Bull. Math. Soc. Sci. Math. Roumanie
Tome 58(106) No. 1, 2015, 49–58

An efficient Steffensen-like iterative method with memory

by
Alicia Cordero[1], Taher Lotfi[2], Arezoo Khoshandi[3]
and
Juan R. Torregrosa[4]

Abstract

Some methods with memory for solving nonlinear equations are designed from known methods without memory. We increase the convergence order from 4 to 6 by using a free parameter accelerator by Newton's interpolatory polynomial of the third degree. So, its efficiency index is even better than optimal sixteenth-order methods without memory. Dynamical behavior on low-degree polynomials is analyzed, highly improving the stability properties of the original schemes. Numerical test problems are given to prove its competitiveness with methods of the same class.

Key Words:Iterative methods, R-order, Steffensen-like methods with memory, computational efficiency, Herzberger's matrix, stability, basin of attraction.
2010 Mathematics Subject Classification: Primary 65H05, Secondary 65D05, 37M99.

1 Introduction

Root finding is a great task in mathematics, both historically and practically. It has attracted the attention of great mathematicians like Gauss and Newton. It has real major applications and because of these real features, it is still alive as a research field.

Kung and Traub's conjecture is the basic fact to construct optimal multipoint methods without memory [9]. This conjecture establish that the order of convergence of any multipliont method without memory using $n+1$ functional evaluations per step is, at most, 2^n. When this bound is reached, the scheme is called optimal. On the other hand, multipoint methods with memory can increase the efficiency index of an optimal method without memory, without consuming any new functional evaluations and merely using accelerator parameter(s). This great power of methods with memory has not been much considered until very recently. So, we have been motivated to develop a modified version with memory of a known optimal fourth-order methods recently published in [4].

Traub, in [16], introduced methods with and without memory for the first time. Moreover, he constructed a Steffensen-type method with memory using secant approach. In fact, he increased the order of convergence of the Steffensen method (see [15]) from 2 to 2.41. This is the first method with memory based on our best knowledge. Džunić and Petković have improved this scheme and increased its order of convergence from 2 to 3 without any new functional evaluation in [5]. However, a two-step (three-point) class with memory was created in [13] with R-order at least 4.44. Later on, this method has been modified totally in [12] to reach the R-order 6.

Derivative free iterative methods for approximating a root α of nonlinear equations $f(x) = 0$ are important in the sense that in many practical situations it is preferable to avoid the calculation of the derivative of f. Usually, the starting scheme is the Steffensen's method,

$$x_{k+1} = x_k - \frac{f(x_k)^2}{f(x_k + f(x_k)) - f(x_k)}, \quad k = 0, 1, 2, \ldots, \tag{1.1}$$

which is obtained from Newton's method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \qquad k = 0, 1, 2, \ldots$$

by approximating the derivative $f'(x_k)$ by the quotient $\frac{f(x_k+f(x_k))-f(x_k)}{f(x_k)}$.

If a real parameter γ is included in this estimation, a family of Steffensen-like methods is obtained

$$x_{k+1} = x_k - \frac{\gamma f(x_k)^2}{f(x_k + \gamma f(x_k)) - f(x_k)}, \qquad \gamma \in \mathbb{R}, \tag{1.2}$$

with the same order and efficiency index as that of Newton's method, for any value of parameter γ different from zero.

Cordero and Torregrosa [4] designed a family of fourth-order optimal derivative free without memory schemes, whose iterative expression is

$$\begin{cases} y_k = x_k - \frac{f(x_k)^2}{f(z_k)-f(x_k)}, \\ x_{k+1} = y_k - \frac{f(y_k)}{\frac{f(y_k)-\beta f(z_k)}{y_k-z_k} + \frac{f(y_k)-(1-\beta)f(x_k)}{y_k-x_k}}, \qquad \beta \in \mathbb{R}, \end{cases} \tag{1.3}$$

where $z_k = x_k + f(x_k)$.

