Bull. Math. Soc. Sci. Math. Roumanie
Tome 58(106) No. 2, 2015, 137–145

Ultrametric q-difference equations and q-Wronskian

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Abstract

Let $\mathbb{K}$ be an ultrametric complete and algebraically closed field and let q be an element of $\mathbb{K}$ which is not a root of unity and is such that $|q| = 1$. In this article, we establish some inequalities linking the growth of generalized q-wronskians of a finite family of elements of $\mathbb{K}[[x]]$ to the growth of the ordinary q-wronskian of this family of power series.

We then apply these results to study some q-difference equations with coefficients in $\mathbb{K}[x]$. Specifically, we show that the solutions of such equations are rational functions.

Key Words: Ultrametric, p-adic, q-difference equation, q-wronskian.
2010 Mathematics Subject Classification: Primary 12J25, Secondary 46S10.

1 Introduction

For every prime number p, we denote by $\mathbb{Q}_p$ the field of p-adic numbers and by $\mathbb{C}_p$ the completion of an algebraic closure of $\mathbb{Q}_p$, (cf. [1] for further details). More generally, in the sequel, $\mathbb{K}$ is a complete ultrametric algebraically closed field.

Given $R > 0$, we denote by $d(0, R^-)$ and $d(0, R)$ the disks: $\{x \in \mathbb{K} \ / \ |x| < R\}$ and $\{x \in \mathbb{K} \ / \ |x| \leq R\}$ respectively. We denote by $\mathcal{A}(\mathbb{K})$ the $\mathbb{K}$-algebra of entire functions in $\mathbb{K}$ and by $\mathcal{M}(\mathbb{K})$ the field of meromorphic functions in $\mathbb{K}$. In the same way, we denote by $\mathcal{A}(d(0, R^-))$ the $\mathbb{K}$-algebra of analytic functions inside the disk $d(0, R^-)$ and by $\mathcal{M}(d(0, R^-))$ the field of meromorphic functions in $d(0, R^-)$.

For every $r \in]0, R[$ we define a multiplicative norm $| \ |(r)$ on $\mathcal{A}(d(0, R^-))$ by $|f|(r) = \sup_{n\geq 0} |a_n|r^n$ for every function $f(x) = \sum_{n\geq 0} a_n x^n$ of $\mathcal{A}(d(0, R^-))$. We extend this to $\mathcal{M}$ $(d(0, R^-))$ by setting $|f|(r) = |g|(r)/|h|(r)$ for every element $f = g/h$ of $\mathcal{M}(d(0, R^-))$, (cf. [5]).

Let q be an element of $\mathbb{K}$ which is not a root of unity and is such that $|q| = 1$. In this work, we will first prove some inequalities linking the growth of a generalized q-Wronskian to the growth of the "ordinary" q-Wronskian.

We then apply this result to study some q-difference equations and show that:
If a linear q-difference equation (E) with coefficients in $\mathbb{K}[x]$ has a complete system of solutions consisting of elements of $\mathcal{M}(\mathbb{K})$, then any solution of (E) is a rational function.

This work has its origins in the articles [3] and [4] where it is established that, in general, a differential equation with coefficients in $\mathbb{K}[x]$ could admit transcendental entire solutions. This study is continued in [2], where J. P. Bézivin gets rationality criteria for solutions of some p-adic differential equations. Here, we study some q-difference equations and show that several types of such equations have no solution except rational functions. The method used is based on a comparison of the growth of q-Wronskians and closely follows the one used in [2].

2 q-difference operators and q-wronskian.

For $n \in \mathbb{N}^*$, we set $[n] = (q^n - 1)/(q-1)$ and $[n]! = \prod_{i=1}^{n}[i]$, (we agree that $[0]! = 1$). For $k \in \mathbb{N}$ such that $k \leq n$, we set $\left[\begin{matrix} n \\ k \end{matrix} \right] = [n]!/([k]!)([n-k]!)$. We easily check that: $\left[\begin{matrix} n+1 \\ k \end{matrix} \right] = \left[\begin{matrix} n \\ k-1 \end{matrix} \right] + q^k \left[\begin{matrix} n \\ k \end{matrix} \right]$. We finally define the operators σ_q and D_q in $\mathbb{K}[[x]]$ by: $\sigma_q(f)(x) = f(qx)$ and $D_q(f)(x) = (\sigma_q - Id)(f)(x)/(q-1)x$. The operator D_q is an endomorphism of the K-vector space $\mathbb{K}[[x]]$. The operator σ_q is an automorphism of the K-algebra $\mathbb{K}[[x]]$ and we have $\sigma_q^{-1} = \sigma_{\frac{1}{q}}$. For $k \in \mathbb{N}^*$, we denote by $\sigma_q^k(f)$ (resp. $D_q^k(f)$) the application $\mathbb{K}$ times of the operator σ_q (resp. D_q) to the formal power series f. We agree that $\sigma_q^0 = D_q^0 = Id$, where Id is the identity mapping in $\mathbb{K}[[x]]$. Some properties of these operators are summarized in the following Lemma:

