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Laplace's method for the case of a countably infinite set of global maximum points

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Abstract

We propose an extension of the well-known method of Laplace which provides the asymptotic expansion of exponential integrals. We study the situation where the corresponding exponent is a function with a countably infinite set of global maximum points. We establish conditions for a certain asymptotic behaviour of this kind of integrals. Finally, we illustrate the theoretical results with two examples.

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1 Introduction

Laplace's method provides an asymptotic evaluation of an exponential integral depending on a positive parameter. Asymptotic analysis, including this very important method, is treated in comprehensive monographs, as Dingle [2], Olver [6], Murray [5] and Miller [4], among many others. Laplace's method has attracted the attention of many researchers, who provide various extensions with appropriate applications and computational algorithms. We mention here some very recent contributions in this area: Aygar and Bairamov [1], Hanna and Davis [3], Ozalp and Bairamov [7] and Paris [8].

In the classical context of real Laplace's integrals, the exponent function has an unique global maximum point on the integration interval. The asymptotic behavior provided by Laplace's method can further be extended to the case of a finite set of global maximum points. In this paper, we study the case where the exponent function has a countably infinite number of saddle points of global maximum. We think this approach may find interesting applications in optimization problems.

Let us recall the classical version of the Laplace's method. Consider the Laplace integrals

$$I_\lambda(f) = \int_a^b e^{\lambda R(t)} f(t)\, dt,\ \ \lambda > 0,$$

where $f \in C[a,b]$ and $R \in C^2[a,b]$. We shall denote

$$I_\lambda = \int_a^b e^{\lambda R(t)}\, dt,\ \ \lambda > 0,$$

the particular case of $f = 1$.

Assume that $\max_{t\in[a,b]} R(t) = 0$. If R attains its global maximum at a unique point t_0 (i.e. $R(t_0) = 0$), such that $t_0 \in (a,b)$ and $R''(t_0) < 0$, then

$$\lim_{\lambda\to\infty} \sqrt{\lambda} I_\lambda(f) = f(t_0)\sqrt{\frac{-2\pi}{R''(t_0)}},$$

i.e.

$$I_\lambda(f) = f(t_0)\sqrt{\frac{-2\pi}{\lambda R''(t_0)}} + o\left(\lambda^{-1/2}\right),\ \ \lambda \to \infty.$$

More generally, if R has a finite set $T_R = \{t_1, \ldots, t_n\}$ of global maximum points (i.e. $R(t_k) = 0$, for $k = 1, \cdots, n$), such that $T_R \subset (a,b)$ and $R''(t_k) < 0$, for $k = 1, \ldots, n$, then

$$I_\lambda(f) = \sum_{k=1}^{n} f(t_k)\sqrt{\frac{-2\pi}{\lambda R''(t_k)}} + o\left(\lambda^{-1/2}\right),\ \ \lambda \to \infty.$$

Also, if R attains its global maximum at an endpoint of the interval $[a,b]$, say at b, with $R'(b) = 0 > R''(b)$ then

$$I_\lambda(f) = \frac{f(b)}{2}\sqrt{\frac{-2\pi}{\lambda R''(b)}} + o\left(\lambda^{-1/2}\right),\ \ \lambda \to \infty.$$

Our aim is to study the asymptotic behaviour of $I_\lambda(f),\ \lambda \to \infty$, in the case when the set of global maximum points of R,

$$T_R = \{t \in [a,b] | R(t) = 0\},$$

is countably infinite.

Throughout this paper, we shall assume that the function R satisfies the following conditions denoted $\mathbf{C_0}$:

1) $R : [a,b] \to (-\infty, 0]$, with $\max\limits_{t\in[a,b]} R(t) = 0$;

2) $T_R \setminus \{b\} = \{t_k | k \geq 1\} \subset (a,b)$, *so that the sequence* $(t_k)_{k\geq1}$ *is strictly increasing, with* $\lim\limits_{k\to\infty} t_k = b$;

3) R *is continuous on* $[a,b)$ *and there are two strictly increasing sequences* $(u_k)_{k\geq1}$ and $(v_k)_{k\geq1}$ *in the interval* (a,b), with $t_k \in (u_k, v_k)$ and $v_k \leq u_{k+1}$, for $k \geq 1$, *such that*

a. $R \in C^2[u_k, v_k]$ *and* $R''(t_k) < 0$, *for all* $k \geq 1$;

b. $R'(t) > 0,\ \forall\, t \in [u_k, t_k)$, *and* $R'(t) < 0,\ \forall\, t \in (t_k, v_k]$, for $k = 1, 2, \cdots$.