If we replace z_k by $w_k = x_k + \gamma f(x_k)$, including a real parameter γ, the resulting family has order of convergence four, for any non-zero value of γ,

$$\begin{cases} y_k = x_k - \gamma \frac{f(x_k)^2}{f(w_k)-f(x_k)}, \\ x_{k+1} = y_k - \frac{f(y_k)}{\frac{f(y_k)-\beta f(w_k)}{y_k-w_k} + \frac{f(y_k)-(1-\beta)f(x_k)}{y_k-x_k}}, \qquad \beta \in \mathbb{R}, \end{cases} \tag{1.4}$$

and its error equation is

$$e_{k+1} = (1 + \gamma f'(\alpha))^2 c_2(c_2^2 - c_3)e_k^4 + O(e_k^5), \tag{1.5}$$

where $e_k = x_k - \alpha$ and $c_k = \frac{1}{k!}\frac{f^{(k)}(\alpha)}{f'(\alpha)}$, $k = 2, 3, \ldots$. This family is denoted by CT.

This paper is organized as follows: in Section 2 the proposed iterative scheme is designed and its convergence order is proven. In Section 3 we analyze the stability and dynamical behavior of the rational functions associated to an element of the family applied on low degree polynomials and finally, numerical examinations and comparisons are presented in the last section.

2 Development of the method with memory

We can observe from (1.5) that the order of convergence of the family (1.4) is four when $\gamma \neq -1/f'(\alpha)$. This order could be increased taking $\gamma = -1/f'(\alpha)$ but, in practice we have no information on the exact value of $f'(\alpha)$. So, we could use an approximation $\tilde{f}'(\alpha)$ of $f'(\alpha)$, based on available information. Then, setting $\gamma = -1/\tilde{f}'(\alpha)$ in (1.4), we achieve that the order of convergence of the modified methods exceeds four without using any new functional evaluation. An idea in constructing methods with memory consists of the calculation of the parameter $\gamma = \gamma_k$ as the iteration proceeds by the formula $\gamma_k = -1/\tilde{f}'(\alpha)$, $k = 1, 2, \ldots$

We consider the following accelerator for approximating γ_k:

$$\gamma_k = \frac{-1}{\tilde{f}'(\alpha)} = \frac{-1}{N_3'(x_k)}, \tag{2.1}$$

where

$$N_3(t) = N_3(t; x_k, y_{k-1}, w_{k-1}, x_{k-1}),$$

is Newton's interpolatory polynomial of third degree, set through four *best* available approximation (nodes) x_k, y_{k-1}, w_{k-1} and x_{k-1}. So,

$$\begin{aligned} N_3'(x_k) &= \left[\frac{d}{dt}N_3(t)\right]_{t=x_k} \\ &= f[x_k, y_{k-1}] + f[x_k, x_{k-1}] - f[y_{k-1}, x_{k-1}] + (f[x_k, y_{k-1}, w_{k-1}] \\ &\quad -f[y_{k-1}, x_{k-1}, w_{k-1}])(x_k - y_{k-1}), \end{aligned}$$

where $f[x, y]$ and $f[x, y, z]$ are divided differences of order one and two, respectively. It should be noted that if one uses lower Newton's interpolation, lower accelerators are obtained.

It is assumed that the initial value γ_0 should be chosen before starting the iterative process. Replacing the fixed parameter γ in the iterative family (1.4) by the varying γ_k one calculated by (2.1), the following derivative-free family of two-point methods with memory is achieved:

$$\begin{cases} \gamma_0 \text{ is given}, \quad w_k = x_k + \gamma_k f(x_k), \quad \gamma_k = -\frac{1}{N_3'(x_k)}, \\ y_k = x_k - \frac{f(x_k)}{f[w_k, x_k]}, \\ x_{k+1} = y_k - \frac{f(y_k)}{\frac{f(y_k) - \beta f(z_k)}{y_k - z_k} + \frac{f(y_k) - (1-\beta) f(x_k)}{y_k - x_k}} \qquad \beta \in \mathbb{R}. \end{cases} \tag{2.2}$$

The methods of this family are denoted by CT6.