Lemma 1. *i)* $\sigma_q = (q-1)xD_q + Id$, $D_q = (1/q)D_{(1/q)} \circ \sigma_q$,
ii) $D_q^k \circ \sigma_q^\ell = q^{k\ell}\sigma_q^\ell \circ D_q^k$, $\forall k, \ell \in \mathbb{N}$,
iii) $D_q x - xD_q = \sigma_q$, *and* $D_q x - qxD_q = Id$,
iv) $D_q(fg) = (D_q f)(\sigma_q g) + f(D_q g)$, $\forall f, g \in \mathbb{K}[[x]]$,
v) $D_q((f/g)) = (gD_q f - fD_q g)/g\sigma_q g$, $\forall f, g \in \mathbb{K}[[x]]$,
vi) $D_q^n(fg)(x) = \sum_{k=0}^{n} \left[\begin{matrix} n \\ k \end{matrix} \right] D_q^k(f)\sigma_q^k D_q^{n-k}(g)(x)$, $\forall f, g \in \mathbb{K}[[x]]$.

Let $f_1, \cdots, f_s$, $(s \geq 1)$, be elements of $\mathbb{K}[[x]]$ and let $k_1, \cdots, k_s \in \mathbb{N}$.

Definition 1. *We call q-wronskian (or ordinary q-wronskian) of* $\underline{f} = (f_1, \cdots, f_s)$ *and we denote by* $W_q(\underline{f})$ *the determinant of the matrix* $(D_q^j(f_i))_{1 \leq i \leq s, 0 \leq j \leq s-1}$.

Definition 2. *We call generalized q-wronskian of* $\underline{f} = (f_1, \cdots, f_s)$ *relatively to* $\underline{k} = (k_1, \cdots, k_s)$ *and we denote by* $W_q(\underline{f}; \underline{k})$ *the determinant of the matrix* $(D_q^{k_j}(f_i))_{1 \leq i \leq s, 1 \leq j \leq s}$.

Remark 1. *1) The ordinary q-wronskian of* $\underline{f} = (f_1, \cdots, f_s)$ *is equal to the generalized q-wronskian* $W_q(\underline{f}; \underline{k}_s)$ *of* $\underline{f}$ *relatively to* $\underline{k}_s = (0, 1, ..., s-1)$.
2) More generally, let $\underline{k}_j = (0, \cdots, \hat{j}, \cdots, s) = (0, \cdots, j-1, j+1, \cdots, s)$ *for every* $j \in \{0, \cdots, s\}$. *If we consider the usual derivation* $D = d/dx$, *we obtain a family of (usual) generalized wronksians* $W(\underline{f}; \underline{k}_j)$ *of* $\underline{f} = (f_1, \cdots, f_s)$, *for* $0 \leq j \leq s$. *And we easily check :* $DW(\underline{f}; \underline{k}_s) = W(\underline{f}; \underline{k}_{(s-1)})$.

Now, consider the family of generalized q-wronksians $W_q(\underline{f};\underline{k}_j)$ *of* $\underline{f}$ *for* $0 \leq j \leq s$. *We see that, for* $D_qW_q(\underline{f};\underline{k}_s)$, *we do not have an expression as simple as the one above. However, the following lemma allows us to express* $D_qW_q(\underline{f};\underline{k}_s)$ *as a combination of all the generalized q-wronksians* $W_q(\underline{f};\underline{k}_j)$ *of* $\underline{f}$ *for* $0 \leq j \leq s-1$.

Lemma 2. *With the notations above, we have:*

$$D_q(W_q(\underline{f};\underline{k}_s)) = \sum_{j=0}^{s-1}[(q-1)x]^{s-1-j}W_q(\underline{f};\underline{k}_j).$$

Let $s \geq 2$ and let $f_1, \cdots, f_s$ be elements of $KK[[x]]$, linearly independent over $\mathbb{K}$. Let us set $\underline{f} = (f_1, \cdots, f_s)$ and $\underline{k}_j = (0, \cdots, \hat{j}, \cdots, s)$ for $0 \leq j \leq s$. Let us also set $\underline{g} = (f_1, \cdots, f_{s-1})$, $\underline{\ell}_i = (0, \cdots, \hat{i}, \cdots, s-1)$ for $0 \leq i \leq s-1$ and $\underline{\ell}_{(i,s-1)} = (0, \cdots, i-1, i+1, \cdots, s-2, s)$ for $0 \leq i \leq s-2$. Recall that $W_q(\underline{f};\underline{k}_j)$ is the generalized q-wronskian of $\underline{f}$ relatively to $\underline{k}_j$, for $0 \leq j \leq s$. In the same way, $W_q(\underline{g};\underline{\ell}_i)$ is the generalized q-wronskian of $\underline{g}$ relatively to $\underline{\ell}_i$, for $0 \leq i \leq s-1$. Finally, $W_q(\underline{g};\underline{\ell}_{(i,s-1)})$ is the generalized q-wronskian of $\underline{g}$ relatively to $\underline{\ell}_{(i,s-1)}$, for $0 \leq i \leq s-2$. In the following lemma, the q-derivative of the q-wronskian is given by an expression which is better suited for the comparison of the growth of q-wronskians.