We find a general characterization of R for having an asymptotic formula of the following type

$$I_\lambda(f) = \frac{1}{\sqrt{\lambda}} \left(\sum_{k=1}^{\infty} a_k f(t_k) + B f(b) \right) + o\left(\lambda^{-1/2}\right), \ \lambda \to \infty, \tag{1.1}$$

where $f \in C[a,b],\ a_k := \sqrt{\frac{-2\pi}{R''(t_k)}},\ k \geq 1$, and $B \geq 0$. Furthermore, by using a set of appropriate functions (namely, the *indicator functions*), we introduce an auxiliary function g which provides another characterization for (1.1). In this context, we further indicate some sufficient conditions to have $B = 0$ in (1.1). These results are illustrated by examples.

2 Main results

Firstly, we formulate necessary and sufficient conditions for the asymptotic formula (1.1), respectively.

Theorem 1. *Assume that a function R satisfies the conditions $\mathbf{C_0}$. Denote $I_\lambda = \int_a^b e^{\lambda R(t)}\,dt$, for $\lambda > 0$, and $a_k = \sqrt{\frac{-2\pi}{R''(t_k)}}$, for $k = 1, 2, \ldots$.*

If the function $\sqrt{\lambda} I_\lambda,\ \lambda > 0$, has a finite limit at infinity, then

1. *the series of positive numbers $\sum_{k=1}^{\infty} a_k$ is convergent;*
2. *the function*
$$\sqrt{\lambda} \int_{b-\delta}^{b} e^{\lambda R(t)}\,dt,\ \lambda > 0,$$
has a finite limit at infinity, for all $\delta \in (0, b-a]$;
3. *the following finite limits exist*
$$B := \lim_{\delta \to 0+} \lim_{\lambda \to \infty} \left(\sqrt{\lambda} \int_{b-\delta}^{b} e^{\lambda R(t)}\,dt \right)$$
and
$$\lim_{\lambda \to \infty} \sqrt{\lambda} I_\lambda(f) = \sum_{k=1}^{\infty} a_k f(t_k) + B f(b),\ \forall\, f \in C[a,b]. \tag{2.1}$$

Conversely, if the series $\sum_{k=1}^{\infty} a_k$ is convergent and

$$\inf_{\delta \in (0,b-a]} \limsup_{\lambda \to \infty} \left(\sqrt{\lambda} \int_{b-\delta}^{b} e^{\lambda R(t)}\,dt \right) = \inf_{\delta \in (0,b-a]} \liminf_{\lambda \to \infty} \left(\sqrt{\lambda} \int_{b-\delta}^{b} e^{\lambda R(t)}\,dt \right) < \infty$$

then the function $\sqrt{\lambda} I_\lambda,\ \lambda > 0$, has a finite limit at infinity.

Proof: Suppose that $\lim_{\lambda\to\infty}\sqrt{\lambda}I_\lambda = L \geq 0$.
1. There exists $M > 0$ such that

$$\sqrt{\lambda}\int_a^{v_k} e^{\lambda R(t)}\,dt \leq \sqrt{\lambda}I_\lambda \leq M,\ \forall\,\lambda > 0,\ \forall\,k \geq 1.$$

Then, Laplace's method ensures

$$\sum_{i=1}^{k} a_i = \lim_{\lambda\to\infty}\left(\sqrt{\lambda}\int_a^{v_k} e^{\lambda R(t)}\,dt\right) \leq M,\ \forall\,k \geq 1.$$

As follows, the series of positive numbers $\sum\limits_{k=1}^{\infty} a_k$ is convergent.
2. Let δ be a positive number, with $\delta \leq b-a$. The function $\sqrt{\lambda}\int_a^{b-\delta} e^{\lambda R(t)}\,dt$ has a finite limit $L(\delta)$ at infinity, where

$$L(\delta) = \begin{cases} 0, & \text{if } b-\delta < t_1; \\ \sum_{i=1}^{k} a_i, & \text{if } b-\delta \in (t_k, t_{k+1}),\ k \geq 1; \\ \sum_{1\leq i<k} a_i + a_k/2, & \text{if } b-\delta = t_k,\ k \geq 1; \end{cases}$$

Then, from the relation

$$\sqrt{\lambda}\int_{b-\delta}^{b} e^{\lambda R(t)}\,dt = \sqrt{\lambda}I_\lambda - \sqrt{\lambda}\int_a^{b-\delta} e^{\lambda R(t)}\,dt,\ \lambda > 0,$$

we obtain $\lim_{\lambda\to\infty}\sqrt{\lambda}\int_{b-\delta}^{b} e^{\lambda R(t)}\,dt = L - L(\delta) =: J(\delta)$.
3. The function $J(\delta) = \lim_{\lambda\to\infty}\left(\sqrt{\lambda}\int_{b-\delta}^{b} e^{\lambda R(t)}\,dt\right)$ is positive and monotone increasing on $(0, b-a]$, so $\lim_{\delta\to 0+} J(\delta) = \inf_{\delta\in(0,b-a]} J(\delta)$. Let us denote this limit by B. Consider now $f \in C[a,b]$. Since there exists $m > 0$ such that $|f(x)| \leq m,\ \forall\,x \in [a,b]$, the series $\sum_{k=1}^{\infty} a_k f(t_k)$ is absolutely convergent. For $\delta \in (0, b-a]$, we have