2.1 Convergence Analysis

In order to obtain the order of convergence of the family of two-point methods with memory (2.2), where γ_k is calculated by using (2.1), we will use the concept of the R-order of convergence introduced by Ortega and Rheinboldt [11]. Now we state the following convergence result.

Theorem 1. *If an initial approximation x_0 is sufficiently close to the zero α of $f(x)$, then the R-order of convergence of the two-point method (2.2), with γ_k calculated as (2.1), is at least 6.*

Proof: We will use Herzberger's matrix method (see [7]) to determine the R-order of convergence (see [6]). The lower bound of order of a single step 4-point method

$$x_k = G(x_{k-1}, x_{k-2}, x_{k-3}, x_{k-4})$$

is the spectral radius of a 4×4 real matrix $M = (m_{i,j})$, associated to this method, with elements

- $m_{1,j}$ = amount of information required at point x_{k-j}, $j = 1, 2, 3, 4$;
- $m_{i,i-1} = 1$, $i = 2, 3, 4$;
- $m_{i,j} = 0$, otherwise.

The lower bound of order of a 3-step method $G = G_1 \circ G_2 \circ G_3$ is the spectral radius of the product of matrices $M = M_1M_2M_3$.

We can express each approximation x_{k+1}, y_k, and w_k as a function of available information $f(y_k)$, $f(w_k)$ and $f(x_k)$ from the kth iteration and $f(y_{k-1})$, $f(w_{k-1})$ and $f(x_{k-1})$ from the previous iteration, depending on the accelerating technique. According to the relations (2.2) and (2.1) we form the respective matrices. More details and illustrations about the construction of Herzberger's matrices can be found in [10].

We use the following matrices to express informational dependence:

$$x_{k+1} = G_1(y_k, w_k, x_k, y_{k-1}); \qquad M_1 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$y_k = G_2(w_k, x_k, y_{k-1}, w_{k-1}); \qquad M_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$w_k = G_3(x_k, y_{k-1}, w_{k-1}, x_{k-1}); \qquad M_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The matrix M corresponding to the multi-point method CT6 is

$$M = M_1M_2M_3 = \begin{bmatrix} 4 & 2 & 2 & 2 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

an its eigenvalues are $(6, 0, 0, 0)$. Since the spectral radius of the matrix M is 6, we conclude that the R-order of the methods with memory CT6 is at least six. □

Let us remark that the obtained R-order suppose a 50% improvement of the order of convergence of the family without memory (1.4).

3 Dynamical behavior

From the numerical point of view, the dynamical behavior of the rational function associated with an iterative method give us important information about its stability and reliability. In these terms, Varona in [17] and Amat et al. in [1], among others, described the dynamical behavior of several iterative methods. In [3], a deep dynamical study was made of some derivative-free iterative schemes, specifically Steffensen's method.

We are going to recall now some dynamical concepts of complex dynamics (see [2]) that we use in this work. Given a rational function $R : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$, where $\hat{\mathbb{C}}$ is the Riemann sphere, the *orbit of a point* $z_0 \in \hat{\mathbb{C}}$ is defined as:

$$\{z_0,\, R\left(z_0\right),\, R^2\left(z_0\right), ..., R^n\left(z_0\right), ...\}.$$

We analyze the phase plane of the map R by classifying the starting points from the asymptotic behavior of their orbits. A $z_0 \in \hat{\mathbb{C}}$ is called a *fixed point* if $R\left(z_0\right) = z_0$. Moreover, a fixed point z_0 is called *attractor* if $|R'(z_0)| < 1$, *superattractor* if $|R'(z_0)| = 0$, *repulsor* if $|R'(z_0)| > 1$ and *parabolic* if $|R'(z_0)| = 1$. Then, the *basin of attraction* of an attractor α is defined as:

