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# Stanley depth of monomial ideals

by
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**Abstract**

Let $I \supsetneq J$ be two monomial ideals of a polynomial algebra over a field generated in degree $\geq d$, resp. $\geq d+1$ . We study when the Stanley Conjecture holds for $I/J$ using the recent result of [6] concerning the polarization.

**Key Words**: Monomial Ideals, Depth, Stanley depth.
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## Introduction

Let $K$ be a field and $S = K[x_1, \ldots, x_n]$ be the polynomial $K$-algebra in $n$ variables. Let $I \supsetneq J$ be two monomial ideals of $S$ and suppose that $I$ is generated by monomials of degrees $\geq d$ for some positive integer $d$. Using a multigraded isomorphism we may assume either that $J = 0$, or $J$ is generated in degrees $\geq d+1$.

If $I, J$ are squarefree monomial ideals then $d$ is a lower bound of $\operatorname{depth}_S I/J$ by [3, Proposition 3.1] (see also [15, Lemma 1.1]). Proposition 2 gives a lower bound of $\operatorname{depth}_S I/J$ in terms of degrees also in the case when $I, J$ are not squarefree using the polarization and the so called the canonical form of $I/J$ (see [10]).

A Stanley decomposition of a multigraded $S$-module $M$ is a finite family $\mathcal{D} = (S_l, u_l)_{l \in L}$ in which $u_l$ are homogeneous elements of $M$ and $S_l$ are multigraded $K-$algebra retract of $S$ for all $l \in L$ such that $S_l \cap \operatorname{Ann}_S u_l = 0$ and $M = \oplus_{l \in L} S_l u_l$ as a multigraded $K-$vector space. The Stanley depth of $\mathcal{D}$, denoted by sdepth($\mathcal{D}$), is the depth of the $S-$module $\oplus_{l \in L} S_l u_l$. The Stanley depth of $M$ is defined as

$$\text{sdepth } (M) := \max\{\text{sdepth } (\mathcal{D}) \mid \mathcal{D} \text{ is a Stanley decomposition of } M\}.$$

Depth and Stanley depth behave in a different way for instance $\operatorname{depth}_S(M \oplus M') = \min\{\operatorname{depth}_S M, \operatorname{depth}_S M'\}$ while for sdepth it can happen $\operatorname{sdepth}_S(M \oplus M') > \min\{\operatorname{sdepth}_S M, \operatorname{sdepth}_S M'\}$ sometimes as seen in [7, Examples 14, 16] with the help of [9]. These results were obtained using the so called the Hilbert depth (see [1], [23]). The same notion is important also in other properties of depth and Stanley depth (see [21, Proposition 2.4]).

An upper bound for $\operatorname{depth}_S M$ could be given by the following conjecture.

**Conjecture 1.** *(Stanley [22])* $\operatorname{depth}_S M \leq \operatorname{sdepth}_S M$.

It will be very nice if this conjecture holds for $M = I/J$. Recently Ichim, Katthän and Moyano-Fernández proved that Stanley's Conjecture holds for all factors $I/J$ as above if and only if it holds for their polarizations [6, Theorem 4.3]. Thus we may restrict to the case when $I, J$ are squarefree monomial ideals. Unfortunately, there are few results in this case in spite of the many papers appeared on this subject (see [3], [12], [13], [5], [14], [2], [15]). It is the purpose of our paper to study what these few results say in the non squarefree case using [6, Theorem 4.3]. We use here the lower bound given by Proposition 2 (see Theorems 5, 6 and Proposition 4).

A particular case of this conjecture is the following one.

**Conjecture 2.** *Suppose that $I \subset S$ is minimally generated by some squarefree monomials $f_1, \ldots, f_r$ of degrees $d$, and a set $E$ of squarefree monomials of degree $\geq d+1$. If* $\operatorname{sdepth}_S I/J = d+1$ *then* $\operatorname{depth}_S I/J \leq d+1$.

This conjecture is studied in [17], [18], [19], [11], [16] when $r \leq 4$ and some cases when $r = 5$ (see Theorems 3, 4). Proposition 3 proves Conjecture 2 also when $r = 6$ but $d = 1$ and $E = \emptyset$.

