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Laguerre characterization and rigidity of hypersurfaces in $\mathbb{R}^n$

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Abstract

Let $x : M \to \mathbb{R}^n$ be an $(n-1)$-dimensional hypersurface in $\mathbb{R}^n$, $\mathbf{L}$ be the Laguerre tensor, $\mathbf{B}$ be the Laguerre second fundamental form and $\mathbf{C}$ be the Laguerre form of the immersion x. The purpose of this paper is to investigate Laguerre characterization and rigidity of hypersurfaces in $\mathbb{R}^n$. We firstly obtain the classification of Laguerre isoparametric hypersurfaces with three distinct Laguerre principal curvatures one of which is simple and then we obtain a Laguerre rigidity result of hypersurfaces in $\mathbb{R}^n$.

Key Words: Laguerre tensor, Laguerre second fundamental form, Laguerre form, Laguerre metric.
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1 Introduction

In Laguerre geometry, T. Li and C. Wang [4] studied invariants of hypersurfaces in Euclidean space $\mathbb{R}^n$ under the Laguerre transformation group. The Laguerre transformations are the Lie sphere transformations which take oriented hyperplanes in $\mathbb{R}^n$ to oriented hyperplanes and preserve the tangential distance.

Let $U\mathbb{R}^n$ be the unit tangent bundle over $\mathbb{R}^n$. An oriented sphere in $\mathbb{R}^n$ centered at p with radius r can be regarded as the oriented sphere $\{(x,\xi)|x-p=r\xi\}$ in $U\mathbb{R}^n$, where x is the position vector and ξ the unit normal vector of the sphere. An oriented hyperplane in $\mathbb{R}^n$ with constant unit normal vector ξ and constant real number c can be regarded as the oriented hyperplane $\{(x,\xi)|x\cdot\xi=c\}$ in $U\mathbb{R}^n$. A diffeomorphism $\phi : U\mathbb{R}^n \to U\mathbb{R}^n$ which takes oriented spheres to oriented spheres, oriented hyperplanes to oriented hyperplanes, preserving the tangential distance of any two spheres, is called a Laguerre transformation. All Laguerre transformations in $U\mathbb{R}^n$ form a group of dimension $(n+1)(n+2)/2$, called Laguerre transformation group. An oriented hypersurface $x : M \to \mathbb{R}^n$ can be identified as the submanifold $(x,\xi) : M \to U\mathbb{R}^n$, where ξ is the unit normal of x. Two hypersurfaces $x, x^* : M \to \mathbb{R}^n$ are called Laguerre equivalent, if there is a Laguerre transformation $\phi : U\mathbb{R}^n \to U\mathbb{R}^n$ such that $(x^*,\xi^*) = \phi\circ(x,\xi)$ (see [5]).

In [4], T. Li and C. Wang gave a complete Laguerre invariant system for hypersurfaces in $\mathbb{R}^n$. They proved that two umbilical free oriented hypersurfaces in $\mathbb{R}^n$ with non-zero principal curvatures are Laguerre equivalent if and only if they have the same Laguerre metric g and Laguerre second fundamental form $\mathbf{B}$. We should notices that the Laguerre geometry of surfaces in $\mathbb{R}^3$ has been studied by Blaschke in [1] and other authors in [2], [3], [6].

Let $\mathbb{R}_2^{n+3}$ be the space $\mathbb{R}^{n+3}$ equipped with the inner product $\langle X, Y\rangle = -X_1Y_1 + X_2Y_2 + \cdots + X_{n+2}Y_{n+2} - X_{n+3}Y_{n+3}$. Let C^{n+2} be the light-cone in $\mathbb{R}^{n+3}$ given by $C^{n+2} = \{X \in \mathbb{R}_2^{n+3} | \langle X, X\rangle = 0\}$. Let $L\mathbb{G}$ be the subgroup of the orthogonal group $O(n+1,2)$ on $\mathbb{R}_2^{n+3}$ given by $L\mathbb{G} = \{T \in O(n+1,2) | \zeta T = \zeta\}$, where $\zeta = (1, -1, \mathbf{0}, 0)$ and $\mathbf{0} \in \mathbb{R}^n$ is a light-like vector in $\mathbb{R}_2^{n+3}$.

Let $x : M \to \mathbb{R}^n$ be an umbilic free hypersurface with non-zero principal curvatures, and $\xi : M \to S^{n-1}$ be its unit normal vector. Let $\{e_1, e_2, \ldots, e_{n-1}\}$ be the orthonormal basis for TM with respect to $dx \cdot dx$, consisting of unit principal vectors. The structure equations of $x : M \to \mathbb{R}^n$ are (see [5])

$$e_j(e_i(x)) = \sum_k \Gamma_{ij}^k e_k(x) + k_i\delta_{ij}\xi, \quad e_i(\xi) = -k_ie_i(x), \quad i, j, k = 1, \ldots, n-1, \tag{1.1}$$

where $k_i \neq 0$ is the principal curvature corresponding to e_i. Let

$$r_i = \frac{1}{k_i}, \quad r = \frac{r_1 + r_2 + \cdots + r_{n-1}}{n-1}, \tag{1.2}$$