$$\left|\sqrt{\lambda}I_\lambda(f) - \left(\sum_{k=1}^{\infty} a_k f(t_k) + Bf(b)\right)\right| \leq \left|\sqrt{\lambda}\int_a^{b-\delta} e^{\lambda R(t)} f(t)\,dt - \sum_{k\geq 1;\ t_k\leq b-\delta} a_k f(t_k)\right|$$

$$+\left|\sum_{k\geq 1;\ t_k> b-\delta} a_k f(t_k)\right| + \sqrt{\lambda}\int_{b-\delta}^{b} e^{\lambda R(t)}|f(t)-f(b)|\,dt$$

$$+|f(b)|\left(\left|\sqrt{\lambda}\int_{b-\delta}^{b} e^{\lambda R(t)}\,dt - J(\delta)\right| + |J(\delta) - B|\right).$$

Let ε be an arbitrary positive number.
We can find a positive integer p such that $\left|\sum_{i=k+1}^{\infty} a_i f(t_i)\right| < \varepsilon/5,\ \forall\,k \geq p$. Also there are $\delta_1, \delta_2 \in (0, b-a]$ such that $|f(t) - f(b)| < \varepsilon/(5M),\ \forall\,t \in (b-\delta_1, b]$ and $J(\delta) - B = |J(\delta) - B| <$

$\varepsilon/(5m)$, $\forall\,\delta \in (0,\delta_2)$. Let $k \geq p$ be a fixed positive integer, such that $t_k > \max_{i=1,2}(b-\delta_i)$. Then, for $\delta = b - v_k$, we have

$$\left|\sqrt{\lambda} I_\lambda(f) - \left(\sum_{k=1}^{\infty} a_k f(t_k) + B f(b)\right)\right| < \left|\sqrt{\lambda}\int_a^{v_k} e^{\lambda R(t)} f(t)\,dt - \sum_{i=1}^{k} a_i f(t_i)\right|$$

$$+\frac{\varepsilon\sqrt{\lambda}}{5M}\int_{v_k}^{b} e^{\lambda R(t)}\,dt + |f(b)|\left|\sqrt{\lambda}\int_{v_k}^{b} e^{\lambda R(t)}\,dt - J(b-v_k)\right| + \frac{2\varepsilon}{5}$$

$$< \left|\sqrt{\lambda}\int_a^{v_k} e^{\lambda R(t)} f(t)\,dt - \sum_{i=1}^{k} a_i f(t_i)\right| + m\left|\sqrt{\lambda}\int_{v_k}^{b} e^{\lambda R(t)}\,dt - J(b-v_k)\right| + \frac{3\varepsilon}{5}.$$

According to the Laplace's method and the definition of $J(\delta)$, there are $\lambda_1, \lambda_2 > 0$ so that

$$\left|\sqrt{\lambda}\int_a^{v_k} e^{\lambda R(t)} f(t)\,dt - \sum_{i=1}^{k} a_i f(t_i)\right| < \frac{\varepsilon}{5},\ \forall\,\lambda > \lambda_1,$$

and

$$\left|\sqrt{\lambda}\int_{v_k}^{b} e^{\lambda R(t)}\,dt - J(b-v_k)\right| < \frac{\varepsilon}{5m},\ \forall\,\lambda > \lambda_2,$$

respectively. Therefore,

$$\left|\sqrt{\lambda} I_\lambda(f) - \left(\sum_{k=1}^{\infty} a_k f(t_k) + B f(b)\right)\right| < \varepsilon,\ \forall\,\lambda > \max_{i=1,2} \lambda_i.$$

So, (2.1) is proved.

For the converse statement, let us denote $S = \sum\limits_{k=1}^{\infty} a_k$ and

$$B = \inf_{\delta\in(0,b-a]} \limsup_{\lambda\to\infty}\left(\sqrt{\lambda}\int_{b-\delta}^{b} e^{\lambda R(t)}\,dt\right) = \inf_{\delta\in(0,b-a]} \liminf_{\lambda\to\infty}\left(\sqrt{\lambda}\int_{b-\delta}^{b} e^{\lambda R(t)}\,dt\right).$$

We have

$$\limsup_{\lambda\to\infty}\sqrt{\lambda} I_\lambda = L(\delta) + \limsup_{\lambda\to\infty}\left(\sqrt{\lambda}\int_{b-\delta}^{b} e^{\lambda R(t)}\,dt\right),\ \delta \in (0, b-a),$$

hence, passing to limit with $\delta \to 0+$, we find

$$\limsup_{\lambda\to\infty}\sqrt{\lambda} I_\lambda = S + B.$$

Similarly,

$$\liminf_{\lambda\to\infty}\sqrt{\lambda} I_\lambda = S + B.$$

So we obtain the conclusion. ∎

For the rest of this section, we will assume that R satisfies $\mathbf{C_0}$ and:

$\mathbf{C_1}$: *There is* $m > 0$ *such that* $R(u_k) = R(v_k) = -m^2$, $\forall\, k \geq 1$, *and* $\sup_{t\in[a,b]\setminus\bigcup_{k\geq 1}[u_k,v_k]} R(t) = -m^2$.