## 1 A lower bound of depth and Stanley depth

Let $S = K[x_1, \ldots, x_n]$ be the polynomial $K$-algebra over a field $K$ and $J \subsetneq I \subset S$ two monomial ideals. Denote by $G(I)$, respectively $G(J)$, the minimal monomial system of generators of $I$, respectively $J$.

A very important result concerning the Stanley depth is given by [6, Corollary 4.4] and we recall it below.

**Theorem 1.** *(Ichim, Katthän, Moyano-Fernández) Let $J \subsetneq I$ be monomial ideals of $S$, and let $I^p \subset J^p$ be their (complete) polarizations. Then*

$$\operatorname{sdepth}_S I/J - \operatorname{depth}_S I/J = \operatorname{sdepth} I^p/J^p - \operatorname{depth} I^p/J^p.$$

For $i \in [n]$ let $e_i = \max_{u \in G(I) \cup G(J)} \deg_{x_i} u$ and set $e_{I/J} = \sum_{i \in [n], e_i > 0} (e_i - 1)$. We have $e_{I/J} = \operatorname{depth} I^p/J^p - \operatorname{depth}_S I/J$, that is $e_{I/J}$ is the number of the new variables necessary for polarization. Suppose that $I$ is generated by some monomials $f_1, \ldots, f_r$ of degrees $d_{I/J}$ and a set of monomials $E$ of degrees $\geq d_{I/J} + 1$. Then

**Proposition 1.** $\operatorname{depth}_S I/J \geq d_{I/J} - e_{I/J}$ *and* $\operatorname{sdepth}_S I/J \geq d_{I/J} - e_{I/J}$.

**Proof**: By [3, Proposition 3.1] (see also [15, Lemma 1.1]) we have $\operatorname{depth} I^p/J^p \geq d_{I/J}$ because $I^p, J^p$ are squarefree monomial ideals. Note that by polarization the degrees of monomials are preserved. It follows that $\operatorname{depth}_S I/J = \operatorname{depth} I^p/J^p - e_{I/J} \geq d_{I/J} - e_{I/J}$. The inequality concerning sdepth is similar but easier since obviously the sdepth is $\geq d_{I/J}$ in the case of a factor of some squarefree monomial ideals. □

**Example 1.** Let $n = 3$, $d = 12$, $I = (x_1^3x_2^4x_3^5, x_1^{10}x_2^2)$. Note that $e_1 = 10$, $e_2 = 4$, $e_3 = 5$ and $e_I = 16$.

**Remark 1.** In the above example Proposition 1 gives $\operatorname{depth}_S I \geq -4$ which is obvious. This situation will be next improved considering the so called the canonical form of $I$ given by [10].

We recall some definitions and results from [10].

**Definition 1.** The power $x_n^r$ *enters in a monomial* $u$ if $x_n^r | u$ but $x_n^{r+1} \nmid u$. We say that $I/J$ is *of type* $(k_1, \ldots, k_s)$ *with respect to* $x_n$ if $x_n^{k_i}$ are all the powers of $x_n$ which enter in a monomial of $G(I) \cup G(J)$ for $i \in [s]$ and $1 \leq k_1 < \ldots < k_s$. $I/J$ is *in the canonical form with respect to* $x_n$ if $I/J$ is of type $(1, \ldots, s)$ for some $s \in \mathbb{N}$ and we say that $I/J$ is *the canonical form* if it is in the canonical form with respect to all variables $x_1, \ldots, x_n$.

**Remark 2.** Suppose that $I/J$ is of type $(k_1, \ldots, k_s)$ with respect to $x_n$. It is easy to get the *canonical form* $I'/J'$ of $I/J$ *with respect to* $x_n$ replacing $x_n^{k_i}$ by $x_n^i$ whenever $x_n^{k_i}$ enters in a generators of $G(I) \cup G(J)$. Applying by recurrence this procedure for other variables we get the *canonical form* of $I/J$, that is with respect to all variables.

**Theorem 2.** *(A. Popescu [10, Theorems 1, 2]) Let $I'/J'$ be the canonical form of $I/J$. Then* $\operatorname{sdepth}_S I'/J' = \operatorname{sdepth}_S I/J$ *and* $\operatorname{depth}_S I'/J' = \operatorname{depth}_S I/J$.