be the curvature radii and mean curvature radius of x respectively. We define $Y = \rho(x\cdot\xi, -x\cdot\xi, \xi, 1) : M \to C^{n+2} \subset \mathbb{R}_2^{n+3}$, where $\rho = \sqrt{\sum_i (r_i - r)^2} > 0$. From [4], we know that the *Laguerre metric* g of the immersion x can be defined by $g = \langle dY, dY\rangle$. Let $\{E_1, E_2, \ldots, E_{n-1}\}$ be an orthonormal basis for g with dual basis $\{\omega_1, \omega_2, \ldots, \omega_{n-1}\}$. *The Laguerre tensor* $\mathbf{L}$, *the Laguerre second fundamental form* $\mathbf{B}$ and *the Laguerre form* $\mathbf{C}$ of the immersion x are defined by

$$\mathbf{L} = \sum_{i,j=1}^{n-1} L_{ij}\omega_i \otimes \omega_j, \quad \mathbf{B} = \sum_{i,j=1}^{n-1} B_{ij}\omega_i \otimes \omega_j, \quad \mathbf{C} = \sum_{i=1}^{n-1} C_i\omega_i, \tag{1.3}$$

respectively, where L_{ij}, B_{ij} and C_i are defined by formulas (2.10)–(2.12) in Section 2. We should notices that g, $\mathbf{L}$, $\mathbf{B}$ and $\mathbf{C}$ are Laguerre invariants (see [4]).

From [7], we know that an eigenvalue of the Laguerre tensor is called *a Laguerre eigenvalue* of x. A hypersurface with vanishing Laguerre form is called *a Laguerre isotropic hypersurface* if the Laguerre eigenvalues of x are equal. An eigenvalue of the Laguerre second fundamental form is called *a Laguerre principal curvature* of x. An umbilic free hypersurface $x : M \to \mathbb{R}^n$ with non-zero principal curvatures and vanishing Laguerre form $\mathbf{C} \equiv 0$ is called *a Laguerre isoparametric hypersurface* if the Laguerre principal curvatures of x are constants.

We define the Laguerre embedding $\tau : U\mathbb{R}_0^n \to U\mathbb{R}^n$(see [4]). Let $\mathbb{R}_1^{n+1}$ be the Minkowski space with the inner product $\langle X, Y\rangle = X_1Y_1 + \cdots + X_nY_n - X_{n+1}Y_{n+1}$. Let $\nu = (1, \mathbf{0}, 1)$ be the light-like vector in $\mathbb{R}_1^{n+1}$, $\mathbf{0} \in \mathbb{R}^{n-1}$. Let $\mathbb{R}_0^n$ be the degenerate hyperplane in $\mathbb{R}_1^{n+1}$ defined by $\mathbb{R}_0^n = \{X \in \mathbb{R}_1^{n+1} | \langle X, \nu\rangle = 0\}$. We define

$$U\mathbb{R}_0^n = \{(x, \xi) \in \mathbb{R}_1^{n+1} \times \mathbb{R}_1^{n+1} | \langle x, \nu\rangle = 0, \langle \xi, \xi\rangle = 0, \langle \xi, \nu\rangle = 1\}. \tag{1.4}$$

The Laguerre embedding $\tau : U\mathbb{R}_0^n \to U\mathbb{R}^n$ is defined by

$$\tau(x, \xi) = (x', \xi') \in U\mathbb{R}^n, \tag{1.5}$$

where $x = (x_1, x_0, x_1) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}$, $\xi = (\xi_1 + 1, \xi_0, \xi_1) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}$ and

$$x' = \left(-\frac{x_1}{\xi_1}, x_0 - \frac{x_1}{\xi_1}\xi_0\right), \quad \xi' = \left(1 + \frac{1}{\xi_1}, \frac{\xi_0}{\xi_1}\right). \tag{1.6}$$

Let $x : M \to \mathbb{R}^n_0$ be a space-like oriented hypersurface in the degenerate hyperplane $\mathbb{R}^n_0$. Let ξ be the unique vector in $\mathbb{R}^{n+1}_1$ satisfying $\langle \xi, dx\rangle = 0$, $\langle \xi, \xi\rangle = 0$, $\langle \xi, \nu\rangle = 1$. From $\tau(x, \xi) = (x', \xi') \in U\mathbb{R}^n$, we may obtain a hypersurface $x' : M \to \mathbb{R}^n$.

We should notice that it is one of the important aims to characterize hypersurfaces in terms of Laguerre invariants. Concerning this topic, recently, T. Li, H. Li and C. Wang [5] and [7] studied the Laguerre geometry of hypersurfaces with parallel Laguerre second fundamental form or constant Laguerre eigenvalues in $\mathbb{R}^n$.

Theorem 1.1 ([5]) *Let $x : M \to \mathbb{R}^n$ be an umbilic free hypersurface with non-zero principal curvatures. If the Laguerre second fundamental form of x is parallel, then x is Laguerre equivalent to an open part of one of the following hypersurfaces:*
(1) *the oriented hypersurface $x : S^{k-1} \times H^{n-k} \to \mathbb{R}^n$ given by Example 2.1; or*
(2) *the image of τ of the oriented hypersurface $x : \mathbb{R}^{n-1} \to \mathbb{R}^n_0$ given by Example 2.2.*

In this paper, we firstly classify completely the Laguerre isoparametric hypersurfaces with three distinct Laguerre principal curvatures one of which is simple and then we obtain a Laguerre rigidity result of hypersurfaces in $\mathbb{R}^n$. More precisely, we obtain the following:

Theorem 1.2 *Let $x : M \to \mathbb{R}^n (n \geq 4)$ be an $(n-1)$-dimensional Laguerre isoparametric hypersurface with three distinct Laguerre principal curvatures one of which is simple. Then x is Laguerre equivalent to an open part of the image of τ of the oriented hypersurface $x : \mathbb{R}^{n-1} \to \mathbb{R}^n_0$ given by Example 2.2.*

Theorem 1.3 *Let $x : M \to \mathbb{R}^n (n \geq 4)$ be an $(n-1)$-dimensional umbilic free hypersurface with non-zero principal curvatures and vanishing Laguerre form. If the square of the norm of Laguerre tensor satisfies*

$$|\mathbf{L}|^2 \leq \frac{(n-1)R^2}{4(n-2)^2(n-3)^2}, \tag{1.7}$$

then x is Laguerre equivalent to an open part of the oriented hypersurface $x : S^{n-2} \times H^1 \to \mathbb{R}^n$ given by Example 2.1, where $R \geq 0$ is the Laguerre scalar curvature of x.

Remark 1.4 From Example 2.1 in section 2, we see that the pinching constant for $|\mathbf{L}|^2$ in Theorem 1.3 is optimal.

2 Laguerre fundamental formulas and examples

In this section, we review the Laguerre invariants and fundamental formulas on Laguerre geometry of hypersurfaces in $\mathbb{R}^n$, for more details, see [4].

Let $x : M \to \mathbb{R}^n$ be an $(n-1)$-dimensional umbilical free hypersurface with vanishing Laguerre form in $\mathbb{R}^n$. Let $\{E_1, \dots, E_{n-1}\}$ denote a local orthonormal frame for Laguerre metric $g = \langle dY, dY\rangle$ with dual frame $\{\omega_1, \dots, \omega_{n-1}\}$. Putting $Y_i = E_i(Y)$, then we have

$$N = \frac{1}{n-1}\Delta Y + \frac{1}{2(n-1)^2}\langle \Delta Y, \Delta Y\rangle Y, \tag{2.1}$$

$$\langle Y, Y\rangle = \langle N, N\rangle = 0, \quad \langle Y, N\rangle = -1, \tag{2.2}$$

and the following orthogonal decomposition:

$$\mathbb{R}^{n+3}_2 = \mathrm{Span}\{Y, N\} \oplus \mathrm{Span}\{Y_1, \dots, Y_{n-1}\} \oplus \mathbb{V}, \tag{2.3}$$

where $\{Y, N, Y_1, \dots, Y_{n-1}, \eta, \wp\}$ forms a moving frame in $\mathbb{R}^{n+3}_2$ and $\mathbb{V} = \{\eta, \wp\}$ is called *Laguerre normal bundle* of x . We use the following range of indices throughout this paper:

$$1 \le i, j, k, l, m \le n-1.$$

The structure equations on x with respect to the Laguerre metric g can be written as

$$dY = \sum_i \omega_i Y_i, \tag{2.4}$$

$$dN = \sum_i \psi_i Y_i + \varphi\eta, \tag{2.5}$$

$$dY_i = -\psi_i Y - \omega_i N + \sum_j \omega_{ij} Y_j + \omega_{in+1}\eta, \tag{2.6}$$

$$d\wp = -\varphi Y - \sum_i \omega_{in+1} Y_i, \tag{2.7}$$

where $\{\psi_i, \omega_{ij}, \omega_{in+1}, \varphi\}$ are 1-forms on x with

$$\omega_{ij} + \omega_{ji} = 0, \quad d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \tag{2.8}$$

and

$$\psi_i = \sum_j L_{ij}\omega_j, \quad L_{ij} = L_{ji}, \quad \omega_{in+1} = \sum_j B_{ij}\omega_j, \quad B_{ij} = B_{ji}, \quad \varphi = \sum_i C_i\omega_i. \tag{2.9}$$

We define $\tilde{E}_i = r_i e_i$, $1 \le i \le n-1$, then $\{\tilde{E}_1, \dots, \tilde{E}_{n-1}\}$ is an orthonormal basis for $III = d\xi \cdot d\xi$ and $\{E_i = \rho^{-1}\tilde{E}_i\}$ is an orthonormal basis for the Laguerre metric g with dual frame $\{\omega_1, \dots, \omega_{n-1}\}$. L_{ij}, B_{ij} and C_i are locally defined functions and satisfy

$$L_{ij} = \rho^{-2}\left\{\mathrm{Hess}_{ij}(\log\rho) - \tilde{E}_i(\log\rho)\tilde{E}_j(\log\rho) + \frac{1}{2}\left(|\nabla \log\rho|^2 - 1\right)\delta_{ij}\right\}, \tag{2.10}$$

$$B_{ij} = \rho^{-1}(r_i - r)\delta_{ij}, \tag{2.11}$$

$$C_i = -\rho^{-2}\left\{\tilde{E}_i(r) - \tilde{E}_i(\log\rho)(r_i - r)\right\}, \tag{2.12}$$

where $g = \sum_i (r_i - r)^2 III = \rho^2 III$, r_i and r are defined by (1.2), Hess_{ij} and ∇ are the Hessian matrix and the gradient with respect to the third fundamental form $III = d\xi \cdot d\xi$ of x(see [4]).