Remark that, under the assumption $\mathbf{C_0}$, the above condition $\mathbf{C_1}$ can be reduced to the assumption

$$\sup_{t\in[a,b]\setminus\bigcup_{k\geq 1}[u_k,v_k]} R(t) < 0.$$

Indeed, let m be a positive number such that $-m^2 \geq \sup_{t\in[a,b]\setminus\bigcup_{k\geq 1}[u_k,v_k]} R(t)$. We have $R(u_k), R(v_k) \leq -m^2$ and $R(t_k) = 0$, for $k = 1, 2, \ldots$. Then, the continuity of R on $[a, b)$ ensures the existence of the numbers $u'_k \in [u_k, t_k)$ and $v'_k \in (t_k, v_k]$ such that $R(u'_k) = R(v'_k) = -m^2$, for all $k \geq 1$. From $\mathbf{C_0}$, item 3), we obtain $\sup_{t\in[a,b]\setminus\bigcup_{k\geq 1}[u'_k,v'_k]} R(t) = -m^2$.

Clearly, under the condition $\mathbf{C_1}$, R is discontinuous at b.

In what follows we use a technique similar to the one described in Miller [4, §3.4]. For each $k \geq 1$, let us define the bijection $\tau_k : [-m, m] \to [u_k, v_k]$ by

$$\tau_k(s) = \begin{cases} \left(R|_{[u_k,t_k]}\right)^{-1}(-s^2), & s \in [-m, 0] \\ \left(R|_{[t_k,v_k]}\right)^{-1}(-s^2), & s \in [0, m] \end{cases}. \tag{2.2}$$

Note that $\tau_k(-m) = u_k,\ \tau_k(m) = v_k$ and $\tau_k(0) = t_k,\ \forall\, k \geq 1$. Let us consider the *indicator functions* $r_k : [-m, m] \to \mathbb{R},\ k \geq 1$, of R, defined by

$$r_k(s) = R'(\tau_k(s)),\ s \in [-m, m].$$

The function τ_k $(k \geq 1)$ is continuous differentiable on $[-m, m]$, with a positive derivative:

$$\tau'_k(s) = \begin{cases} \frac{-2s}{r_k(s)} = \frac{2|s|}{|R'(\tau_k(s))|}, & s \in [-m, m] \setminus \{0\} \\ \\ \sqrt{\frac{-2}{R''(t_k)}} = \frac{a_k}{\sqrt{\pi}}, & s = 0 \end{cases}. \tag{2.3}$$

See Miller [4, §3.4] for details.

For every positive integer p, let us define the function $g_p : [-m, m] \to (0, \infty)$,

$$g_p(s) = \sum_{k=1}^{p} \tau'_k(s),\ s \in [-m, m].$$

Let $g : [-m, m] \to (0, \infty]$ be the function defined by

$$g(s) = \lim_{p\to\infty} g_p(s) \in (0, \infty],\ s \in [-m, m],$$

i.e. g is the sum of the series of continuous positive functions $\sum_{k=1}^{\infty} \tau'_k$.

We obtain the following general result.

Theorem 2. *Assume that R satisfies $\mathbf{C_0}$ and $\mathbf{C_1}$, with $\sum_{k=1}^{\infty} a_k < \infty$. Consider a real number B. Then*

$$\lim_{\lambda\to\infty}\left(\sqrt{\lambda}\int_{-m}^{m} e^{-\lambda s^2}(g(s)-g(0))ds\right) = B \tag{2.4}$$

if and only if

$$\lim_{\lambda\to\infty}\sqrt{\lambda}I_\lambda(f) = \sum_{k=1}^{\infty} a_k f(t_k) + Bf(b), \ \textit{for } f \in C[a,b]. \tag{2.5}$$

Proof: Firstly, observe that

$$g(0) = \sum_{k=1}^{\infty}\tau_k'(0) = \frac{1}{\sqrt{\pi}}\sum_{k=1}^{\infty} a_k \in (0,\infty).$$

Denote

$$D_R = \bigcup_{k\geq 1}[u_k, v_k] \subset (a,b).$$

The contribution of the set $[a,b]\setminus D_R$ to the value of the function $\sqrt{\lambda}I_\lambda$ is asymptotically negligible (for $\lambda\to\infty$). Thus, under the assumption $\mathbf{C_1}$, we have

$$0 < \sqrt{\lambda}\int_{[a,b]\setminus D_R} e^{\lambda R(t)}\,dt \leq \sqrt{\lambda}e^{-\lambda m^2}\int_{[a,b]\setminus D_R} dt \leq \sqrt{\lambda}e^{-\lambda m^2}(b-a).$$

Therefore, $\lim_{\lambda\to\infty}\left(\sqrt{\lambda}\int_{[a,b]\setminus D_R} e^{\lambda R(t)}\,dt\right) = 0$.