**Definition 2.** Let $I'/J'$ be the canonical form of $I/J$ and set $t_{I/J} = \max\{d_{I'/J'} - e_{I'/J'}, 0\}$ (we may have $d_{I'/J'} < e_{I'/J'}$ as shows Example 3). We call $t_{I/J}$ the *index of* $I/J$. When $J = 0$ we write $t_I$ instead $t_{I/J}$ for simplicity. If $I, J$ are squarefree monomial ideals then $t_{I/J} = d_{I/J}$.

Using the terminology defined above we get a better lower bound for sdepth and depth as in Proposition 1.

**Proposition 2.** $\operatorname{depth}_S I/J \geq t_{I/J}$ *and* $\operatorname{sdepth}_S I/J \geq t_{I/J}$.

**Proof**: By Theorem 2 we have $\operatorname{depth}_S I/J = \operatorname{depth}_S I'/J' \geq \max\{d_{I'/J'} - e_{I'/J'}, 0\} = t_{I/J}$. The second inequality holds similarly. □

**Remark 3.** This lower bound is easy to describe but it is not the best known lower bound. For example, when $J = 0$ then a better lower bound is given by $1 + \operatorname{size}(I)$ in the terminology of [8], [4]. More precisely, if $n = 3$, $d_I = 1$, $I = (x_1, x_2x_3) = (x_1, x_2) \cap (x_1, x_3)$ then $\operatorname{size}(I) = 1$ and $t_I = d_I$ since $I$ is squarefree. Thus $1 + \operatorname{size}(I) > t_I$.

**Remark 4.** In Example 1 note that $I$ is of type $(3, 10)$ with respect to $x_1$, of type $(2, 4)$ with respect to $x_2$ and of type $(5)$ with respect to $x_3$. Then the canonical form of $I$ is $I' = (x_1x_2^2x_3, x_1^2x_2)$. Note that $I$ is generated by monomials of degrees 12 but in $I'$ one generator has degree 4 and the other 3. Clearly, $e_{I'} = 2$, $d_{I'} = 3$ and so the index $t_I$ of $I$ is 1. Thus Proposition 2 says that $\operatorname{depth}_S I \geq 1$, which is also trivial but anyway better than what follows from Proposition 1 (see Remark 1).

## 2 Stanley depth of monomial ideals which are not necessarily squarefree

Suppose that $I$ is minimally generated by some squarefree monomials $f_1, \dots, f_r$ of degrees $d$ for some $d \in \mathbb{N}$ and a set of squarefree monomials $E$ of degree $\geq d+1$. Let $B$ be the set of the squarefree monomials of degrees $d+1$ of $I \setminus J$.

We start recalling two results of [16] (see also [19] and [11]).

**Theorem 3.** *Conjecture 2 holds for $r \leq 4$.*

**Theorem 4.** *Conjecture 2 holds for $r = 5$ if there exists $j \notin \cup_{i\in[5]} \operatorname{supp} f_i$, $j \in [n]$ such that $(B \setminus E) \cap (x_j) \neq \emptyset$ and $E \subset (x_j)$.*

For simplicity we denote $t = t_{I/J}$, that is the index of $I/J$.

**Theorem 5.** *Let $J \subsetneq I$ be monomial ideals of $S$ not necessarily squarefree and $I'/J'$ the canonical form of $I/J$. Suppose that $I'$ is generated by $r'$ monomials $f_1, \dots, f_{r'}$ of degree $d_{I'/J'}$ and a set $E'$ of monomials of degree $\geq d_{I'/J'} + 1$. Let $B'$ be the set of monomials of degree $d_{I'/J'} + 1$ from $I' \setminus J'$. Assume that $\operatorname{sdepth}_S I/J = t+1$. Then the following statements hold:*

1. *If $r' \leq 4$ then $\operatorname{depth}_S I/J \leq t+1$,*
2. *If $r' = 5$ and there exists $j \notin \cup_{i\in[5]} \operatorname{supp} f_i$, $j \in [n]$ such that $(B' \setminus E') \cap (x_j) \neq \emptyset$ and $E' \subset (x_j)$, then $\operatorname{depth}_S I/J \leq t+1$.*

