : L] = [\overline{L} : F][F : L] = 2$. Since F is not algebraically closed (see the hypotheses), one observes that $[\overline{L} : F] = 2$ and so, $F = L$ is this case. Since (F, v) is complete, it is Henselian (see [4], or [5]), thus (L, v) would be Henselian in this last case and the statement is completely proved.

Assume now that $\overline{L}$ is not algebraically closed. Since $(\overline{L}, \overline{v})$ and $(\overline{L}, w)$ are complete valued fields, using the famous F. K. Schmidt result (see [7], or [2]), we have to conclude that w and $\overline{v}$ are equivalent as Krull valuations of rank 1, thus $w = s\overline{v}$ for a positive real number s (see [7], or [2]). Now, since w and $\overline{v}$ are extensions of the same nontrivial valuation v_L, $s = 1$ and so, $w = \overline{v}$. Let $\overline{F}$ be an algebraic closure of F (and also of L). Since v_L*does* not split in $\overline{L}$, it does not also split in F. But (F, v) is complete, so Henselian, thus v_L does not split in $\overline{F}$, i.e. (L, v_L) is a Henselian field. Hence (F, v) is a strongly Henselian field (see definition 1).

Example 1. *Let $(F, v) = (K((X)), ord)$ be the common rational Laurent power series with coefficients in a field K. Here, for $f = \sum_{n>-\infty}^{\infty} a_n X^n$, $v(f) = \min\{n : a_n \neq 0\}$ is the usual* *order valuation on $F = K((X))$. We know that (F, v) is a complete nonalgebraically closed field (see [4]). Thus, Theorem 1 says that (F, v) is a strongly Henselian field (see also [6] for a particular case).*

Example 2. *Let $(\mathbb{Q}, v_p)$ be the rational number field with the p-adic valuation on it and let $(\mathbb{Q}_p, \widetilde{v}_p)$ be the completion of $(\mathbb{Q}, v_p)$, i.e. the field of p-adic numbers. Let $\mathbb{Q}^h$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{Q}_p$, i.e. the "henselianization" of $(\mathbb{Q}, v_p)$, i.e the least Henselian field which contains $(\mathbb{Q}, v_p)$ (see [4]). It is clear enough that $\mathbb{Q} \neq \mathbb{Q}^h$ and that $\mathbb{Q}^h$ is a Henselian field (Hensel's Lemma works in it) with respect to the restriction of the valuation $\widetilde{v}_p$ on it. Thus, for any subfield $L \subset \mathbb{Q}^h$, $L \neq \mathbb{Q}^h$, the valued field $(L, \widetilde{v}_{p,L})$ is not a Henselian field. Thus $\left(\mathbb{Q}^h, \widetilde{v}_{p,\mathbb{Q}^h}\right)$ is Henselian but not strongly Henselian. Here $\widetilde{v}_{p,L}$ is the restriction of $\widetilde{v}_p$ to L and $\widetilde{v}_{p,\mathbb{Q}^h}$ is the restriction of $\widetilde{v}_p$ to $\mathbb{Q}^h$.*

Corollary 1. *Let (F, v) be a strongly Henselian field and let $L \subseteq F$ be a finite separable coextension of F, i.e. L is a subfield of F, F/L is separable and $[F : L] = n < \infty$. Then F is topologically closed in F.*

Proof: Let $\left(\widetilde{L}, \widetilde{v}_L\right)$ be the topological closure of L in (F, v) and let $(\overline{L}, \overline{v})$ be the least (finite) Galois extension of (L, v_L) which contains (F, v). Here $\overline{v}$ is the unique extension of v to $\overline{L}$. Since (L, v_L) is Henselian, the restriction of $\overline{v}$ to F is equal to v.

Let us assume that $L \subsetneq \widetilde{L}$ and take $\alpha \in \widetilde{L} \setminus L$. Thus, the constant of Krasner

$$\omega(\alpha) = \max\left\{\overline{v}(\alpha - \sigma(\alpha)) : \sigma \in Gal(\overline{L}/L)\right\}$$

is not equal to zero. Since (L, v) is dense in $\left(\widetilde{L}, \widetilde{v}_L\right)$, we can find an element $\beta \in L$ with $\overline{v}(\beta - \alpha) > \omega(\alpha)$. Since (L, v_L) is a Henselian valued field ((F, v) is strongly Henselian), we can apply Krasner's Lemma (see [7], [9], or [2]) and conclude that $L[\alpha] \subset L[\beta] = L$, i.e. $\alpha \in L$, a contradiction. Thus $L = \widetilde{L}$, i.e. L is closed in (F, v). □

We give in the following a generalization of a result of Schilling (see [10] and also [6]).