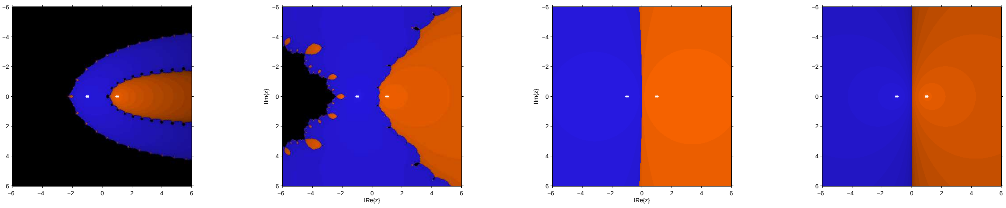
$$\mathcal{A}\left(\alpha\right) = \{z_0 \in \hat{\mathbb{C}} \ :\ R^n\left(z_0\right) \rightarrow \alpha,\ n \rightarrow \infty\}.$$

Also, it is well-known that the basin of attraction of any fixed point belongs to the so called *Fatou set* and the boundaries of these basins of attraction are the *Julia set*.

In the following, we study the stability properties of the proposed methods for low-degree polynomials. In fact, we obtain the dynamical planes associated to the rational function obtained when the method is applied on a given polynomial $p(z)$. These planes are obtained in the following way: in the rectangle $[-6, 6] \times [-6, 6]$ of the complex plane, a mesh of 400×400 initial estimations is defined. If the sequence generated by the iterative method reaches a zero of the polynomial (superattracting fixed point) with an error estimation lower than 10^{-3} and a maximum of 40 iterations, we decide that the initial point is in the basin of attraction of these zero and we paint it in a color previously selected for this root. In the same basin of attraction, the number of iterations needed to achieve the solution is showed in darker or brighter colors. Black color denotes lack of convergence to any of the roots (with the maximum of iterations established) or convergence to the infinity.

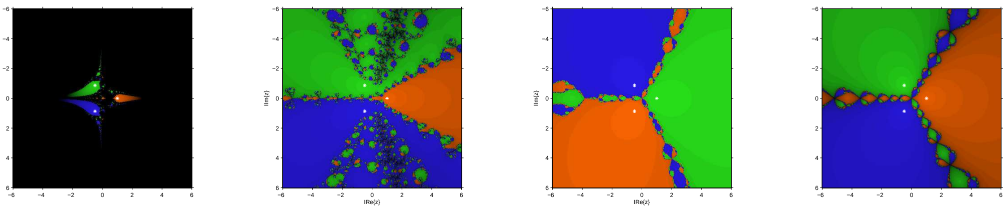
For Steffensen's and proposed methods, it can be proved that an Scaling Theorem is not satisfied. So, we must study the dynamics of the schemes on specific polynomials. The behavior of each method is analyzed on three different polynomials: $p_2(z) = z^2 - 1$, $p_3(z) = z^3 - 1$ and $p_4(z) = z^4 - 1$. The dynamical planes for Steffensen, CT (for $\beta = 1$, $\gamma = 1$), CT6 (for $\beta = 1$, $\gamma_0 = -0.01$) and Newton's schemes are showed in Figures 1 to 3.

Let us note that the widest black regions of Figures 2a and 3a correspond the basin of the infinity in Steffensen's method. This basin of the infinity does not exist in the other dynamical



(a) Steffensen (b) CT (c) CT6 (d) Newton

Figure 1: Dynamical planes of different schemes on $p_2(z) = z^2 - 1$.



(a) Steffensen (b) CT (c) CT6 (d) Newton

Figure 2: Dynamical planes of different schemes on $p_3(z) = z^3 - 1$.

planes (except for CT method on $p_2(z)$). Although the behavior of CT scheme is much more stable than Steffensen's one, it is highly improved when memory factor is introduced (CT6 method). In fact, it can be observed that they "tend" to the Newton's dynamical planes, for each polynomial.

4 Numerical results

The described process for designing CT6 can be applied on other fourth-order methods depending on parameter γ and with a factor in the term of e^4 in the error equation such as $1+\gamma f'(\alpha)$. For example, we have selected two families of methods of order four displayed below.

