Bull. Math. Soc. Sci. Math. Roumanie
Tome 58(106) No. 2, 2015, 129–136

On the Forcing Dimension of a Graph

by
Behrooz Bagheri Ghavam Abadi[1] and Ali Zaghian[2]

Abstract

A set $W \subseteq V(G)$ is called a resolving set, if for each two distinct vertices $u, v \in V(G)$ there exists $w \in W$ such that $d(u,w) \neq d(v,w)$, where $d(x,y)$ is the distance between the vertices x and y. A resolving set for G with minimum cardinality is called a metric basis. The forcing dimension $f(G, dim)$ (or $f(G)$) of G is the smallest cardinality of a subset $S \subset V(G)$ such that there is a unique basis containing S. The forcing dimensions of some well-known graphs are determined. In this paper, among some other results, it is shown that for large enough integer n and all integers a, b with $0 \leq a \leq b \leq n$ and $b \geq 1$, there exists a nontrivial connected graph G of order n with $f(G) = a$ and $dim(G) = b$ if $\{a, b\} \neq \{0, 1\}$.

Key Words: Resolving set, metric basis, forcing dimension, basis number.
2010 Mathematics Subject Classification: Primary 05C12, Secondary 05C15, 05C62.

1 Introduction

Throughout the paper, $G = (V, E)$ is a finite, simple, and connected graph of order n. The distance between two vertices u and v, denoted by $d(u,v)$, is the length of a shortest path between u and v in G. The diameter of G is diam(G) $= \max\{d(u,v) \mid u, v \in V(G)\}$. The girth of G is the length of a shortest cycle in G. The set of all adjacent vertices to a vertex v is denoted by $N(v)$ and $|N(v)|$ is the degree of a vertex v, deg(v). The maximum degree and the minimum degree of a graph G, are denoted by $\Delta(G)$ and $\delta(G)$, respectively. The notations $u \sim v$ and $u \nsim v$ denote the adjacency and non-adjacency relations between u and v, respectively. The Cartesian product of two graphs G and H, denoted by $G \square H$, is the graph with vertex set $V(G) \times V(H)$ and two vertices (u,v) and (x,y) are adjacent if and only if either $u = x$ and $vy \in E(H)$ or $ux \in E(G)$ and $v = y$.

For an ordered set $W = \{w_1, w_2, \ldots, w_k\} \subseteq V(G)$ and a vertex v of G, the k-vector

$$r(v|W) = (d(v,w_1), d(v,w_2), \ldots, d(v,w_k))$$

is called the metric representation of v with respect to W. The set W is called a resolving set for G if distinct vertices have different metric representations. A resolving set for G with minimum cardinality is called a metric basis, and its cardinality is the metric dimension of G, denoted by $dim(G)$. If $dim(G) = k$, then G is said to be k-dimensional. A basis number of G, $bas(G)$, is the maximum r such that every r-set S of vertices of G is a subset of some basis of G.

In [16], Slater introduced the idea of a resolving set and used a locating set and the location number for what we call a resolving set and the metric dimension, respectively. He described the usefulness of these concepts when working with U.S. Sonar and Coast Guard Loran stations. Independently, Harary and Melter [10] discovered the concept of the location number as well and called it the metric dimension. For more results related to these concepts see [5, 6, 8, 13]. The concept of a resolving set has various applications in diverse areas including coin weighing problems [15], network discovery and verification [3], robot navigation [13], mastermind game [5], problems of pattern recognition and image processing [14], and combinatorial search and optimization [15].

It is obvious that to see whether a given set W is a resolving set, it is sufficient to consider the vertices in $V(G)\backslash W$, because $w \in W$ is the unique vertex in G for which $d(w, w) = 0$. When W is a resolving set for G, we say that W resolves G. In general, we say an ordered set W resolves a set $T \subseteq V(G)$, if for each two distinct vertices $u, v \in T$, $r(u|W) \neq r(v|W)$.

The following bound is the known upper bound for the metric dimension.

Theorem 1. [7] *If G is a connected graph of order n and diameter d, then $dim(G) \leq n - d$.*

For a basis W of G, a subset S of W is called a forcing subset of W if W is the unique basis containing S. The forcing number $f_G(W, dim)$ of W in G is the minimum cardinality of a forcing subset for W, while the forcing dimension $f(G, dim)$ (or $f(G)$) of G is the smallest forcing number among all bases of G.