**Proof**: Let $I'/J'$ be the canonical form of $I/J$ and suppose for a moment that $d_{I'/J'} \geq e_{I'/J'}$. By Theorem 1 we have

$$\operatorname{sdepth} I'^p/J'^p = \operatorname{sdepth}_S I'/J' + \operatorname{depth} I'^p/J'^p - \operatorname{depth}_S I'/J' = d_{I'/J'} + 1,$$

because $\operatorname{sdepth}_S I'/J' = d_{I'/J'} - e_{I'/J'} + 1$ and $e_{I'/J'} = \operatorname{depth} I'^p/J'^p - \operatorname{depth}_S I'/J'$. Since $I'^p$ is generated by $r'$ squarefree monomials of degree $d_{I'/J'}$ and a set $E'^p$ of squarefree monomials of degree $d_{I'/J'} + 1$ we get by Theorem 3 that $\operatorname{depth} I'^p/J'^p \leq d_{I'/J'} + 1$. Hence $\operatorname{depth}_S I/J = \operatorname{depth}_S I'/J' \leq t+1$ by Theorem 2, that is (1) holds. If $d_{I'/J'} < e_{I'/J'}$ then $t = 0$, $\operatorname{sdepth}_S I/J = 1 = d_{I/J}$ and so $\operatorname{depth}_S I/J \leq 1$ by [15, Theorem 4.3]. The proof of (2) is the same using Theorem 4 instead Theorem 3. □

**Remark 5.** Let $I$ be generated by some monomials $h_1, \dots, h_r$ of degree $d$ and a set of monomials $E$ of monomials of degree $\geq d+1$. It is possible that $I'$ is generated by $f_1, \dots, f_{r'}$ of degrees $d_{I'/J'}$ with $r' > r$ and a set $E'$ of monomials of degree $\geq d_{I'/J'} + 1$. For example when $n = 2$ and $I = (x_1^3 x_2^4, x_1^{11} x_2)$ we have $r = 1$ and we see that $I' = (x_1 x_2^2, x_1^2 x_2)$ has $r' = 2$.

**Example 2.** Let $n = 2$, $d = 1$, $I = (x_1)$, $J = (x_1 x_2^2)$. Then $e_1 = 1$, $e_2 = 2$, $e_{I/J} = 1$, $t = 0$ and $I^p/J^p = (x_1)/(x_1 x_2 y_2)$, where $y_2$ is the new variable from polarization. We have $I/J \cong x_1 K[x_1] \oplus x_1 x_2 K[x_1]$ as graded $K$-vector spaces. Thus $\operatorname{sdepth}_S I/J = 1 = t+1$. By (1) of the above theorem we get $\operatorname{depth}_S I/J \leq 1$, the inequality being in fact an equality.

**Theorem 6.** *Let $J \subsetneq I$ be monomial ideals of $S$ not necessarily squarefree. Assume that $\operatorname{sdepth}_S I/J = t$. Then $\operatorname{depth}_S I/J = t$*

The proof is similar to the proof of Theorem 5 using now [15, Theorem 4.3] instead Theorem 3.

**Example 3.** Let $n = 2$, $d = 1$, $I = (x_2)$, $J = (x_1^2x_2, x_1x_2^2)$. Then $e_1 = e_2 = e_{I/J} = 2$, $t = \max\{-1, 0\} = 0$ and $I^p/J^p = (x_2)/(x_1y_1x_2, x_1x_2y_2)$, where $y_1, y_2$ are the new variables from polarization. Since $x_2$ induces a nonzero element of the socle of $I/J$ we see that $\operatorname{sdepth}_S I/J = 0$. Thus $\operatorname{sdepth}_S I/J = t = 0$. By the above theorem we get $\operatorname{depth}_S I/J = 0$.

## 3 Stanley depth of a factor of squarefree monomial ideals

Theorem 4 has the following consequence.