Derivative-free Kung-Traub's family, [9], denoted by KT,

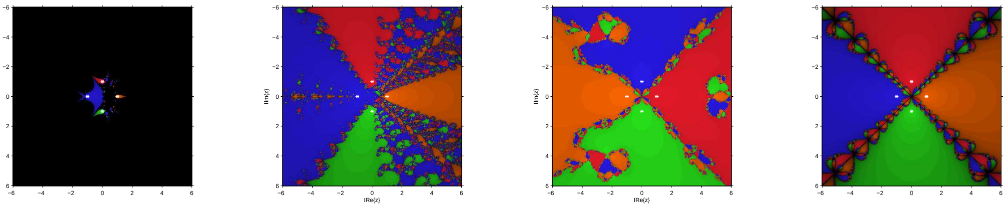
$$\begin{cases} y_k = x_k - \frac{f(x_k)}{f[x_k,w_k]}, \qquad w_k = x_k + \gamma f(x_k), \quad \gamma \neq 0 \\ x_{k+1} = y_k - \frac{f(y_k)f(w_k)}{(f(w_k)-f(y_k))f[x_k,y_k]}, \end{cases} \tag{4.1}$$

whose error equation is

$$e_{k+1} = (1+\gamma f'(\alpha))^2(2c_2^3 - c_2c_3)e_k^4 + O(e_k^5),$$

Derivative-free Soleymani et al. method, [14], denoted by SSLT,

$$\begin{cases} y_k = x_k - \frac{f(x_k)}{f[x_k,w_k]}, \qquad w_k = x_k + \gamma f(x_k), \quad \gamma \neq 0 \\ x_{k+1} = x_k - \frac{f(x_k)+f(y_k)}{f[x_k,w_k]} - \left(\frac{2f(x_k)+af(y_k)}{f[x_k,w_k]}\left(\frac{f(y_k)}{f(x_k)}\right)^2\right)\left(1 - \frac{\gamma f[x_k,w_k]}{2+2\gamma f[x_k,w_k]}\right). \end{cases} \tag{4.2}$$



(a) Steffensen (b) CT (c) CT6 (d) Newton

Figure 3: Dynamical planes of different schemes on $p_4(z) = z^4 - 1$.

Its error equation is

$$\begin{aligned} e_{k+1} &= -\frac{1}{2}(c_2(1+\gamma f'(\alpha))(2c_3(1+\gamma f'(\alpha)) + c_2^2(a(1+\gamma f'(\alpha))(2+\gamma f'(\alpha)) \\ &\quad -2(5+\gamma f'(\alpha)(5+\gamma f'(\alpha))))))e_k^4 + O(e_k^5). \end{aligned}$$

If we replace in these methods γ in a similar way as in Section 2, we obtain a derivative-free schemes with memory and order of convergence six in case of Kung-Traub's method (which is denoted by KT6), and order 5.2 in Soleymani et al. scheme, as its error equation has not the common factor $(1+\gamma f'(\alpha))^2$. It is denoted by SSLT5.

In the following we will compare the numerical results obtained by the proposed sixth-order schemes with memory with the ones used as starting methods, that is, CT, KT and SSLT.

The errors $|x_k - \alpha|$ of approximations to the sought zeros, produced by the different methods, are given in Tables 1-5, where $A(-h)$ stands for $A \times 10^{-h}$. In Table 2, nc means that the corresponding method does not converge. These tables include the values of the computational order of convergence r_c computed by the expression (see [8])

$$r_c = \frac{\log(|f(x_k)/f(x_{k-1})|)}{\log(|f(x_{k-1})/f(x_{k-2})|)}.$$

The software Mathematica 8, with 2000 arbitrary precision arithmetic has been used in our computations. The results alongside the test functions are given in Tables 1-5, while $\gamma = \gamma_0 = -0.01$ in the schemes without memory, $\beta = 1$ in the family CT, and $a = 5$ in the family SSLT.

The last example corresponds to a non-smooth function. In this case the results obtained are similar to those of smooth functions.