It is immediate that $f(G) = 0$ if and only if G has a unique basis. If G has no unique basis but contains a vertex belonging to only one basis, then $f(G) = 1$. Moreover, if for every basis W of G and every proper subset S of W, the set W is not the unique basis containing S, then $f(G) = dim(G)$.

Theorem 2. [9] *Let G be a nontrivial connected graph. If G is a complete graph, cycle, or tree, then $f(G) = dim(G)$.*

Theorem 3. [9] *Let G be a connected graph of order $n \geq 2$ with $dim(G) = n - 2$. If $G = K_{r,s}$ $(r, s \geq 1)$ or $G = K_r + \overline{K_s}$ $(r \geq 1, s \geq 2)$, then $f(G) = dim(G)$. If $G = K_r + (K_1 \cup K_s)$ $(r, s \geq 1)$, then $f(G) = dim(G) - 1$.*

Obviously, if G is a graph with $f(G) = a$ and $dim(G) = b$, then $0 \leq a \leq b$ and $b \geq 1$. In [9], it is determined which pairs a, b of integers with $0 \leq a \leq b$ and $b \geq 1$ are realizable as the forcing dimension and dimension of some nontrivial connected graph.

Theorem 4. [9] *For all integers a, b with $0 \leq a \leq b$ and $b \geq 1$, there exists a nontrivial connected graph G with $f(G) = a$ and $dim(G) = b$ if and only if $\{a, b\} \neq \{0, 1\}$.*

In this paper, some lower bounds for the forcing dimension of graphs are obtained. It is shown that for all integers a, b with $0 \leq a \leq b$ and $b \geq 1$, and for every sufficiently large n, there exists a nontrivial connected graph G of order n with $f(G) = a$ and $dim(G) = b$ if $\{a, b\} \neq \{0, 1\}$.

2 Some lower bounds

In this section we obtain some lower bounds for the forcing dimension of graphs.

Two vertices $u, v \in V(G)$ are called twin vertices if $N(u) \setminus \{v\} = N(v) \setminus \{u\}$. It is known that, if u and v are twin vertices, then every resolving set W for G contains at least one of the vertices u and v. Moreover, if $u \notin W$ then $(W \setminus \{v\}) \cup \{u\}$ is also a resolving set for G [11]. A twin class of a vertex v, T_v, is $\{u \in V(G) \mid u \text{ and } v \text{ are twin vertices.}\}$. Note that, for every $u, v \in V(G)$, $v \in T_v$ and if $u \in T_v$, then $T_v = T_u$. Let $\mathcal{T}(G)$ be a set of all twin classes of G with at least two elements, $\mathcal{T}(G) = \{T_v \mid v \in V(G),\ |T_v| \geq 2\}$.

Lemma 1. *For every twin class T of G with at least two elements, there is a basis B of G, such that $|B \cap T| = |T| - 1$.*

Proof: We know that for every twin class T and every basis B of G, $|B \cap T| \geq |T| - 1$. Let $v \in V(G)$, $T = T_v$, $|T_v| \geq 2$ and B_0 be a basis of G such that $B_0 \cap T_v = T_v$. By the definition of basis and property of twin vertices, since $|T_v| \geq 2$, there is a unique vertex $u \in V(G) \setminus B_0$ such that $r(u|B_0 \setminus \{v\}) = r(v|B_0 \setminus \{v\})$. Therefore, $B = (B_0 \setminus \{v\}) \cup \{u\}$ is a basis of G containing exactly $|T_v| - 1$ elements of T_v. Thus, $|B \cap T| = |T| - 1$. □

Theorem 5. *Let G be a connected graph.*

$$\sum_{T \in \mathcal{T}(G)} (|T| - 1) \leq f(G).$$

Proof: Let G be a connected graph, $\sum_{T \in \mathcal{T}(G)} (|T| - 1) > f(G)$ and $S \subset V(G)$ of cardinality $f(G)$ such that there is a unique basis B of G containing S. Twin classes being pairwise disjoint, there exists a vertex $v \in V(G)$ such that $|T_v| \geq 2$ and $|S \cap T_v| \leq |T_v| - 2$. There are at least two vertices $u_1, u_2 \in T_v$ such that $u_i \notin S$ for $i = 1, 2$. Since B is the unique basis containing S and by the proof of Lemma 1, we have $|B \cap T_v| = |T_v| - 1$. Suppose that $T_v \setminus B = \{u_3\}$. Therefore, $(B \setminus \{u_1\}) \cup \{u_3\}$ or $(B \setminus \{u_2\}) \cup \{u_3\}$ is a basis of G containing S, which is a contradiction. □