Lemma 3. *With the notations above, we have for* $s \geq 2$*:*

i) $\frac{W_q(\underline{f};\underline{k}_j)}{W_q(\underline{f};\underline{k}_s)} = \frac{W_q(\underline{g};\underline{\ell}_j)}{W_q(\underline{g};\underline{\ell}_{(s-1)})}\frac{W_q(\underline{f};\underline{k}_{(s-1)})}{W_q(\underline{f};\underline{k}_s)} - \frac{W_q(\underline{g};\underline{\ell}_{(j,s-1)})}{W_q(\underline{g};\underline{\ell}_{(s-1)})}, \quad \forall\ 0 \leq j \leq s-2;$

ii) $\frac{W_q(\underline{f};\underline{k}_{(s-1)})}{W_q(\underline{f};\underline{k}_s)} = \frac{W_q(\underline{g};\underline{\ell}_{(s-1)})}{\sigma_q W_q(\underline{g};\underline{\ell}_{(s-1)})}\frac{D_qW_q(\underline{f};\underline{k}_s)}{W_q(\underline{f};\underline{k}_s)} + (\sum_{i=0}^{s-2}[(q-1)x]^{s-1-i}\frac{W_q(\underline{g};\underline{\ell}_{(i,s-1)})}{\sigma_q W_q(\underline{g};\underline{\ell}_{(s-1)})}).$

Here, we only consider the case $|q| = 1$. Indeed the case $|q| \neq 1$ is more difficult and will be treated later. Hence, from now on, we make this assumption: q is an element of $\mathbb{K}$ which is not a root of unity and is such that $|q| = 1$.

3 Growth of the q-wronskians

In the following result, we give inequalities linking the growth of generalized q-wronskians of a family of analytic functions to that of the of ordinary q-wronskian of this family of functions.

Theorem 1. *Let* s *be an integer* ≥ 1 *and let* $f_1, \cdots, f_s$ *be* s *elements of* $\mathcal{A}(d(0,R^-))$. *Let* $k_1, \cdots, k_s$ *be integers* ≥ 0. *Let* $\underline{k} = (k_1, \cdots, k_s)$, *and* $\underline{k}_s = (0, 1, \cdots, s-1)$. *For every* $\rho \in]0, R[$, *we have:*

i) $|W_q(\underline{f};\underline{k})|(\rho) \leq |W_q(\underline{f};\underline{k}_s)|(\rho)/\rho^{k_1+k_2+\cdots+k_s-\frac{s(s-1)}{2}}$.

Particularly, for $\underline{k}_j = (0, \cdots, \hat{j}, \cdots, s)$, $j = 0, \cdots, s$, *we have:*

ii) $|W_q(\underline{f};\underline{k}_j)|(\rho) \leq |W_q(\underline{f};\underline{k}_s)|(\rho)/\rho^{s-j}$.

In order to prove Theorem 1, we will first deal with the case $s \leq 2$ and then proceed by induction. The following lemma is easily shown by using Lemma 1.

Lemma 4. *Let* $R > 0$ *and let* f *be an element of* $\mathcal{M}(d(0,R^-))$. *For every* $\rho \in]0, R[$ *and every* $s \in \mathbb{N}$, *we have:* $|\sigma_q^s(f)|(\rho) = |f|(\rho)$ *and* $|D_q^s(f)|(\rho) \leq |f|(\rho)/\rho^s$.

We also have:

Lemma 5. *Let $f_1, f_2 \in \mathcal{A}(d(0,R^-))$ be linearly independent over $\mathbb{K}$. Let us set: $\underline{f} = (f_1, f_2), \underline{k}_2 = (0,1)$, $\underline{k}_1 = (0,2)$, and $\underline{k}_0 = (1,2)$. Then, for every $\rho \in]0,R[$, we have: $|W_q(\underline{f};\underline{k}_0)|(\rho) \leq |W_q(\underline{f};\underline{k}_2)|(\rho)/\rho^2$ and $|W_q(\underline{f};\underline{k}_1)|(\rho) \leq |W_q(\underline{f};\underline{k}_2)|(\rho)/\rho$.*

Proof: We apply Lemma 3 with $s = 2$ and the same notations. As $|q| = 1$, we complete the proof by using Lemma 4. □