5 Conclusion

We provide a new derivative-free iterative method with memory, whose efficiency index is $6^{1/3} \approx 1.8$, which is greater even than optimal sixteen methods without memory, with efficiency index $16^{1/5} \approx 1.7$. Under the dynamical point of view, method with memory are much more stable than the original ones. Moreover, the used technique can be extended to many methods

Table 1: $f(x) = \Pi_{i=1}^{5}(x-i), \quad \alpha = 2, \quad x_0 = 1.5, \quad \gamma_0 = -0.01$

method	$\|x_1-\alpha\|$	$\|x_2-\alpha\|$	$\|x_3-\alpha\|$	r_c
CT without memory	1.317(−1)	8.725(−4)	8.251(−13)	4.253
CT6 with memory	1.317(−1)	3.376(−6)	2.354(−34)	6.209
KT without memory	1.453(−1)	2.700(−3)	1.086(−10)	4.439
KT6 with memory	1.453(−1)	1.525(−5)	6.943(−30)	6.217
SSLT without memory	2.514(−3)	2.655(−11)	0.000	-
SSLT5 with memory	2.514(−3)	1.866(−12)	9.905(−60)	4.999

Table 2: $f(x) = (x-1)^3 - 1, \quad \alpha = 2, \quad x_0 = 1.5, \quad \gamma_0 = -0.01$

method	$\|x_1-\alpha\|$	$\|x_2-\alpha\|$	$\|x_3-\alpha\|$	r_c
CT without memory	1.393(−1)	1.708(−4)	5.341(−16)	3.873
CT6 with memory	1.393(−1)	2.884(−6)	3.840(−34)	5.877
KT without memory	4.630(−1)	1.834(−2)	1.667(−7)	3.195
KT6 with memory	4.630(−1)	2.485(−3)	3.884(−16)	5.216
SSLT without memory	*nc*	*nc*	*nc*	-
SSLT5 with memory	*nc*	*nc*	*nc*	-

Table 3: $f(x) = e^{x^2+x\cos x-1}\sin x + \log(x^2+1)$, $\alpha = 0$, $x_0 = 0.35$, $\gamma_0 = -0.01$

method	$\|x_1-\alpha\|$	$\|x_2-\alpha\|$	$\|x_3-\alpha\|$	r_c
CT without memory	5.817(−3)	3.180(−9)	2.470(−34)	4.006
CT6 with memory	5.817(−3)	1.627(−17)	5.902(−100)	5.667
KT without memory	5.701(−3)	5.220(−9)	3.881(−33)	3.999
KT6 with memory	5.701(−3)	5.733(−16)	4.482(−96)	5.653
SSLT without memory	4.262(−2)	9.507(−5)	3.430(−16)	4.213
SSLT5 with memory	4.262(−2)	1.408(−5)	1.055(−22)	5.000

Table 4: $f(x) = \|x^2-9\|$, $\alpha = 3$, $x_0 = 2.8$, $\gamma_0 = -0.01$

method	$\|x_1-\alpha\|$	$\|x_2-\alpha\|$	$\|x_3-\alpha\|$	r_c
CT without memory	1.639(−2)	2.918(−10)	2.964(−41)	3.999
CT6 with memory	1.639(−2)	5.658(−12)	4.221(−72)	6.354
KT without memory	1.637(−2)	5.800(−10)	9.260(−40)	3.999
KT6 with memory	1.637(−2)	1.123(−11)	5.164(−70)	6.365
SSLT without memory	1.640(−2)	5.521(−11)	0.000	-
SSLT5 with memory	1.640(−2)	4.511(−11)	0.441(−55)	5.000

Table 5: $f(x) = |x^2 - 9|$, $\alpha = -3$, $x_0 = -10$, $\gamma_0 = -0.01$

method	$\lvert x_1 - \alpha\rvert$	$\lvert x_2 - \alpha\rvert$	$\lvert x_3 - \alpha\rvert$	r_c
CT without memory	0.621	5.206(−4)	3.821(−16)	3.890
CT6 with memory	6.207(−1)	4.076(−6)	5.894(−37)	5.902
KT without memory	0.799	2.290(−3)	2.857(−13)	3.813
KT6 with memory	7.999(−1)	2.974(−5)	1.780(−31)	5.848
SSLT without memory	0.733	7.945(−4)	0.000	-
SSLT5 with memory	0.733	9.517(−5)	6.025(−24)	5.000

of order four, depending on the expression of the error equation. Numerical test are made that confirm the theoretical results.