Defining the covariant derivative of C_i, L_{ij}, B_{ij} by

$$\sum_j C_{i,j}\omega_j = dC_i + \sum_j C_j \omega_{ji}, \tag{2.13}$$

$$\sum_k L_{ij,k}\omega_k = dL_{ij} + \sum_k L_{ik}\omega_{kj} + \sum_k L_{kj}\omega_{ki}, \tag{2.14}$$

$$\sum_k B_{ij,k}\omega_k = dB_{ij} + \sum_k B_{ik}\omega_{kj} + \sum_k B_{kj}\omega_{ki}. \tag{2.15}$$

We have from [4] that

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l, \quad R_{ijkl} = -R_{jikl}, \tag{2.16}$$

$$\sum_i B_{ii} = 0, \quad \sum_{i,j} B_{ij}^2 = 1, \quad \sum_i B_{ij,i} = (n-2)C_j, \quad \mathrm{tr}\mathbf{L} = -\frac{R}{2(n-2)}. \tag{2.17}$$

$$L_{ij,k} = L_{ik,j}, \tag{2.18}$$

$$C_{i,j} - C_{j,i} = \sum_k (B_{ik}L_{kj} - B_{jk}L_{ki}), \tag{2.19}$$

$$B_{ij,k} - B_{ik,j} = C_j\delta_{ik} - C_k\delta_{ij}, \tag{2.20}$$

$$R_{ijkl} = L_{jk}\delta_{il} + L_{il}\delta_{jk} - L_{ik}\delta_{jl} - L_{jl}\delta_{ik}, \tag{2.21}$$

where R_{ijkl} and R denote the Laguerre curvature tensor and the Laguerre scalar curvature with respect to the Laguerre metric g on x. Since the Laguerre form $\mathbf{C} \equiv 0$, we have for all indices i, j, k

$$B_{ij,k} = B_{ik,j}, \quad \sum_k B_{ik}L_{kj} = \sum_k B_{kj}L_{ki}. \tag{2.22}$$

Defining the second covariant derivative of B_{ij} by

$$\sum_l B_{ij,kl}\omega_l = dB_{ij,k} + \sum_l B_{lj,k}\omega_{li} + \sum_l B_{il,k}\omega_{lj} + \sum_l B_{ij,l}\omega_{lk}, \tag{2.23}$$

we have the Ricci identity

$$B_{ij,kl} - B_{ij,lk} = \sum_m B_{mj}R_{mikl} + \sum_m B_{im}R_{mjkl}. \tag{2.24}$$

We recall the following examples of hypersurfaces in $\mathbb{R}^n$ and calculate their Laguerre invariants.

Example 2.1([5]) Let $x : S^{k-1} \times H^{n-k} \to \mathbb{R}^n$ be an umbilic free hypersurface in $\mathbb{R}^n$ defined by

$$x(u,v,w) = \left(\frac{u}{w}(1+w), \frac{v}{w}\right),$$

where $H^{n-k} = \{(v,w) \in \mathbb{R}_1^{n-k+1} | v \cdot v - w^2 = -1, w > 0\}$ denotes the hyperbolic space embedded in the Minkowski space $\mathbb{R}_1^{n-k+1}$. From [5], we know that x has two distinct Laguerre principal curvatures $B_1 = -\sqrt{\frac{n-k}{(k-1)(n-1)}}$, $B_2 = \sqrt{\frac{k-1}{(n-k)(n-1)}}$, the Laguerre form is zero and the Laguerre second fundamental form of x is parallel. The Laguerre metric is

$$g = \frac{(k-1)(n-k)}{n-1}\left(du \cdot du + dv \cdot dv - dw^2\right) = g_1 + g_2,$$

where $g_1 = \frac{(k-1)(n-k)}{n-1} du \cdot du$ and $g_2 = \frac{(k-1)(n-k)}{n-1}\left(dv \cdot dv - dw^2\right)$. We know that the sectional curvatures of g_1 and g_2 are $\frac{n-1}{(k-1)(n-k)}$ and $-\frac{n-1}{(k-1)(n-k)}$, respectively. Thus, from (2.21), we see that

$$L_{ij} = -\frac{n-1}{2(k-1)(n-k)}\delta_{ij}, \quad 1 \le i,j \le k-1,$$
$$L_{ij} = \frac{n-1}{2(k-1)(n-k)}\delta_{ij}, \quad k \le i,j \le n-1,$$

that is $x : S^{k-1} \times H^{n-k} \to \mathbb{R}^n$ has two constant distinct Laguerre eigenvalues $-\frac{n-1}{2(k-1)(n-k)}$ and $\frac{n-1}{2(k-1)(n-k)}$ with multiplicities $k-1$ and $n-k$, respectively. We see that $x : S^{n-2} \times H^1 \to \mathbb{R}^n$ has two distinct Laguerre eigenvalues $-\frac{n-1}{2(n-2)}$ and $\frac{n-1}{2(n-2)}$ with multiplicities $n-2$ and 1, by (2.17) and a direct calculation, we have $|\mathbf{L}|^2 = \frac{(n-1)R^2}{4(n-2)^2(n-3)^2}$.