**Proposition 3.** *Suppose that $I \subset S$ is minimally generated by 6 variables $\{x_1, \ldots, x_6\}$ and $J \subsetneq I$ a squarefree monomial ideal. If $\operatorname{sdepth}_S I/J = 2$ then $\operatorname{depth}_S I/J \leq 2$.*

**Proof**: By [17, Proposition 1.3] we see that there exists $c = x_6x_kx_q \notin J$ for $6 < k < q \leq n$. Let $B$ be the set of all squarefree monomials from $I \setminus J$ and $\tilde{I}$ be the ideal generated by $x_1, \ldots, x_5$ and $\tilde{E} = B \setminus ((x_1, \ldots, x_5) \cup [x_6, c])$. Set $\tilde{J} = J \cap \tilde{I}$. Then for $j = 6$ we have $\tilde{E} \subset (x_j)$. In the following exact sequence

$$0 \to \tilde{I}/\tilde{J} \to I/J \to I/J + \tilde{I} \to 0$$

the last term is isomorphic with $(x_6)/(x_6) \cap (J + \tilde{I})$ and has depth $\geq 2$ and sdepth 3 because it has just the interval $[x_6, c]$. Suppose that $\operatorname{sdepth}_S I/J = 2$. By [20, Lemma 2.2] we get $\operatorname{sdepth}_S \tilde{I}/\tilde{J} \leq 2$. When $\operatorname{sdepth}_S \tilde{I}/\tilde{J} = 1$ then it is enough to apply [15, Theorem 4.3]. If $\operatorname{sdepth}_S \tilde{I}/\tilde{J} = 2$ and $(B \setminus \tilde{E}) \cap (x_j) \neq \emptyset$ then it is enough to apply Theorem 4.

Now suppose that $(B \setminus \tilde{E}) \cap (x_j) = \emptyset$, that is $B \cap (x_6) \cap (x_1, \ldots, x_5) = \emptyset$. In the following exact sequence

$$0 \to (x_6)/(x_6) \cap J \to I/J \to I/(J, x_6) \to 0$$

if the last term has sdepth $\geq 3$ then the first term has sdepth $\leq 2$ as above and so also depth $\leq 2$. Otherwise, the last term has sdepth $\leq 2$. But the last term is isomorphic with $(x_1, \ldots, x_5)/(x_1, \ldots, x_5) \cap J$ because $B \cap (x_6) \cap (x_1, \ldots, x_5) = \emptyset$. Thus in the exact sequence

$$0 \to (x_1, \ldots, x_5)/(x_1, \ldots, x_5) \cap J \to I/J \to I/(J, x_1, \ldots, x_5) \to 0$$

the first term has sdepth $\leq 2$ and so its depth $\leq 2$ by Theorem 4 when there exists $k > 6$ such that $B \cap (x_1, \ldots, x_5) \cap (x_k) \neq \emptyset$. Otherwise, $J \geq (x_1, \ldots, x_5)(x_6, \ldots, x_n)$ and we get

$$\operatorname{depth}_S (x_1, \ldots, x_5)/(x_1, \ldots, x_5) \cap J = \operatorname{depth}_{\tilde{S}} (x_1, \ldots, x_5)\tilde{S}/(x_1, \ldots, x_5) \cap J \cap \tilde{S} \leq 1$$

for $\tilde{S} = K[x_1, \ldots, x_5]$. Since the last term is isomorphic with $(x_6)/J \cap (x_6)$ it has depth $\geq 2$ and the Depth Lemma ends the proof. □

**Proposition 4.** *Suppose that $I \subset S$ is minimally generated by 6 variables $\{x_1, \ldots, x_6\}$ and $J \subsetneq I$ is a monomial ideal not necessarily squarefree. Suppose that $\operatorname{sdepth}_S I/J = t + 1$. Then $\operatorname{depth}_S I/J \leq t + 1$.*

The proof is similar to the proof of Theorem 5 using now Proposition 3 instead Theorem 3.

**Example 4.** Let $n = 7$, $I = (x_1, \ldots, x_6)$, $J = (x_1^2, x_1x_2, \ldots, x_1x_5, x_1x_7)$. Then $t = 0$. The element $\hat{x}_1 \in I/J$ induced by $x_1$ is annihilated by all variables but $x_6$. It follows that $\operatorname{sdepth}_S I/J \leq 1$. Thus $\operatorname{sdepth}_S I/J \leq t+1$ and so $\operatorname{depth}_S I/J \leq 1$ by Proposition 4. Note that $I^p/J^p = (x_1, \ldots, x_6)/(x_1y, x_1x_2, \ldots, x_1x_5, x_1x_7)$ has sdepth $\leq 2$ because now the element of $I^p/J^p$ induced by $x_1$ is annihilated by all variables but $x_6, y$.

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