Theorem 6. *If $G = K_{n_1, n_2, \ldots, n_r}$, then $dim(G) = f(G) = \sum_{i=1}^{r} n_i - r$.*

Proof: It is easy to check that $\sum_{T \in \mathcal{T}(G)} (|T| - 1) = \sum_{i=1}^{r} n_i - r$. Therefore by Theorem 5, $f(G) \leq \sum_{i=1}^{r} n_i - r$. For every i, $1 \leq i \leq r$, let V_i be the i^{th} part of $V(G)$, then we remove an arbitrary vertex of V_i, call new subset as V_i'. Obviously, $B = \cup_{i=1}^{r} V_i'$ is a metric basis of G. Thus, $dim(G) = f(G) = \sum_{i=1}^{r} n_i - r$. □

Theorem 7. *Let M be a matching in a K_n, $n \geq 3$. If $G = K_n \setminus M$, then $dim(G) = f(G)$.*

Proof: Let M be a matching in K_n of size r and $G = K_n \setminus M$. Assume that $M = \{u_iv_i \mid 1 \le i \le r\}$ and $w \in V(G)$ be a vertex that is unsaturated by M. Therefore, $B = V(G) \setminus \{w, u_1, u_2, \dots, u_r\}$ is a metric basis of G, so $dim(G) = n - r - 1$. Note that $\mathcal{T}(G) = \{\{u_1, v_1\}, \{u_2, v_2\}, \dots, \{u_r, v_r\}, V(G)\setminus V(M)\}$. Thus by Theorem 5, $f(G) \ge n-(r+1)$. Since $f(G) \le dim(G)$, we have $dim(G) = f(G) = n - r - 1$. □

Theorem 8. [1] *Let M be a perfect matching in a $K_{n,n}$, $n \ge 2$. If $G = K_{n,n} \setminus M$, then $dim(G) = n - 1$.*

Theorem 9. *Let M be a perfect matching in a $K_{n,n}$, $n \ge 3$. If $G = K_{n,n} \setminus M$, then $dim(G) = f(G)$.*

Proof: Let G be a regular bipartite graph with $V_1(G) = \{x_1, \dots, x_n\}$, $V_2(G) = \{y_1, \dots, y_n\}$, and $E(G) = \{x_iy_j \mid i \ne j\}$. Let $B \subset V(G)$ be an arbitrary metric basis of G, so $|W| = n - 1$. It is easily seen that, $|B \cap \{x_i, y_i, x_j, y_j\}| \ge 1$ and $|B \cap \{x_i, y_i\}| \le 1$, for each i, j, $1 \le i < j \le n$. Therefore, $dim(G) = f(G)$. □

Theorem 10. *Let n_1 and n_2 be two positive integers.*

$$dim(P_{n_1} \Box P_{n_2}) = f(P_{n_1} \Box P_{n_2}) = 2.$$

Proof: Let $V(P_1) = \{u_1, u_2, \dots, u_{n_1}\}$, $V(P_2) = \{v_1, v_2, \dots, v_{n_2}\}$, $E(P_{n_1}) = \{u_iu_{i+1} \mid i = 1, \dots, n_1-1\}$, and $E(P_{n_2}) = \{v_iv_{i+1} \mid i = 1, \dots, n_2-1\}$. It is easily seen that, $dim(P_{n_1} \Box P_{n_2}) = 2$ and $P_{n_1} \Box P_{n_2}$ has exactly four metric bases as follows: $\{(u_1, v_1), (u_1, v_{n_2})\}$, $\{(u_1, v_1), (u_{n_1}, v_1)\}$, $\{(u_{n_1}, v_{n_2}), (u_1, v_{n_2})\}$, and $\{(u_{n_1}, v_{n_2}), (u_{n_1}, v_1)\}$.
Therefore, $dim(P_{n_1} \Box P_{n_2}) = f(P_{n_1} \Box P_{n_2}) = 2$. □