Example 2.2([5]) For any positive integers $m_1, \ldots, m_s$ with $m_1 + \cdots + m_s = n-1$ and any non-zero constants $\lambda_1, \ldots, \lambda_s$, we define $x : \mathbb{R}^{n-1} \to \mathbb{R}_0^n$ to be a spacelike oriented hypersurface in $\mathbb{R}_0^n$ given by

$$x = \left\{\frac{\lambda_1|u_1|^2 + \cdots + \lambda_s|u_s|^2}{2}, u_1, u_2, \ldots, u_s, \frac{\lambda_1|u_1|^2 + \cdots + \lambda_s|u_s|^2}{2}\right\},$$

where $(u_1, \ldots, u_s) \in \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_s} = \mathbb{R}^{n-1}$ and $|u_i|^2 = u_i \cdot u_i, i = 1, \ldots, s$. Then $\tau \circ (x, \xi) = (x', \xi') : \mathbb{R}^{n-1} \to U\mathbb{R}^n$, and we obtain the hypersurfaces $x' : \mathbb{R}^{n-1} \to \mathbb{R}^n$. From [5], we know that x has $s(s \ge 3)$ distinct Laguerre principal curvatures $B_i = \frac{r_i - r}{\sqrt{\sum_i (r_i - r)^2}}$, $1 \le i \le s$, where $r_i = \frac{1}{k_i}$, $r = \frac{k_1 r_1 + k_2 r_2 + \cdots + k_s r_s}{n-1}$ and $k_i \ne 0$ is the principal curvature corresponding to e_i. Also from [5], we know that the Laguerre form is zero, $L_{ij} = 0$ for $1 \le i,j \le n-1$ and the Laguerre second fundamental form of x is parallel.

Lemma 2.3([9]) Let A and B be $m \times m$-symmetric matrices satisfying $\mathrm{tr}A = 0$, $\mathrm{tr}B = 0$ and $AB - BA = 0$. Then,

$$|\mathrm{tr}B^2A| \le \frac{m-2}{\sqrt{m(m-1)}}(\mathrm{tr}B^2)(\mathrm{tr}A^2)^{1/2},$$

and the equality in the right (left) hand side holds if and only if $(m-1)$ of the eigenvalues x_i of B and the corresponding eigenvalues y_i of A satisfy $|x_i| = \frac{(\mathrm{tr}B^2)^{1/2}}{\sqrt{m(m-1)}}$, $x_ix_j \geq 0$, $y_i = -\frac{(\mathrm{tr}A^2)^{1/2}}{\sqrt{m(m-1)}}$, $(y_i = \frac{(\mathrm{tr}A^2)^{1/2}}{\sqrt{m(m-1)}})$.

3 Proofs of theorems

Let L and B denote the $(n-1)\times(n-1)$-symmetric matrices (L_{ij}) and (B_{ij}), respectively, where L_{ij} and B_{ij} are defined by (2.10), (2.11). From (2.22), we know that $BL = LB$. Thus, we may choose a local orthonormal basis $\{E_1, E_2, \ldots, E_{n-1}\}$ such that

$$L_{ij} = L_i\delta_{ij}, \quad B_{ij} = B_i\delta_{ij},$$

where L_i and B_i are the Laguerre eigenvalues and the Laguerre principal curvatures of the immersion x.

Throughout this section, we shall make the following convention on the ranges of indices:

$$\begin{aligned}&1 \leq a, b \leq m_1, \quad m_1+1 \leq p, q \leq m_1+m_2,\\ &m_1+m_2+1 \leq \alpha, \beta \leq m_1+m_2+m_3 = n-1, \quad 1 \leq i, j, k \leq n-1.\end{aligned}$$

We may prove the following Proposition firstly.

Proposition 3.1 *Let $x : M \to \mathbb{R}^n (n \geq 4)$ be an $(n-1)$-dimensional Laguerre isoparametric hypersurface with three distinct Laguerre principal curvatures one of which is simple. If the Laguerre second fundamental form of x is not parallel, then there is no such hypersurface in $\mathbb{R}^n$.*

Proof: Let B_1, B_2 and B_3 be the three constant Laguerre principal curvatures of x with multiplicities m_1, m_2 and m_3. From (2.15), we have

$$B_{ij,k} = \Gamma^j_{ik}(B_i - B_j), \tag{3.1}$$

where Γ^j_{ik} is the Levi-Civita connection for the Laguerre metric g given by

$$\omega_{ij} = \sum_k \Gamma^j_{ik}\omega_k, \quad \Gamma^j_{ik} = -\Gamma^i_{jk}.$$

It follows that

$$B_{ab,k} = B_{pq,k} = B_{\alpha\beta,k} = 0 \ \text{ for any } \ a, b, p, q, \alpha, \beta, k. \tag{3.2}$$

If the Laguerre second fundamental form of x is not parallel, we see that the only possible nonzero elements in $\{B_{ij,k}\}$ are of the form $\{B_{ap,\alpha}\}$. Since $n \geq 4$, without loss of generality, we may assume that $m_1 \geq m_2 \geq m_3$ and $m_3 = 1$.

From (2.16), (2.8) and $\omega_{ij} = \sum_k \Gamma^j_{ik}\omega_k$, the curvature tensor of x may be given by (see [8])

$$\begin{aligned}R_{ijkl} =& E_l(\Gamma^j_{ik}) - E_k(\Gamma^j_{il}) + \sum_m \Gamma^j_{im}\Gamma^m_{lk}\\ &- \sum_m \Gamma^j_{im}\Gamma^m_{kl} + \sum_m \Gamma^m_{ik}\Gamma^j_{ml} - \sum_m \Gamma^m_{il}\Gamma^j_{mk}.\end{aligned} \tag{3.3}$$