Acknowledgement *This research was supported by Ministerio de Ciencia y Tecnología of Spain MTM2011-28636-C02-02. The second and third authors would like to appreciate Hamedan Branch of Islamic Azad University for their financial support. Moreover, we express our sincere thanks to the respected referees for their constructive suggestions and comments.*

References

[1] Amat S., Busquier S., Bermudez C., Plaza S., On two families of high order Newton type methods, *Applied Mathematics Letters* **25** (2010), 2209–2217

[2] Blanchard P., Complex Analytic Dynamics on the Riemann Sphere, *Bull. of the AMS* **11**(1) (1984), 85–141

[3] Chicharro F., Cordero A., Gutiérrez J.M., Torregrosa J.R., Complex dynamics of derivative-free methods for nonlinear equations, *Applied Mathematics and Computation* **219** (2013), 7023–7035

[4] Cordero A., Torregrosa J.R., A class of Steffensen type methods with optimal order of convergence, *Appl. Math. Comput.* **217** (2011), 7653–7659

[5] Džunić J., Petković M.S., A cubically convergent Steffensen-like method for solving nonlinear equations, *Applied Mathematics Letters* **25** (2012), 1881–1886

[6] Feldtein A., Traub J.F., *Order of vector recurrences with applications to nonlinear iteration, parallel algoritms, and the power method*, Carnegie Mellon University, Computer Science Department, paper 1578. Academic Press, 1970

[7] Herzberger J., Über Matrixdarstellungen für Iterationverfahren bei nichtlinearen Gleichungen, *Computing* **12** (1974), 215–222

[8] Jay I.O., A note on Q-order of convergence, *BIT Numerical Mathematics* **41** (2011), 422–429

[9] Kung H.T., Traub J.F., Optimal order of one-point and multipoint iteration, *J. Assoc. Comput. Math.* **21** (1974), 634–651

[10] Neta B. , A new family of higher order methods for solving equations, *International J. Computer Mathematics* **14** (1983), 191–195

[11] Ortega J.M., Rheinboldt W.C., *Iterative solution of nonlinear equations in several variables*, Academic Press, New York, 1970

[12] Petković M.S., Džunić J., Petković L.D., A family of two-point methods with memory for solving nonlinear equations, *Applicable Analysis and Discrete Mathematics* **5** (2011), 298–317

[13] Petković M.S., Ilic S., Džunić J., Derivative free two-point methods with and without memory for solving nonlinear equations, *Applied Mathematics and Computation* **217** (2010), 1887–1895

[14] Soleymani F., Sharma R., Li X., Tohidi E., An optimizated derivative-free form of the Potra-Pták method, *Mathematical and Computer Modelling* **56** (2012), 97–104

[15] Steffensen J.F., Remarks on iteration, *Skand Aktuar Tidsr* **16** (1933), 64–72

[16] Traub J.F., *Iterative Methods for the Solution of Equations*, Chelsea Publishing Company, New York, 1982

[17] Varona J.L., Graphic and numerical comparison between iterative methods, *Math. Intelligencer* **24**(1) (2002), 37–46

Received: 14.04.2014

Accepted: 30.09.2014

Instituto de Matemáticas Multidisciplinar, Universitat Politècnica de València,
46022, Valencia, Spain
E-mail: {[1]`acordero`, [4]`jrtorre`}`@mat.upv.es`

Department of Applied Mathematics, Hamedan Branch, Islamic Azad University,
Hamedan, Iran
E-mail: {[2]`lotfitaher`, [3]`arezookhoshandi`} `@yahoo.com`