Let $y = y(x) \in \mathcal{A}(d(0,R^-))$. We define two sequences $A_{n,k} = A_{n,k}(x)$, $(k = 0;\ 1)$ of elements of $\mathcal{M}(d(0,R^-))$ in the following way: $A_{1,0} = 0$, $A_{0,0} = 1$, $A_{1,1} = 1$, $A_{0,1} = 0$, and $D_q^n y(x) = A_{1,n}(x)D_q y(x) + A_{0,n}(x)y(x)$. We further define A_0 and A_1 by: $D_q^2 y(x) = A_1(x)D_q y(x) + A_0(x)y(x)$.
If $f_1, f_2 \in \mathcal{A}(d(0,R^-))$ are two solutions of the above equation linearly independent over $\mathbb{K}$, we have: $A_0 = W_q(\underline{f};\underline{k}_1)/W_q(\underline{f};\underline{k}_2)$ and $A_1 = -W_q(\underline{f};\underline{k}_0)/W_q(\underline{f};\underline{k}_2)$.

The following formulas are easily checked.

Lemma 6. *We have the following induction relations for every integer $n \geq 0$:*
i) $A_{1,n+1} = A_1\sigma_q A_{1,n} + \sigma_q A_{0,n} + D_q A_{1,n}$;
ii) $A_{0,n+1} = A_0\sigma_q A_{1,n} + D_q A_{0,n}$.

Proposition 1. *Let f_1, f_2 be two elements of $\mathcal{A}(d(0,R^-))$. Let $\underline{\ell} = (0,1)$ and let $\underline{k} = (k_1,k_2)$ be a pair of positive integers. We have, for every $\rho \in]0,R[$, the inequality: $|W_q(\underline{f};\underline{k})|(\rho) \leq |W_q(\underline{f};\underline{\ell})|(\rho)/\rho^{k_1+k_2-1} = |W_q(\underline{f})|(\rho)/\rho^{k_1+k_2-1}$.*

Proof: Using Lemma 5, we show first that for every $n \geq 0$ and every $\rho \in]0,R[$, we have: $|A_{1,n}|(\rho) \leq 1/\rho^{n-1}$ and $|A_{0,n}|(\rho) \leq 1/\rho^n$. Then we can write:

$$\begin{pmatrix} D_q^{k_1}(f_1) & D_q^{k_2}(f_1) \\ D_q^{k_2}(f_2) & D_q^{k_2}(f_2) \end{pmatrix} = \begin{pmatrix} f_1 & D_q f_1 \\ f_2 & D_q f_2 \end{pmatrix} \begin{pmatrix} A_{0,k_1} & A_{0,k_2} \\ A_{1,k_1} & A_{1,k_2} \end{pmatrix}.$$

Taking the determinant of both sides, we express $W_q(\underline{f},\underline{k})$ as a function of the $A_{m,j}$'s and $W(\underline{f})$, and we deduce the result.

□

We are now able to prove Theorem 1.

Proof: (**of Theorem 1**)

We proceed by induction. By Lemma 4 and Proposition 1, the inequalities *i)* and *ii)* are true for $s \leq 2$. Suppose that these inequalities are true up to a rank $s \geq 2$.
Now, let $f_1, \cdots, f_s, f_{s+1} \in \mathcal{A}(d(0,R^-))$ be linearly independent over $\mathbb{K}$. Let us set $\underline{f} = (f_1, \cdots, f_s, f_{s+1})$, and $\underline{k}_j = (0, \cdots, \hat{j}, \cdots, s+1)$ for $0 \leq j \leq s+1$. Let us also set $\underline{g} = (f_1, \cdots, f_s)$, $\underline{\ell}_i = (0, \cdots, \hat{i}, \cdots, s)$ for $0 \leq i \leq s$ and $\underline{\ell}_{(i,s)} = (0, \cdots, \hat{i}, \cdots, s-1, s+1)$ for $0 \leq i \leq s-1$.
By Lemma 3, we have:

(1) $\dfrac{W_q(\underline{f};\underline{k}_j)}{W_q(\underline{f};\underline{k}_{(s+1)})} = \dfrac{W_q(\underline{g};\underline{\ell}_j)}{W_q(\underline{g};\underline{\ell}_s)}\dfrac{W_q(\underline{f};\underline{k}_s)}{W_q(\underline{f};\underline{k}_{(s+1)})} - \dfrac{W_q(\underline{g};\underline{\ell}_{(j,s)})}{W_q(\underline{g};\underline{\ell}_s)}, \quad \forall\ 0 \le j \le s-1;$