Thus, from (3.1) and (3.2), we have

$$\Gamma^p_{ab} = \Gamma^\alpha_{ab} = 0, \quad \Gamma^a_{pq} = \Gamma^\alpha_{pq} = 0, \quad \Gamma^a_{\alpha\beta} = \Gamma^p_{\alpha\beta} = 0, \tag{3.4}$$

$$\Gamma^p_{a\alpha} = \frac{B_{ap,\alpha}}{B_1 - B_2}, \quad \Gamma^a_{\alpha p} = \frac{B_{\alpha a,p}}{B_3 - B_1}, \quad \Gamma^\alpha_{pa} = \frac{B_{p\alpha,a}}{B_2 - B_3}. \tag{3.5}$$

From (3.4) and (3.5), we have

$$\Gamma^a_{n-1n-1} = \Gamma^p_{n-1n-1} = 0, \quad \Gamma^{n-1}_{aa} = \Gamma^{n-1}_{pp} = 0. \tag{3.6}$$

$$\Gamma^p_{an-1} = \frac{B_{ap,n-1}}{B_1 - B_2}, \quad \Gamma^p_{n-1b} = \frac{B_{bp,n-1}}{B_3 - B_2}, \quad \Gamma^{n-1}_{bq} = \frac{B_{bq,n-1}}{B_1 - B_3}, \tag{3.7}$$
$$\Gamma^{n-1}_{qb} = \frac{B_{bq,n-1}}{B_2 - B_3}.$$

Thus, we have

$$\begin{aligned} R_{apbq} =& E_q(\Gamma^p_{ab}) - E_b(\Gamma^p_{aq}) + \sum_m \Gamma^p_{am}\Gamma^m_{qb} \\ & - \sum_m \Gamma^p_{am}\Gamma^m_{bq} + \sum_m \Gamma^m_{ab}\Gamma^p_{mq} - \sum_m \Gamma^m_{aq}\Gamma^p_{mb} \\ =& \Gamma^p_{an-1}\Gamma^{n-1}_{qb} - \Gamma^p_{an-1}\Gamma^{n-1}_{bq} - \Gamma^{n-1}_{aq}\Gamma^p_{n-1b} \\ =& \frac{B_{ap,n-1}B_{bq,n-1} + B_{aq,n-1}B_{bp,n-1}}{(B_1 - B_3)(B_2 - B_3)}. \end{aligned} \tag{3.8}$$

On the other hand, from (2.21), we have

$$R_{apbq} = -(L_a + L_p)\delta_{ab}\delta_{pq}. \tag{3.9}$$

It follows from (3.8) and (3.9) that

$$\begin{aligned} & B_{ap,n-1}B_{bq,n-1} + B_{aq,n-1}B_{bp,n-1} \\ & \quad = -(L_a + L_p)(B_1 - B_3)(B_2 - B_3)\delta_{ab}\delta_{pq}. \end{aligned} \tag{3.10}$$

If $a = b$, we have

$$2B_{ap,n-1}B_{aq,n-1} = -(L_a + L_p)(B_1 - B_3)(B_2 - B_3)\delta_{pq}. \tag{3.11}$$

If $p = q$, we have

$$2B_{ap,n-1}B_{bp,n-1} = -(L_a + L_p)(B_1 - B_3)(B_2 - B_3)\delta_{ab}. \tag{3.12}$$

If $m_1 = 1$, it follows that $2B_{1p,n-1}B_{1q,n-1} = -(L_1 + L_p)(B_1 - B_3)(B_2 - B_3)\delta_{pq}$. Since the Laguerre second fundamental form of x is not parallel, we may prove that there exists exactly one p, such that $B_{1p,n-1} \neq 0$. In fact, if there exists more than one p, for example p_1, p_2, $(p_1 \neq p_2)$ such that $B_{1p_1,n-1} \neq 0$, $B_{1p_2,n-1} \neq 0$, this is a contradiction with $B_{1p_1,n-1}B_{1p_2,n-1} = 0$.

If $m_2 = 1$, it follows that $2B_{am_1+1,n-1}B_{bm_1+1,n-1} = -(L_a + L_{m_1+1})(B_1 - B_3)(B_2 - B_3)\delta_{ab}$. The same reason implies that there exists exactly one a, such that $B_{am_1+1,n-1} \neq 0$.

If $m_1 \geq 2$ and $m_2 \geq 2$, we may prove that there exists exactly one a and exactly one p such that $B_{ap,n-1} \neq 0$. In fact, if there exists more than one a, for example a_1, a_2, $(a_1 \neq a_2)$ such that $B_{a_1p,n-1} \neq 0$, $B_{a_2p,n-1} \neq 0$. From (3.12), we see that $B_{a_1p,n-1}B_{a_2p,n-1} = 0$, a contradiction. The same reason implies that there exists exactly one p, such that $B_{ap,n-1} \neq 0$. Thus, we conclude.

Combining with the above three cases, we see that if $m_1 \geq 1$ and $m_2 \geq 1$, there exists exactly one a and exactly one p, say a_1 and p_1, such that

$$B_{a_1p_1,n-1} \neq 0, \quad B_{ap,n-1} = 0, \quad \text{for} \quad a \neq a_1, \forall p, \quad \text{or} \quad \text{for} \quad p \neq p_1, \forall a. \tag{3.13}$$