Proposition 1. *Let G be a connected graph.*

$$bas(G) \le f(G).$$

Proof: Suppose that there exists a connected graph G with $f(G) < bas(G)$. Let S be a subset of $V(G)$ of size $f(G)$ such that there is a unique basis containing S. Since $f(G) < bas(G)$, for every subset S' of $V(G) \setminus S$ such that $|S \cup S'| = bas(G)$, there is a basis B such that $S \cup S' \subset B$, which is a contradiction. □

Theorem 11. *Let G be a connected graph and $f(G) = k \ge 2$. If $T = \{v_1, v_2, \dots, v_k, v_{k+1}\}$ is a twin class, then $f(G \setminus \{v_i\}) = k - 1$ for $1 \le i \le k + 1$.*

Proof: By Theorem 5, G has just one twin class with at least two elements, that is T. Without loss of generality, let $S = T \setminus \{v_{k+1}\}$ be a forcing subset of a basis of G. First, we show that for every basis B of G, $T \nsubseteq B$. On the contrary, suppose that there exists a basis B of G such that $T \subseteq B$. Since T is a twin class, there exists a unique vertex $u \in V(G) \setminus B$ such that

$r(u|B\setminus\{v_{k+1}\}) = r(v_{k+1}|B\setminus\{v_{k+1}\})$. Therefore, $B' = (B\setminus\{v_{k+1}\})\cup\{u\}$ is a basis of G. Thus, $S \subseteq B \cap B'$, which is a contradiction.

For every k-subset of T, there is a unique basis of G containing that subset. On the contrary and without loss of generality, assume that there are two bases B_1 and B_2 of G such that $\{v_2, v_3, \ldots, v_{k+1}\} \subset B_1 \cap B_2$. Since $T \nsubseteq B_i$, $B_i' = (B_i \setminus \{v_{k+1}\}) \cup \{v_1\}$, for $i = 1, 2$, is a basis of G. Thus, $S \subset B_1 \cap B_2$, which is a contradiction.

Note that by Theorem 5, $f(G \setminus \{v_i\}) \geq \sum_{T' \in \mathcal{T}(G\setminus\{v_i\})}(|T'| - 1) = k - 1$, for $1 \leq i \leq k+1$.

Claim 1: For every basis B of $G \setminus \{v_i\}$, $|B \cap (T \setminus \{v_i\})| = k - 1$. Suppose that B_0 is a basis of $G \setminus \{v_i\}$ for some $1 \leq i \leq k+1$ such that $T \setminus \{v_i\} \subset B_0$. Since $dim(G \setminus \{v_i\}) = dim(G) - 1$ and $k \geq 2$, $B' = B_0 \cup \{v_i\}$ is a basis of G containing T, which is a contradiction.

On the contrary, suppose that $f(G \setminus \{v_i\}) \geq k$ for some $1 \leq i \leq k+1$. Therefore, for every $T' \subset T_v \setminus \{v_i\}$ of cardinality $k-1$, there are at least two basis B_1 and B_2 of $G \setminus \{v_i\}$ that $T' \subset B_j$, for $j = 1, 2$. By Claim 1, $|B_j \cap T \setminus \{v_i\}| = k - 1$, therefore, $B_j \cup \{v_i\}$ for $j = 1, 2$ are two basis of G containing $T' \cup \{v_i\}$ and $|T' \cup \{v_i\}| = k$, which is a contradiction. □

3 Graphs with prescribed dimensions, forcing dimensions, basis numbers and orders

In Theorem 4, it is shown that for every $0 \leq a \leq b$ and $b \geq 1$, there exists a connected graph G with $f(G) = a$ and $dim(G) = b$. By using the similar technique of Theorem 4, we prove that for every $0 \leq a \leq b$, $b \geq 1$, and for every sufficiently large n, there is a connected graph G of order n with $f(G) = a$ and $dim(G) = b$. First we need the following theorem.

Let G be a k-dimensional graph such that it has a unique basis. Such graph is called uniquely k-dimensional graph [2]. Obviously, for every uniquely k-dimensional graph G, $bas(G) = f(G) = 0$. It is proved that for every k, $k \geq 2$, and every $n \geq \lceil \frac{5k}{2} + 1 \rceil$, there is a uniquely k-dimensional graph of order n.