(2) $\dfrac{W_q(\underline{f};\underline{k}_s)}{W_q(\underline{f};\underline{k}_{(s+1)})} = \dfrac{W_q(\underline{g};\underline{\ell}_s)}{\sigma_q W_q(\underline{g};\underline{\ell}_s)}\dfrac{D_q W_q(\underline{f};\underline{k}_{(s+1)})}{W_q(\underline{f};\underline{k}_{(s+1)})} + \sum_{i=0}^{s-1}[(q-1)x]^{s-i}\dfrac{W_q(\underline{g};\underline{\ell}_{(i,s)})}{\sigma_q W_q(\underline{g};\underline{\ell}_s)}.$

By Lemma 4, we have: $\frac{\left|W_q(\underline{g};\underline{\ell}_s)\right|(\rho)}{\left|\sigma_q W_q(\underline{g};\underline{\ell}_s)\right|(\rho)} = 1$ and $\frac{\left|D_q W_q(\underline{f};\underline{k}_{(s+1)})\right|(\rho)}{\left|W_q(\underline{f};\underline{k}_{(s+1)})\right|(\rho)} \le \frac{1}{\rho}, \ \ \forall \rho \in]0, R[.$
From the hypothesis, we deduce that for every $0 \le i \le s-1$:
$|[(q-1)x]^{s-i}|(\rho)\frac{|W_q(\underline{g};\underline{\ell}_{(i,s)})|(\rho)}{|\sigma_q W_q(\underline{g};\underline{\ell}_s)|(\rho)} \le 1/\rho.$
It follows, by some calculation, that the inequality *ii*) is true for the rank $s+1$ and is therefore true for $s \ge 1$. This completes the proof of Inequality *ii*).
Let us now prove inequality *i*)
The equation verified by $f_1, \cdots, f_{s+1}$ is: $\sum_{j=0}^{s+1}(-1)^j W_q(\underline{f};\underline{k}_j)D_q^j y = 0.$
This equation can be written in the following form:
(3) $D_q^{s+1}y = \sum_{j=0}^{s} A_j D_q^j y,$ where $A_j = (-1)^{s-j}W_q(\underline{f};\underline{k}_j)/W_q(\underline{f};\underline{k}_{(s+1)}).$
More generally, for every $n \ge 0$, let us set:
(4) $D_q^n y = \sum_{j=0}^{s} A_{j,n} D_q^j y$,
where the $A_{j,n}$'s are elements of $\mathcal{M}(d(0, R^-))$ satisfying the following relations:
(5) $A_{j,n} = 0$ if $j \ne n$ and $A_{j,n}1$ if $j = n$, for $0 \le n \le s$;
(6) $A_{j,s+1} = A_j$, , for $0 \le j \le s$;
(7) $A_{0,n+1} = D_q A_{0,n} + A_0\sigma_q A_{s,n}$ and $A_{j,n+1} = D_q A_{j,n} + A_j \sigma_q A_{s,n} + \sigma_q A_{j-1,n}$ for $1 \le j \le s$.
Let us now show that, for every $j \in \{0, \cdots, s+1\}$ and every $\rho \in]0, R[$, we have:
(8) $|A_{j,n}|(\rho) \le 1/\rho^{n-j}, \ \ \forall n \ge 0.$
Inequality (8) is trivial for $0 \le n \le s$ because of the formula (5). Using (6) and (2), we see that Inequality (8) is true for $n = s+1$. Using (7) and Proposition 1, we complete the proof of Inequality (8) by induction on n.
Now, we have the formula:

$$\begin{pmatrix} D_q^{k_1} f_1 & \cdots & D_q^{k_{s+1}} f_1 \\ \vdots & \vdots & \vdots \\ D_q^{k_1} f_{s+1} & \cdots & D_q^{k_{s+1}} f_{s+1} \end{pmatrix} = \begin{pmatrix} f_1 & \cdots & D_q^s f_1 \\ \vdots & \vdots & \vdots \\ f_{s+1} & \cdots & D_q^s f_{s+1} \end{pmatrix} \begin{pmatrix} A_{0,k_1} & \cdots & A_{0,k_{s+1}} \\ \vdots & \vdots & \vdots \\ A_{s,k_1} & \cdots & A_{s,k_{s+1}} \end{pmatrix}.$$

Taking the determinants of both sides, we have: $W_q(\underline{f};\underline{k}) = \Delta W_q(\underline{f};\underline{k}_s)$, where Δ is the determinant of the matrix

$$\begin{pmatrix} A_{0,k_1} & \cdots & A_{0,k_{s+1}} \\ \vdots & \vdots & \vdots \\ A_{s,k_1} & \cdots & A_{s,k_{s+1}} \end{pmatrix}$$

We complete then the proof of *ii*) by showing that $|\Delta|(\rho) \le \dfrac{1}{\rho^{(k_1+\cdots+k_{s+1})-\frac{s(s+1)}{2}}}$ as in Theorem 2.1 of [2].