Let k be a positive integer. With the substitution $t = \tau_k(s)$, we obtain

$$I_{\lambda,k} := \int_{u_k}^{v_k} e^{\lambda R(t)}\,dt = \int_{-m}^{m} e^{-\lambda s^2}\tau_k'(s)\,ds$$

$$= \int_{-m}^{m} e^{-\lambda s^2}\tau_k'(0)\,ds + \int_{-m}^{m} e^{-\lambda s^2}(\tau_k'(s)-\tau_k'(0))\,ds.$$

Denote

$$I_{\lambda,k}^{(1)} = \int_{-m}^{m} e^{-\lambda s^2}\tau_k'(0)\,ds \ \text{ and } \ I_{\lambda,k}^{(2)} = \int_{-m}^{m} e^{-\lambda s^2}(\tau_k'(s)-\tau_k'(0))\,ds.$$

For the first integral, we have:

$$I_{\lambda,k}^{(1)} = 2\tau_k'(0)\int_0^m e^{-\lambda s^2}\,ds = \frac{a_k}{\sqrt{\pi}}\int_0^{m^2} e^{-\lambda u}u^{-1/2}\,du.$$

From Watson's lemma (see, for example, Miller [4, Proposition 2.1]) we obtain

$$\int_0^{m^2} e^{-\lambda u}u^{-1/2}\,du = \sqrt{\frac{\pi}{\lambda}} + \Delta_m(\lambda), \ \text{where } \Delta_m(\lambda) = O\left(\lambda^{-3/2}\right), \ \text{for } \lambda\to\infty.$$

Hence

$$I^{(1)}_{\lambda,k} = \frac{a_k}{\sqrt{\lambda}} + \frac{a_k \Delta_m(\lambda)}{\sqrt{\pi}},\ k = 1, 2, \cdots .$$

Since the series $\sum_{k=1}^{\infty} a_k$ is convergent, we get the following estimation

$$\sum_{k=1}^{\infty} I^{(1)}_{\lambda,k} = \frac{1}{\sqrt{\lambda}} \sum_{k=1}^{\infty} a_k + O\left(\lambda^{-3/2}\right),\ \text{for } \lambda \to \infty.$$

Now we see that (2.5) with $f = 1$ is equivalent to the condition

$$\lim_{\lambda\to\infty} \left(\sqrt{\lambda} \sum_{k=1}^{\infty} I^{(2)}_{\lambda,k} \right) = B.$$

We can apply the Theorem of monotone convergence of Beppo-Levi to the sequence of functions $(g_p)_{p\geq 1}$, to obtain

$$\sum_{k=1}^{\infty} I^{(2)}_{\lambda,k} = \int_{-m}^{m} e^{-\lambda s^2} (g(s) - g(0))\, ds.$$

Hence (2.5) with $f = 1$ is equivalent to (2.4). Then we apply Theorem 1 to complete the proof. In addition, note that we must have $B \geq 0$. □

The following theorem indicates an useful sufficient condition for $B = 0$.

Theorem 3. *Assume that R satisfies $\mathbf{C_0}$ and $\mathbf{C_1}$, with $\sum_{k=1}^{\infty} a_k < \infty$. If the function g is continuous at 0, then*

$$\lim_{\lambda\to\infty} \sqrt{\lambda} I_\lambda(f) = \sum_{k=1}^{\infty} a_k f(t_k),\ \textit{for } f \in C[a,b]. \tag{2.6}$$

In particular, if there is a convergent series $\sum_{k=1}^{\infty} A_k$ of positive numbers, such that

$$\frac{2\sqrt{-R(t)}}{|R'(t)|} \leq A_k,\ t \in [u_k, v_k] \setminus \{t_k\},\ \forall\, k \geq 1,$$

then we have (2.6).