Theorem 12. [2] *For every k, $k \geq 2$, there exists a uniquely k-dimensional graph of order n for every $n \geq \lceil \frac{5k}{2} + 1 \rceil$.*

Theorem 13. *For every a, b, $0 \leq a \leq b$, $b \geq 1$, $\{a, b\} \neq \{0, 1\}$, and for every $n \geq \lceil \frac{5(b-a)}{2} + 1 \rceil + a$, there exists a nontrivial connected graph G of order n with $f(G) = a$ and $dim(G) = b$.*

Proof: By Theorem 12, let H be a uniquely $(b-a)$-dimensional graph of order $n_0 \geq \lceil \frac{5(b-a)}{2} + 1 \rceil$. First we construct a graph G of order $n_0 + a$ with $V(G) = V(H) \cup X$, where $X = \{x_1, \ldots, x_a\}$, such that each x_i $(1 \leq i \leq a)$ has the same neighborhood as u in H and the induced subgraph $\{u, x_1, \ldots, x_a\}$ is complete, where $u \in V(H) \setminus B_0$ and B_0 is the unique basis of H.

We first show that $B_1 = B_0 \cup X$ is a basis of G and therefore, $dim(G) = b$. Since H has a unique basis, $T_u = \{u\}$ in H. Thus, $X \cup \{u\}$ is a twin class and B_0 is the unique basis of H,

I. $r_H(v|B_0) = r_G(v|B_0)$ for every $v \in V(H)$;

II. $|B \cap (X \cup \{u\})| \geq a$, for each basis B of G;

III. $B_0 \subset B$, for each basis B of G. (Since $B \setminus X$ is a resolving set of H.)

Therefore $B_0 \cup X$ is a basis of G and $dim(G) = b$.

We are now prepared to show that $f(G) = a$. Let B be a basis of G. Since B_0 must belong to B, it follows that B is the unique basis containing $B \setminus B_0$. Thus, $f_G(B) \leq |B \setminus B_0| = b - (b - a) = a$. This is true for every basis B of G and so $f(G) \leq a$. On the other hand, by Theorem 5, since $X \cup \{u\}$ is a twin class, $f(G) \geq a$. Therefore, $f(G) = a$. □

In [12], the properties of k-dimensional graphs in which every k subset of vertices is a metric basis are studied. Such graphs are called randomly k-dimensional graphs. It is obvious that for every randomly k-dimensional graph G, $bas(G) = f(G) = dim(G) = k$.

Example 1. *Let $n \geq 2$ be an integer. Every complete graph K_n is a randomly $(n-1)$-dimensional graph and $bas(K_n) = f(K_n) = dim(K_n) = n - 1$.*

Theorem 14. *For every a, n, $0 \leq a \leq n$, there exists a connected graph G of order n with $bas(G) = 0$ and $f(G) = dim(G) = a$.*

Proof: Let H be a cycle of order $n - a + 1$. First we construct a graph G of order n with $V(G) = V(H) \cup X$, where $X = \{x_1, \ldots, x_{a-1}\}$, such that each x_i $(1 \leq i \leq a-1)$ has the same neighborhood as u in H and the induced subgraph $\{u, x_1, \ldots, x_{a-1}\}$ is complete, where $u \in V(H)$. Let $v \in V(G)$ such that $d(u, v) = \lfloor \frac{n-a+1}{2} \rfloor$. It is easy to check that $f(G) = dim(G) = a$ and there is no basis containing v. Thus, $bas(G) = 0$. □

Theorem 15. *For every a, n, $0 \leq a \leq n$, there exists a connected graph G of order n with $bas(G) = 1$ and $f(G) = dim(G) = a$.*

Proof: Let $H = (u_1, u_2, u_3, \ldots, u_{n-a+1})$ be a path of order $n-a+1$. First we construct a graph G of order n with $V(G) = V(H) \cup X$, where $X = \{x_1, \ldots, x_{a-1}\}$, such that each x_i $(1 \leq i \leq a-1)$ has the same neighborhood as u_2 in H and the induced subgraph $\{u_2, x_1, \ldots, x_{a-1}\}$ is complete. Since the twin class T_{u_2} has a vertices, $dim(G) \geq f(G) \geq a - 1$. It is easy to check that every $B \subset V(G)$ of cardinality a with $|B \cap T_{u_2}| = a - 1$ is a resolving set of G. Hence $f(G) = dim(G) = a$ and for every $v \in V(G)$ there is some basis containing v. For each $\{u_i, u_j\} \subset V(H) \setminus \{u_2\}$, $i, j \neq 2$, there is no basis B such that $\{u_i, u_j\} \subset B$. Thus, $bas(G) = 1$. □

The following question is an open problem in this area.