□

Remark 2. *The property $|q| = 1$ is used when it is stated that, for a meromorphic function φ, we have $|\sigma_q(\varphi)|(\rho) = |\varphi|(\rho)$, which is not true in general if $|q| \neq 1$. So, generalizing our results to any $|q|$ is not at all clear and would require a deep change in the method of proof.*

Now, we extend the result of Theorem 1 to meromorphic functions.

Corollary 1. *Let $f_1, \cdots, f_s$, be elements of $\mathcal{M}(d(0, R^-))$ and let $k_1, \cdots, k_s$ be integers ≥ 0. Let $\underline{f} = (f_1, \cdots, f_s)$, $\underline{k} = (k_1, \cdots, k_s)$ and $\underline{k}_s = (0, \cdots, s-1)$. Then, we have for every $\rho \in]0, R[$:* $|W_q(\underline{f}; \underline{k})|(\rho) \leq \dfrac{|W_q(\underline{f}; \underline{k}_s)|(\rho)}{\rho^{(k_1+\cdots+k_s)-\frac{s(s-1)}{2}}}$.

Proof: Let $\rho \in]0, R[$ and let $r \in]\rho, R[$. Then, there exists a nonzero polynomial P such that: $g_1 = Pf_1, \cdots, g_s = Pf_s$ are elements of $\mathcal{A}(d(0, r^-))$. We can easily prove that: $W_q(\underline{f}; \underline{k}_s) = (\prod_{j=0}^{s-1} \sigma_q^j P)^{-1} W_q(\underline{g}; \underline{k}_s)$, and then: $|W_q(\underline{f}; \underline{k}_s)|(\rho) = |W_q(\underline{g}; \underline{k}_s)|(\rho)(|P|(\rho))^{-s}$.
Since the g_i's are analytic functions in $d(0, r^-)$, by Theorem 1 we have:
$|W_q(\underline{g}; \underline{\ell})|(\rho) \leq |W_q(\underline{g}; \underline{k}_s)|(\rho)/\rho^{(\ell_1+\cdots+\ell_s)-\frac{s(s-1)}{2}}$.
From this and the property of the ultrametric inequality we get:
$|W_q(\underline{f}; \underline{k})|(\rho) \leq \frac{|W_q(\underline{g}; \underline{k}_s)|(\rho)}{|P|^s(\rho)} \frac{1}{\rho^{(k_1+\cdots+k_s)-\frac{s(s-1)}{2}}} = \frac{|W_q(\underline{f}, \underline{k}_s)|(\rho)}{\rho^{(k_1+\cdots+k_s)-\frac{s(s-1)}{2}}}$.
That completes the proof of Corollary 1. □

The following result gives an algebraic property of the q-wronskians of polynomials or rational functions. Recall that if $P(x)$, $Q(x)$ are polynomials, then the *algebraic degree* of the rational function $R(x) = P(x)/Q(x)$ is $\deg_a R = \deg P - \deg Q$.

Corollary 2. *Let L be a field and let q be a nonzero element of L different from any root of unity. Let $Q_1, \cdots, Q_s$, $s \geq 1$, be elements of $L(x)$ linearly independent over L. Let $\underline{Q} = (Q_1, \cdots, Q_s)$, $\underline{k}_s = (0, \cdots, s-1)$ and $\underline{k} = (k_1, \cdots, k_s)$, where $k_1, \cdots, k_s$ are integers ≥ 0. Let d_1, d_2 be the algebraic degrees of the rational functions $W_q(\underline{Q}; \underline{k}_s)$ and $W_q(\underline{Q}; \underline{k})$ respectively. Then we have:*

$$d_2 \leq d_1 + \tfrac{s(s-1)}{2} - (k_1 + \cdots + k_s).$$

Proof: We may assume that L is an algebraically closed field equipped with the trivial absolute value $|\ |_0$ defined by $|0|_0 = 0$ and $|x|_0 = 1$ if $x \neq 0$. Then it is clear that L is a complete ultrametric field with respect to this absolute value. Moreover the entire functions (resp. meromorphic functions) on L are just the polynomials (resp. rational functions) on L. On the one hand, we have:
(1) $|W_q(\underline{Q}; \underline{k}_s)|(\rho) = \rho^{d_1}$ and $|W_q(\underline{Q}; \underline{k})|(\rho) = \rho^{d_2}$, for every $\rho > 1$.
On the other hand, by Theorem 1, we have:
(2) $|W_q(\underline{Q}; \underline{k})|(\rho) \leq |W_q(\underline{Q}; \underline{k}_s)|(\rho)/\rho^{(k_1+\cdots+k_s)-\frac{s(s-1)}{2}}$.
From (1) and (2), we have: $1 \leq \rho^{d_1-d_2+\frac{s(s-1)}{2}-(k_1+\cdots+k_s)}$.
The required inequality follows immediately. □