By (3.11) and (3.13), we get

$$-L_{a_1} - L_{p_1} = \frac{2B^2_{a_1p_1,n-1}}{(B_1 - B_3)(B_2 - B_3)}, \tag{3.14}$$

$$-L_a - L_{p_1} = 0, \quad a \neq a_1, \tag{3.15}$$

$$-L_{a_1} - L_p = 0, \quad p \neq p_1, \tag{3.16}$$

$$-L_a - L_p = 0, \quad a \neq a_1, \quad p \neq p_1. \tag{3.17}$$

From (3.1)–(3.3), (2.21) and by reasoning as above, we get

$$-L_{a_1} - L_{n-1} = \frac{2B^2_{a_1p_1,n-1}}{(B_1 - B_2)(B_3 - B_2)}, \tag{3.18}$$

$$-L_a - L_{n-1} = 0, \quad a \neq a_1 \tag{3.19}$$

$$-L_{p_1} - L_{n-1} = \frac{2B^2_{a_1p_1,n-1}}{(B_2 - B_1)(B_3 - B_1)}, \tag{3.20}$$

$$-L_p - L_{n-1} = 0 \quad p \neq p_1. \tag{3.21}$$

By (3.14), (3.18) and (3.20), we obtain that $L_{n-1} = \frac{2B_{a_1p_1,n-1}}{(B_1-B_3)(B_2-B_3)}$.

If $m_2 \geq 2$, then there exists some $p(p \neq p_1)$ such that (3.17) and (3.21) hold. From (3.17), (3.19) and (3.21), we see that $L_{n-1} = 0$, a contradiction. Thus, it follows that $m_2 = 1$. By (3.18)–(3.20), we know that x has Laguerre eigenvalues

$$L_a = -\frac{2B_{a_1p_1,n-1}}{(B_1 - B_3)(B_2 - B_3)}, \quad a \neq a_1, \tag{3.22}$$

$$L_{a_1} = \frac{2B_{a_1p_1,n-1}}{(B_2 - B_1)(B_3 - B_1)}, \tag{3.23}$$

$$L_{p_1} = \frac{2B_{a_1p_1,n-1}}{(B_1 - B_2)(B_3 - B_2)}, \tag{3.24}$$

$$L_{n-1} = \frac{2B_{a_1p_1,n-1}}{(B_1 - B_3)(B_2 - B_3)}. \tag{3.25}$$

We may prove that $B_{a_1p_1,n-1}$ is a constant. In fact, from $\omega_{ij}=\sum_k \Gamma^j_{ik}\omega_k$, (2.23), (3.1) and (3.2), we have

$$\begin{aligned}\sum_l B_{ab,pl}\omega_l =& dB_{ab,p}+\sum_l B_{lb,p}\omega_{la}+\sum_l B_{al,p}\omega_{lb}+\sum_l B_{ab,l}\omega_{lp}\\ =&\sum_l \frac{B_{n-1b,p}B_{n-1a,l}+B_{n-1a,p}B_{n-1b,l}}{B_3-B_1}\omega_l.\end{aligned}$$

Thus

$$B_{ab,pl}=\frac{B_{n-1b,p}B_{n-1a,l}+B_{n-1a,p}B_{n-1b,l}}{B_3-B_1},\quad \forall\ \ a,b,p,l. \tag{3.26}$$

By reasoning as above, we also have

$$B_{pq,al}=\frac{B_{n-1a,p}B_{n-1l,q}+B_{n-1a,q}B_{n-1l,p}}{B_3-B_2},\quad \forall\ \ a,p,q,l, \tag{3.27}$$

$$B_{n-1n-1,ap}=0,\ \forall\ \ a,p. \tag{3.28}$$

From (2.24), we have $B_{ij,kl}-B_{ij,lk}=(B_i-B_j)R_{ijkl}$. By (2.21) and $L_{ij}=L_i\delta_{ij}$, we know that if three of $\{i,j,k,l\}$ are distinct, then $R_{ijkl}=0$. Thus, if three of $\{i,j,k,l\}$ are distinct, we have

$$B_{ij,kl}=B_{ij,lk}. \tag{3.29}$$

From (2.23), (3.1) and (3.2), we have

$$\begin{aligned}dB_{a_1p_1,n-1}=&\sum_l B_{a_1p_1,n-1l}\omega_l=B_{a_1p_1,n-1a_1}\omega_{a_1}\\ &+\sum_{l=a,a\neq a_1} B_{a_1p_1,n-1l}\omega_l+B_{a_1p_1,n-1p_1}\omega_{p_1}\\ &+\sum_{l=p,p\neq p_1} B_{a_1p_1,n-1l}\omega_l+B_{a_1p_1,n-1n-1}\omega_{n-1}.\end{aligned} \tag{3.30}$$

Thus, it follows from (3.26)–(3.29) that $dB_{a_1p_1,n-1}=0$, that is, $B_{a_1p_1,n-1}$ is constant. Thus, we see that x has constant Laguerre eigenvalues

$$L_a(a\neq a_1),\quad L_{a_1},\quad L_{p_1},\quad L_{n-1}.$$

If $m_1\geq 2$, then there exists some $a(a\neq a_1)$ such that (3.15), (3.19) and (3.22) hold. From (3.15) and (3.19), we see that $L_{p_1}=L_{n-1}$, that is, from (3.24) and (3.25), we have

$$2B_1-B_2-B_3=0. \tag{3.31}$$

We may prove that $L_{a_1}\neq L_{p_1}$. In fact, if $L_{a_1}=L_{p_1}$, we have

$$2B_3-B_1-B_2=0. \tag{3.32}$$

Combining with (3.31), we see that $B_1 = B_2 = B_3$, a contradiction. It may be easily checked that $L_a \neq L_{a_1}$, $L_a \neq L_{p_1}$, this shows that x has three distinct constant Laguerre eigenvalues