Proposition 1. *Let (F, v) be a complete nonalgebraically valued field of rank 1 and let K be an arbitrary subfield of F. Then any automorphism $\sigma \in Aut(F/K)$ is continuous with respect to v. Moreover, if v_K, the restriction of v to K is not trivial, then σ is an isometry, i.e. $v \circ \sigma = v$.*

Proof: The new valuation $w = v \circ \sigma$ on F makes F also complete with respect to it (see the proof of Theorem 1). Since (F, v) and (F, w) are complete valued fields and F is not algebraically closed, the F. K. Schmidt theorem on multivalued fields (see [7], or [2]) says that w and v are equivalent as valuations, i.e. $w = sv$ for a positive real number s, i.e. $v \circ \sigma = sv$. Thus σ is continuous relative to v. Moreover, if there exists $z \in K$ with $v(z) \neq 0$, then $s = 1$ and so, σ is an isometry with respect to v. □

The following example says that sometimes in a strongly Henselian field (F, v) one can find infinite algebraic coextensions L which are not closed.

Example 3. *Let again (F, v) be the Laurent power series field $(K((X)), ord)$ from example 1, where we assume that the characteristic of K is equal to zero. Let T be a transcendental basis in $F = K((X))$ over K, which contains the variable X. Then, for $L = K(T)$, the rational function field in variables elements of T, the extension F/L is algebraic and L is dense in F (because $X \in T$). Now, $(1+X)^{\frac{1}{2}} \in F$ and $(1+X)^{\frac{1}{2}} \notin L$ (otherwise, $(1+X)^{\frac{1}{2}} \in K(X)(T \setminus \{X\}) = L$ and it is algebraic over $K(X)$!). Thus we see that (F, v) is an example of a strongly Henselian field (see Theorem 1) with F/L algebraic and L is not closed in F. Hence $[F : L] = \infty$ (see corollary 1).*

Theorem 2 is a characterization of a class of closed subfields (L, v_L) in a large class of valued fields (F, v). Here is the proof of it (it was stated in Section 1).

Proof of Theorem 2. $i) \Rightarrow ii)$ directly comes from the proof of Theorem 1.

For $ii) \Rightarrow iii)$ we can easily use the same idea as in the proof of Corollary 1, because Krasner's Lemma also works for Henselian fields, not only for the complete ones (see [1] and [9] for even a more general frame).

$iii) \Rightarrow i)$ Since (F, v) is complete and (L, v_L) is closed in (F, v), we see that (L, v_L) is also complete. Now, both (F, v) and (L, v_L) are discrete, complete and equal characteristic valued fields, thus F and L are formal power series over their residual fields (see [8]). Hence $L = \widehat{L}((f)) \subset \widehat{F}((\pi)) = F$. Since $[\widehat{F} : \widehat{L}] < \infty$, it remains to prove that $\widehat{F}((f)) \subset \widehat{F}((\pi))$ is finite. To do this, let us firstly observe that for any series $g = \pi^a(a_0 + a_1\pi + ...)$, $a_0 \neq 0$, $a \in \mathbb{N}$, there exists a unique series $h_0 \in \widehat{F}[[\pi]]$ and a unique polynomial $P_0(\pi) = b_0 + b_1\pi + ... + b_{n-1}\pi^{n-1}$, where $n = ord(f)$, such that $g = fh_0 + P_0$ (the division algorithm in $\widehat{F}[[\pi]]$). We repeat this procedure for h instead of g and obtain: $g = P_0 + P_1 f + h_1 f^2$, $h_1 \in \widehat{F}[[\pi]]$ and $\deg h_0 = \deg h_1 \leq n - 1$. Thus

$$g = P_0 + P_1 f + ... + P_k f^k + ...$$

and since $ord(P_k f^k) \geq kn \to \infty$, whenever $n \to \infty$, g is a well defined series in the variable f, with coefficients in the vector space of all polynomials of degree at most $n - 1$ over $\widehat{F}$. Thus $F = \widehat{F}((\pi))$ is a vector space of dimension n over $\widehat{F}((f))$ and the proof of the theorem is now complete.

Remark 2. *Theorem 2 can be viewed as a generalization of the main result of [6]. Namely, if $\widehat{F} = \widehat{L}$, if $\widehat{F}$ contains all the roots of unity and if the characteristic p of F does not divide any $t = [T : L]$ for $L \subset T \subset F$, T finite over L, then $i)$, $ii)$, $iii)$ are also equivalent to the following statement: "F/L is a Galois finite extension". To see this, it is sufficient to change the prime element π of F with another one of the form $\sqrt[n]{fu}$, where u is a unit in L, etc. (see [6] for details).*

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