Problem 1. *For which a, b, c of integers with $0 \leq a \leq b \leq c$, does there exist a nontrivial connected graph G with $bas(G) = a$, $f(G) = b$, and $dim(G) = c$?*

Acknowledgement: Behrooz Bagheri Gh. wishes to thank the National Elites Foundation of Iran, Sharif University of Technology, and Malek Ashtar University of Technology.

References

[1] M. Baca, E.T. Baskoro, A.N.M. Salman, S.W. Saputro and D. Suprijanto, The metric dimension of regular bipartite graphs, *Bull. Math. Soc. Sci. Math. Roumanie*, **54 (102)** (2011), 15–28.

[2] B. Bagheri Gh., M. Jannesari and B. Omoomi, Uniquely dimensional graphs, *Ars Combinatoria*, to appear.

[3] Z. Beerliova, F. Eberhard, T. Erlebach, A. Hall, M. Hoffmann, M. Mihalak and L.S. Ram, Network dicovery and verification, *IEEE Journal On Selected Areas in Communications*, **24 (12)** (2006), 2168–2181.

[4] P. Buczkowski, G. Chartrand, C. Poisson and P. Zhang, On k-dimensional graphs and their bases, *Period. Math. Hungar.*, **46 (1)** (2003), 9–15.

[5] J. Caceres, C. Hernando, M. Mora, I.M. Pelayo, M.L. Puertas, C. Seara and D.R. Wood, On the metric dimension of cartesian products of graphs, *SIAM Journal on Discrete Mathematics*, **21 (2)** (2007), 423–441.

[6] G.G. Chappell, J. Gimbel and C. Hartman, Bounds on the metric and partition dimensions of a graph, *Ars Combinatoria*, **88** (2008), 349–366.

[7] G. Chartrand, L. Eroh, M.A. Johnson and O.R. Ollermann, Resolvability in graphs and the metric dimension of a graph, *Discrete Applied Mathematics*, **105** (2000), 99–113.

[8] G. Chartrand and P. Zhang, The theory and applications of resolvability in graphs, A survey. In *Proc. 34th Southeastern International Conf. on Combinatorics, Graph Theory and Computing* , **160** (2003), 47–68.

[9] G. Chartrand, P. Zhang and Kalamazoo, The forcing dimension of a graph, *Mathematica Bohemica*, **126 (4)** (2001), 711–720.

[10] F. Harary and R.A. Melter, On the metric dimension of a graph, *Ars Combinatoria*, **2** (1976), 191–195.

[11] C. Hernando, M. Mora, I.M. Pelayo, C. Seara and D.R. Wood, Extremal graph theory for metric dimension and diameter, *The Electronic Journal of Combinatorics*, **17** (2010), #R30.

[12] M. Jannesari and B. Omoomi, On randomly k-dimensional graphs, *Applied Mathematics Letters*, **24** (2011), 1625–1629.

[13] S. Khuller, B. Raghavachari and A. Rosenfeld, Landmarks in graphs, *Discrete Applied Mathematics*, **70 (3)** (1996), 217–229.

[14] R.A. Melter and I. Tomescu, Metric bases in digital geometry, *Computer Vision Graphics and Image Processing*, **25** (1984), 113–121.

[15] A. Sebo and E. Tannier, On metric generators of graphs, *Mathematics of Operations Research*, **29 (2)** (2004), 383–393.

[16] P.J. Slater, Leaves of trees, *Congressus Numerantium*, **14** (1975), 549–559.

Received: 12.04.2014
Accepted: 29.10.2014

[1]Department of Mathematical Sciences,
Sharif University of Technology,
P. O. Box 11155–9415, Tehran, Iran
E-mail: b.bagheri@sharif.edu

[2]Department of Mathematics and Cryptography,
Malek Ashtar University of Technology,
P. Box 83145–115, Isfahan, Iran
E-mail: a_zaghian@mut-es.ac.ir