Proof: From Theorem 2, it suffices to show

$$\lim_{\lambda\to\infty} \left(\sqrt{\lambda} \int_{-m}^{m} e^{-\lambda s^2} (g(s) - g(0)) ds \right) = 0.$$

Let $\varepsilon > 0$. Since the function g is assumed to be continuous at 0, we can choose $\delta = \delta_\varepsilon \in (0, m)$ such that $|g(s) - g(0)| < \varepsilon,\ \forall\, s \in [-\delta, \delta]$. Denote $u'_k = \tau_k(-\delta) \in (u_k, t_k)$ and $v'_k = \tau_k(\delta) \in$

(t_k, v_k). Note that $R(u_k') = R(v_k') = -\delta^2$.
We have

$$\left|\int_{-m}^{m} e^{-\lambda s^2}(g(s)-g(0))\,ds\right|$$

$$\leq \int_{-\delta}^{\delta} e^{-\lambda s^2}|g(s)-g(0)|\,ds + \int_{[-m,-\delta]\cup[\delta,m]} e^{-\lambda s^2}|g(s)-g(0)|\,ds$$

$$\leq \varepsilon \int_{-\delta}^{\delta} e^{-\lambda s^2}\,ds + \int_{[-m,-\delta]\cup[\delta,m]} e^{-\lambda s^2} g(s)\,ds + g(0)\int_{[-m,-\delta]\cup[\delta,m]} e^{-\lambda s^2}\,ds.$$

Firstly, recall that

$$\int_{-\delta}^{\delta} e^{-\lambda s^2}\,ds = \sqrt{\frac{\pi}{\lambda}} + O\left(\lambda^{-3/2}\right), \text{ for } \lambda \to \infty.$$

Then

$$\int_{[-m,-\delta]\cup[\delta,m]} e^{-\lambda s^2} g(s)\,ds = \sum_{k=1}^{\infty} \int_{[-m,-\delta]\cup[\delta,m]} e^{-\lambda s^2}\tau_k'(s)\,ds$$

$$= \sum_{k=1}^{\infty} \int_{[u_k,u_k']\cup[v_k',v_k]} e^{\lambda R(t)}\,dt \leq e^{-\lambda\delta^2}(b-a)$$

and

$$\int_{[-m,-\delta]\cup[\delta,m]} e^{-\lambda s^2}\,ds < 2(m-\delta)e^{-\lambda\delta^2}.$$

Therefore

$$\limsup_{\lambda\to\infty}\left|\sqrt{\lambda}\int_{-m}^{m} e^{-\lambda s^2}(g(s)-g(0))\,ds\right| \leq \varepsilon\sqrt{\pi}.$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$\lim_{\lambda\to\infty}\sqrt{\lambda}\int_{-m}^{m} e^{-\lambda s^2}(g(s)-g(0))\,ds = 0.$$

The conclusion follows from Theorem 2.

For the last part, observe that the relations (2.2) and (2.3) provide the following equivalence

$$\frac{2\sqrt{-R(t)}}{|R'(t)|} \leq A_k,\ \forall\, t \in [u_k, v_k]\setminus\{t_k\} \quad\Leftrightarrow\quad 0 < \tau_k'(s) \leq A_k,\ \forall\, s \in [-m, m],$$

for $k \geq 1$. Then, from the Weierstrass M-test, the series of continuous functions $\sum_{k=1}^{\infty}\tau_k'$ is uniformly convergent on $[-m, m]$. So $g = \sum_{k=1}^{\infty}\tau_k'$ is continuous on $[-m, m]$ and we obtain the conclusion. ∎

Let us see now some useful results based on above theorem.

Corollary 1. *Assume that R satisfies $\mathbf{C_0}$ and $\mathbf{C_1}$, with $\sum_{k=1}^{\infty} a_k < \infty$. If the indicator functions $\{r_k\}_{k\geq 1}$ are convex on $[-m,0]$ and concave on $[0,m]$, then the relation (2.6) is satisfied. We have the same conclusion if the functions $\{\tau_k\}_{k\geq 1}$ are convex on $[-m,0]$ and concave on $[0,m]$.*

Proof: Let k be a positive integer. Since $\tau_k \in C^1[-m,m]$, and $r_k(0)=0$, we can use the rule of l'Hôpital in (2.3) to obtain $\tau_k'(0) = \frac{-2}{r_k'(0)}$.

The convexity of r_k on $[-m,0]$ ensures the inequality

$$r_k(s) \geq r_k'(0)s,\ \forall\, s \in [-m,0].$$

Since $r_k'(0) = R''(t_k)\tau_k'(0) < 0$ and $r_k(s) = R'(\tau_k(s)) > 0$, we get the inequality

$$\frac{-2s}{r_k(s)} \leq \frac{-2}{r_k'(0)}.$$

Hence, from (2.3),

$$\tau_k'(s) \leq \tau_k'(0),\ \forall\, s \in [-m,0]. \tag{2.7}$$

Similarly, using the concavity of r_k on $[0,m]$, we obtain

$$\tau_k'(s) \leq \tau_k'(0),\ \forall\, s \in [0,m]. \tag{2.8}$$

Now, from the inequalities $0 < \tau_k'(s) \leq \tau_k'(0),\ \forall\, s \in [-m,m],\ \forall\, k \geq 1$, and the convergence of the series $\sum_{k=1}^{\infty} \tau_k'(0) = \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} a_k$, we obtain the conclusion by applying Theorem 3.