Theorem 2. *Let $f_1, \cdots, f_s,\ s \geq 1$, be elements of $\mathcal{A}(\mathbb{K})$, $\underline{f} = (f_1, \cdots, f_s)$ and $\underline{k}_s = (0, \cdots, s-1)$. Suppose that the q-Wronskian $W_q(\underline{f}, \underline{k}_s)$ is a nonzero polynomial. Then $f_1, \cdots, f_s$ are polynomials.*

Proof: The result is trivial for $s = 1$. Suppose that $s \geq 2$ is such that the result is true for $s-1$. So, by hypothesis, $W_q(\underline{f}, \underline{k}_s)$ is a nonzero polynomial $P(x)$.
Let us first consider the case when $P(x)$ is a constant C. Then, by Theorem 1, we have: $|W_q(\underline{f}; \underline{k}_j)|(\rho) \leq |W_q(\underline{f}; \underline{k}_s)|(\rho)/\rho^{s-j} = |C|/\rho^{s-j}$, for $j = 0, \cdots, s-1$.
The considered functions being entire, this implies that: $W_q(\underline{f}; \underline{k}_j) = 0$, for $j = 0, \cdots, s-1$. The q-difference equation verified by $f_1, \cdots, f_s$ is then reduced to $CD_q^s y = 0$, which implies easily that $f_1, \cdots, f_s$ are polynomials.

We then proceed by induction on the degree of the polynomial $P(x)$. Suppose that the result is true if $P(x)$ is of degree $\leq n$ and consider the case when $P(x)$ is of the degree $n+1$. By Theorem 1, we have: $|W_q(\underline{f}; \underline{k}_0)|(\rho) \leq |P|(\rho)/\rho^s$. Hence, by Liouville ultrametric Theorem, we see that $W_q(\underline{f}; \underline{k}_0)$ is a polynomial of degree $\leq n+1-s < n$. If this polynomial is nonzero, then $D_q f_1, \cdots, D_q f_s$ are polynomials by the induction hypothesis and thus $f_1, \cdots, f_s$ are polynomials.
If the polynomial $W_q(\underline{f}; \underline{k}_0)$ is null, then the system $D_q f_1, \cdots, D_q f_s$ is of rank $r \leq s-1$. We may assume that $D_q f_1, \cdots, D_q f_r$ are linearly independent. Then every $D_q f_j$ is a linear combination of $D_q f_1, \cdots, D_q f_r$ and thus every f_j is a linear combination of $f_1, \cdots, f_r$ and the constant function 1. Hence, the $\mathbb{K}$-vector subspace generated by the functions $f_1, \cdots, f_s$ (of dimension s) is included in the $\mathbb{K}$-vector subspace generated by $f_1, \cdots, f_r, 1$ (of dimension $\leq r+1$) and therefore $s \leq r+1$. Finally, it follows that $r = s-1$. So we may assume that $D_q f_1, \cdots, D_q f_{s-1}$ are linearly independent and that $D_q f_s$ is a linear combination of $D_q f_1, \cdots,\ D_q f_{s-1}$ with coefficients in $\mathbb{K}$: $D_q f_s = a_1 D_q f_1 + a_2 D_q f_2 + \cdots + a_{s-1} D_q f_{s-1}$. We deduce that $f_s = a_1 f_1 + a_2 f_2 + \cdots + a_{s-1} f_{s-1} + b$ with a nonzero constant b. We can easily see that the q-wronskian of $f_1, \cdots, f_s$ is equal (up to sign) to b multiplied by the q-wronskian of $D_q f_1, \cdots, D_q f_{s-1}$. Hence, this last q-wronskian is a nonzero polynomial, and the induction hypothesis on s shows that $D_q f_1, \cdots, D_q f_{s-1}$ are polynomials and then $f_1, \cdots, f_{s-1}$ are polynomials. The formula $f_s = a_1 f_1 + a_2 f_2 + \cdots + a_{s-1} f_{s-1} + b$ then shows that f_s, too, is a polynomial. Thus the proof of Theorem 2. is completed. □

Remark 3. *The previous result does not extend to $\mathcal{M}(\mathbb{K})$. Indeed, let g be a non-polynomial entire function and let h be an entire function such that $D_q h = g \sigma_q g$. Let $f_1 = 1/g$, and $f_2 = h/g$. We see that $f_1,\ f_2$ are non-rational meromorphic functions while the q-wronskian of $f_1,\ f_2$ is equal to 1.*

Theorem 3. *Let $P_0, \cdots, P_s,\ s \geq 1$, be elements of $KK[x]$ such that $P_s \neq 0$. Suppose that the equation: $(E)\quad P_s D_q^s y + \cdots + P_1 D_q y + P_0 y = 0$ has a complete system of solutions in $\mathcal{A}(\mathbb{K})$. Then every entire solution of (E) is a polynomial.*