$$L_a(a \neq a_1), \quad L_{a_1}, \quad L_{p_1} = L_{n-1},$$

and therefore is not a Laguerre isotropic hypersurface. But from a result of [7] (see Proposition 6.1 in [7]), we know that if x has constant Laguerre eigenvalues, then it must have only two distinct constant Laguerre eigenvalues, a contradiction. Thus, it follows that $m_1 = 1$. Combining with $m_3 = 1$ and $m_2 = 1$, we see that $n = 4$. This shows that x has constant Laguerre eigenvalues

$$\begin{aligned} L_1 &= \frac{2B_{12,3}}{(B_2 - B_1)(B_3 - B_1)}, \\ L_2 &= \frac{2B_{12,3}}{(B_1 - B_2)(B_3 - B_2)}, \\ L_3 &= \frac{2B_{12,3}}{(B_1 - B_3)(B_2 - B_3)}. \end{aligned}$$

If $L_1 = L_2 = L_3$, we easily see that $B_1 = B_2 = B_3$, a contradiction. Thus x is not a Laguerre isotropic hypersurface. From Proposition 6.1 in [7], we know that two of L_1, L_2, L_3 must be equal and be equal to the opposite number of the third, that is, x must have two constant Laguerre eigenvalues which are opposite numbers. Without loss of generality, we may assume that $L_1 = L_2 = -L_3$, from the above three equation of L_1, L_2, L_3, we also see that $B_1 = B_2 = B_3$, a contradiction. Thus, if the Laguerre second fundamental form of x is not parallel, there is no Laguerre isoparametric hypersurface with three distinct Laguerre principal curvatures one of which is simple. This completes the proof of Proposition 3.1. □

Proof of Theorem 1.2: If the Laguerre second fundamental form of x is parallel, since x has three distinct constant Laguerre principal curvatures, from Theorem 1.1, Example 2.1 and Example 2.2, we know that x is Laguerre equivalent to an open part of the image of τ of the oriented hypersurface $x : \mathbb{R}^{n-1} \to \mathbb{R}^n_0$ given by Example 2.2.

If the Laguerre second fundamental form of x is not parallel, from Proposition 3.1, we know that there is no Laguerre isoparametric hypersurface with three distinct Laguerre principal curvatures one of which is simple. This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3: Putting $\tilde{L} = (\tilde{L}_{ij})$ with $\tilde{L}_{ij} = L_{ij} - \frac{1}{n-1}\mathrm{tr}L\delta_{ij}$, thus

$$\mathrm{tr}\tilde{L} = 0, \quad B\tilde{L} = \tilde{L}B, \quad \mathrm{tr}(\tilde{L}B^2) = \mathrm{tr}(LB^2) - \frac{1}{n-1}\mathrm{tr}L, \tag{3.33}$$

and

$$|L|^2 = \sum_{i,j}(L_{ij})^2 = \sum_{i,j}(\tilde{L}_{ij})^2 + \frac{1}{n-1}(\mathrm{tr}L)^2 = |\tilde{L}|^2 + \frac{1}{n-1}(\mathrm{tr}L)^2. \tag{3.34}$$

From (1.7), (2.17) and (3.34), we see that

$$|\tilde{L}|^2 = |L|^2 - \frac{1}{n-1}(\mathrm{tr}L)^2 \leq \frac{R^2}{(n-1)(n-2)(n-3)^2}.$$

Since we assume that $R \geq 0$, it follows that

$$|\tilde{L}| \leq \frac{R}{(n-3)\sqrt{(n-1)(n-2)}}. \tag{3.35}$$

From (2.21), (2.22), (2.24), (3.33), (3.35), Lemma 2.3 and by a direct calculation, we have

$$\begin{aligned} 0 =& \frac{1}{2}\Delta \sum_{i,j} B_{ij}^2 = \sum_{i,j,k} B_{ij,k}^2 + \sum_{i,j} B_{ij}\Delta B_{ij} \\ =& \sum_{i,j,k} B_{ij,k}^2 - (n-1)\mathrm{tr}(LB^2) - \mathrm{tr}L \\ =& \sum_{i,j,k} B_{ij,k}^2 - (n-1)\left(\mathrm{tr}(\tilde{L}B^2) + \frac{1}{n-1}\mathrm{tr}L\right) - \mathrm{tr}L \\ \geq& \sum_{i,j,k} B_{ij,k}^2 - (n-1)\left\{\frac{n-3}{\sqrt{(n-1)(n-2)}}|\tilde{L}| + \frac{1}{n-1}\mathrm{tr}L\right\} - \mathrm{tr}L \\ =& \sum_{i,j,k} B_{ij,k}^2 + \left\{\frac{R}{n-2} - (n-3)\sqrt{\frac{n-1}{n-2}}|\tilde{L}|\right\} \geq 0. \end{aligned} \tag{3.36}$$

Thus, the equalities in (3.36) hold. We have $B_{ij,k} = 0$, that is, the Laguerre second fundamental form of x is parallel. Further, the inequality in the right hand side of Lemma 2.3 becomes equality. Thus, we know that x has two distinct constant Laguerre principal curvatures. From Theorem 1.1, Example 2.1 and Example 2.2, we know that x is Laguerre equivalent to an open part of the oriented hypersurface $x : S^{n-2} \times H^1 \to \mathbb{R}^n$ given by Example 2.1. This completes the proof of Theorem 1.3. □

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