If the functions $\{\tau_k\}_{k\geq 1}$ are convex on $[-m,0]$ and concave on $[0,m]$, then the relations (2.7) and (2.8) hold for all positive integers k. So we obtain the same conclusion. □

Corollary 2. *Assume that R satisfies $\mathbf{C_0}$ and $\mathbf{C_1}$, with $\sum_{k=1}^{\infty} a_k < \infty$. If there are two functions $\psi : (a,b) \to (0,\infty)$ and $\varphi : [-m^2, 0] \to [0,\infty)$, where ψ is continuous differentiable and monotone, such that*

$$|R'(t)| = \varphi(R(t))\psi(t),\ \forall\, t \in D_R,$$

then we have (2.6).

Proof: For $k \geq 1$, we have $\varphi(0) = \varphi(R(t_k)) = |R'(t_k)|/\psi(t_k) = 0$. Since

$$\varphi(-s^2) = \frac{R'(\tau_k(s))}{\psi(\tau_k(s))},\ \text{for } s \in [-m,0],$$

we deduce that the function φ is continuous on $[-m^2,0]$ and differentiable on $[-m^2,0)$. From (2.3), we find

$$\frac{2|s|}{\varphi(-s^2)} = \psi(\tau_k(s))\tau_k'(s),\ s \in [-m,m] \setminus \{0\}.$$

Then, the positive continuous function $h(s) := \frac{2|s|}{\varphi(-s^2)},\ s \in [-m,m] \setminus \{0\}$ has a finite limit L at 0, where

$$L = \lim_{s\to 0} \psi(\tau_k(s))\tau_k'(s) = \psi(t_k)\tau_k'(0) > 0,\ \forall\, k \geq 1. \tag{2.9}$$

Thus, there is $M \geq L$ such that $0 < h(s) \leq M,\ \forall\, s \in [-m, m] \setminus \{0\}$. So,

$$0 < \tau_k'(s) \leq M \frac{1}{\psi(\tau_k(s))},\ \forall\, s \in [-m, m]. \tag{2.10}$$

The assumption $\sum_{k=1}^{\infty} a_k < \infty$, where $a_k = \tau_k'(0)\sqrt{\pi} = L\sqrt{\pi}/\psi(t_k)$, for $k \geq 1$ (see (2.3) and (2.9)), ensures $\lim_{k\to\infty} \psi(t_k) = \infty$. As follows, the monotone function ψ must be increasing on (a, b), with $\lim_{t\uparrow b} \psi(t) = \infty$. In this case, from (2.9) and (2.10), we get

$$0 < \tau_k'(s) \leq M \frac{1}{\psi(t_{k-1})} = \frac{M}{L} \tau_{k-1}'(0),\ s \in [-m, m],\ k \geq 2.$$

In addition, the continuous function τ_1' is bounded on the compact $[-m, m]$. Since the series $\sum_{k=1}^{\infty} \tau_k'(0)$ is convergent, we can apply Theorem 3 to finish the proof of (2.6). □

Notice that one can rewrite all above results for the case when the set $T_R \setminus \{a\}$ forms a strictly decreasing sequence $(t_k)_{k\geq 1} \subset (a, b)$, with $\lim_{k\to\infty} t_k = a$.

3 Examples

We illustrate our theoretical results by two examples.

Example 1. *Let*

$$I_\lambda(f) = \int_0^1 e^{\lambda R(t)} f(t)\, dt,\ f \in C[0, 1],\ \lambda > 0,$$

where $R : [0, 1) \to \mathbb{R}$,

$$R(t) = -2^{2k+2} \left[t - \sum_{i=1}^{k-1} \frac{1}{2^i} - \frac{1}{2^{k+1}} \right]^2,\ t \in \left[\sum_{i=1}^{k-1} \frac{1}{2^i}, \sum_{i=1}^{k} \frac{1}{2^i} \right],\ k \geq 1.$$

We have

$$T_R = \left\{ t_k = \sum_{i=1}^{k-1} \frac{1}{2^i} + \frac{1}{2^{k+1}} \Big| k \geq 1 \right\} \subset (0, 1),$$

with $(t_k)_{k\geq 1}$ strictly increasing and $\lim_{k\to\infty} t_k = 1$. We can choose $m = 1$. Thus, we find $u_k = \sum_{i=1}^{k-1} \frac{1}{2^i}$ and $v_k = \sum_{i=1}^{k} \frac{1}{2^i}$, for $k \geq 1$. For every $k \geq 1$, the indicator function of R is

$$r_k(s) = \sum_{i=1}^{k-1} \frac{1}{2^i} + \frac{s+1}{2^{k+1}},\ s \in [-1, 1].$$