Proof: Let $f_1, \cdots, f_s$ be entire functions in $\mathbb{K}$, making a basis of the $\mathbb{K}$-vector space of solutions of Equation (E). Then the q-wronskian $W = W_q(\underline{f}; \underline{k}_s)$ of $f_1, \cdots, f_s$ is a nonzero entire function. An immediate calculation gives:

(1) $\quad P_s D_q W_q(\underline{f};\underline{k}_s) + (\sum_{i=0}^{s-1}[(1-q)x]^{s-1-i}P_i)W_q(\underline{f};\underline{k}_s) = 0.$

If $\sum_{i=0}^{s-1}[(1-q)x]^{s-1-i}P_i = 0$, then $W_q(\underline{f};\underline{k}_s)$ is a nonzero constant. It follows, by Theorem 2, that $f_1, \cdots, f_s$ are polynomials. We therefore assume in the following that $\sum_{i=0}^{s-1}[(1-q)x]^{s-1-i}P_i \neq 0$. Let $R > 0$ be such that all zeros of the polynomials P_s and $\sum_{i=0}^{s-1}[(1-q)x]^{s-1-i}P_i$ lie in the disk $d(0,R)$. Suppose that the function $W_q(\underline{f};\underline{k}_s)$ admits a zero α such that $|\alpha| = \rho > R$. Then, by (1), we have $D_q W_q(\underline{f};\underline{k}_s)(\alpha) = 0$. By Lemma 1, we have $\sigma_q W_q(\underline{f};\underline{k}_s) = (q-1)xD_q W_q(\underline{f};\underline{k}_s) + W_q(\underline{f};\underline{k}_s)$. It follows that $W_q(\underline{f};\underline{k}_s)(q\alpha) = \sigma_q W_q(\underline{f};\underline{k}_s)(\alpha) = 0$. As $|q| = 1$, an immediate induction then shows that the function $W_q(\underline{f};\underline{k}_s)$ has infinitely many zeros in the disk $d(0,\rho)$, which is a contradiction. So $W_q(\underline{f};\underline{k}_s)$ has all its zeros in the disk $d(0,R)$. This means that $W_q(\underline{f};\underline{k}_s)$ has only finitely many zeros and is consequently a polynomial. Theorem 2 then shows that $f_1, \cdots, f_s$ are polynomials, which ends the proof of the theorem. □

We can now generalize the above result to $\mathcal{M}(\mathbb{K})$:

Theorem 4. *Let $P_0, \cdots, P_s$, $s \geq 1$, be elements of $KK[x]$ such that $P_s \neq 0$. Suppose that the equation: (E) $\quad P_s D_q^s y + \cdots + P_1 D_q y + P_0 y = 0$ has a complete system of solutions in $\mathcal{M}(\mathbb{K})$. Then every solution of (E) is a rational function.*

Proof: Let $f_1, \cdots, f_s$ be elements of $\mathcal{M}(\mathbb{K})$, making a basis of the $\mathbb{K}$-vector space of solutions of Equation (E). Using the formula $\sigma_q y = (q-1)xD_q y + y$ we deduce that Equation (E) is equivalent to:

(E') $\quad Q_s(x)\sigma_q^s y(x) + \cdots + Q_0(x)y(x) = 0,$

where $Q_0, \cdots, Q_s$ are elements of $KK[x]$ such that $Q_s = P_s$. We may assume, without loss of generality, that $Q_0 \neq 0$. Let y be a solution of (E') in $\mathcal{M}(\mathbb{K})$ and let ω be a pole of y which is not a zero of Q_0. It follows that there exists $\ell_1 \geq 1$ such that $q^{\ell_1}\omega$ is a pole of y. We can not continue this process indefinitely. So there exists an integer $\ell_\omega \geq 0$ such that for every $j \geq 1$, $q^{\ell_\omega + j}\omega$ is not a pole of y. It follows, from Equation (E'), that the function $Q_0(q^{\ell_\omega}x)y(q^{\ell_\omega}x)$ has no longer ω as a pole. Therefore, $q^{\ell_\omega}\omega$ is a zero of $Q_0(x)$. Let $R > 0$ be such that all zeros of the polynomial $Q_0(x)$ are contained in the disk $d(0,R)$. It follows that all poles of y are in the disk $d(0,R)$. Consequently, y only has finitely many poles. Applying this to $f_1, \cdots, f_s$, we see that there exists a polynomial $H(x)$, such that $g_1(x) = H(x)f_1(x), \cdots, g_s(x) = H(x)f_s(x)$ are entire functions in $\mathbb{K}$. Moreover, these functions are linearly independent and satisfy a q-difference equation of order s with polynomial coefficients. We conclude by using Theorem 3.

□

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Received: 10.01.2014
Revised: 26.06.2014
Accepted: 20.07.2014

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