Then, from Corollary 1 we obtain

$$\lim_{\lambda\to\infty} \sqrt{\lambda} I_\lambda(f) = \sum_{k=1}^{\infty} \frac{\sqrt{\pi}}{2^{k+1}} f(t_k).$$

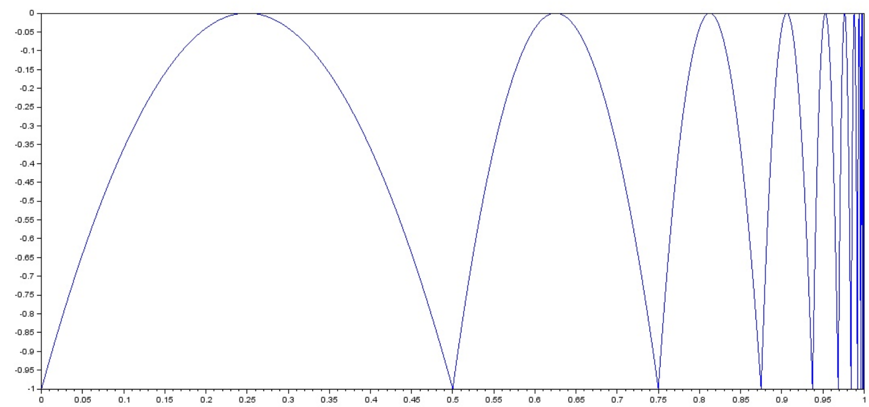

Figure 1: The exponent function $R(t)$

Example 2. *Let*

$$I_\lambda(f) = \int_0^{\frac{1}{\pi}} e^{\lambda\left(\sin\frac{1}{t}-1\right)} f(t)\,dt,\ \ f \in C\left[0, \frac{1}{\pi}\right],\ \ \lambda > 0.$$

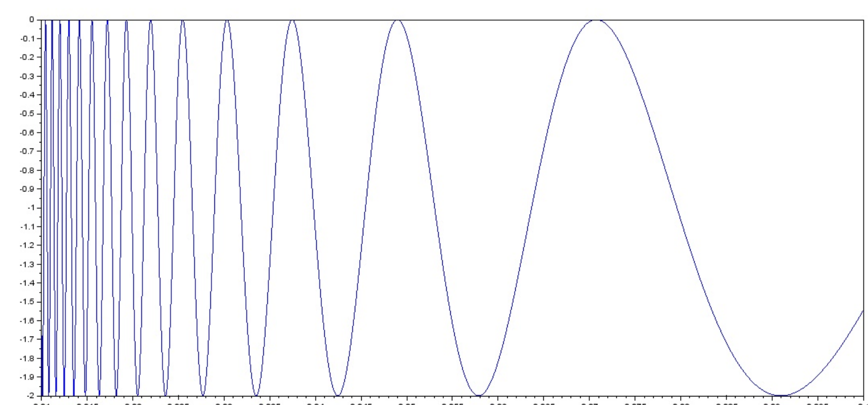

Figure 2: The exponent function $R(t)$

We have

$$R(t) = \sin\frac{1}{t} - 1,\ t \in \left(0, \frac{1}{\pi}\right]$$

with

$$T_R = \left\{ t_k = \frac{2}{(4k+1)\pi} \Big| k \geq 1 \right\} \subset \left(0, \frac{1}{\pi}\right).$$

The sequence $(t_k)_{k\geq 1}$ is strictly decreasing, with $\lim_{k\to\infty} t_k = 0$. We can choose $m = 1$ and we find $u_k = \frac{1}{(2k+1)\pi}$ and $v_k = \frac{1}{2k\pi}$, for $k \geq 1$. For $t \in \left(0, \frac{1}{\pi}\right]$, one has

$$R'(t) = -\cos\left(\frac{1}{t}\right)\frac{1}{t^2} \text{ and } R''(t) = -\frac{1}{t^4}\sin\left(\frac{1}{t}\right) + \frac{2}{t^3}\cos\left(\frac{1}{t}\right).$$

Hence $R''(t_k) = -\frac{1}{t_k^4}$, $k \geq 1$. Thus, $\sum_{k=1}^{\infty} a_k < \infty$. On the other hand,

$$|R'(t)| = \varphi(R(t))\psi(t),\ \forall\, t \in D_R,$$

where

$$\varphi : [-1, 0] \to [0, \infty),\ \varphi(x) = \sqrt{1 - (x+1)^2},\ \text{and}\ \psi : \left(0, \frac{1}{\pi}\right) \to (0, \infty),\ \psi(t) = \frac{1}{t^2}.$$

Thus, from Corollary 2 (adapted to the case $t_k \downarrow a$) we obtain

$$\lim_{\lambda\to\infty} \sqrt{\lambda} I_\lambda(f) = \frac{4\sqrt{2\pi}}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(4k+1)^2} f(t_k).$$

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