Bull. Math. Soc. Sci. Math. Roumanie
Tome 58(106) No. 1, 2015, 117–124

A general form of the Second Main Theorem for hypersurfaces

by
Tran Van Tan[1] and Vu Van Truong[2]

Abstract

We prove a general form of the Second Main Theorem for algebraically nondegenerate holomorphic mappings into a smooth complex projective variety intersecting arbitrary hypersurfaces (rather than just the hypersurfaces in general position) and truncated multiplicities.

Key Words: Nevanlinna theory, Second Main Theorem, Hypersurface.
2010 Mathematics Subject Classification: Primary 32H30,
Secondary 32H04, 32H25, 14J70.

1 Introduction and statements

Let f be a holomorphic mapping of $\mathbb{C}$ into $\mathbb{C}P^N$, with a reduced representation $f = (f_0 : \cdots : f_N)$. The characteristic function $T_f(r)$ of f is defined by

$$T_f(r) := \frac{1}{2\pi}\int_0^{2\pi} \log \|f(re^{i\theta})\| d\theta, \text{ where } \|f\| := \max\{|f_0|, \ldots, |f_N|\}.$$

Let M be a positive integer or $+\infty$, and let ν be a divisor on $\mathbb{C}$. Set $\nu^{[M]}(z) := \min\{\nu(z), M\}$. The truncated counting function to level M of ν is defined by

$$N_\nu^{[M]}(r) := \int_1^r \frac{\sum_{|z|<t} \nu^{[M]}(z)}{t} dt \quad (1 < r < +\infty).$$

Let φ be a nonzero holomorphic function on $\mathbb{C}$. Denote by ν_φ be the zero divisor of φ. Set $N_\varphi^{[M]}(r) := N_{\nu_\varphi}^{[M]}(r)$. For brevity we will omit the character $^{[M]}$ in the counting function and in the divisor if $M = +\infty$.

Let D be a hypersurface in $\mathbb{C}P^N$ of degree $d \geq 1$. Let $Q \in \mathbb{C}[x_0, \ldots, x_N]$ be the homogeneous polynomial of degree d defining D. Set

$$\nu_D^{[M]} := \nu_{Q(f)}^{[M]}, \; N_f^{[M]}(r, D) := N_{Q(f)}^{[M]}(r) \quad \textit{and} \quad \lambda_D(f) := \log \frac{\|f\|^d \cdot \|Q\|}{|Q(f)|},$$

where $\|Q\|$ is the maximum of absolute values of the coefficients of Q.

Let $V \subset \mathbb{C}P^N$ be a smooth complex projective variety of dimension $n \geq 1$. Let $D_1, \dots, D_q$ $(V \not\subset D_j)$, be hypersurfaces in $\mathbb{C}P^N$. We say that the hypersurfaces $D_1, \dots, D_q$ are in general position in V if for any distinct indices $1 \leq j_1, \dots, j_k \leq q$, $(1 \leq k \leq n+1)$ there exist hypersurfaces $S_1, \dots, S_{(n+1-k)}$ in $\mathbb{C}P^N$ such that $D_{j_1} \cap \cdots \cap D_{j_k} \cap S_1 \cap \cdots \cap S_{(n+1-k)} \cap V = \varnothing$.

As usual, by the notation "$\|P$" we mean the assertion P holds for all $r \in [1, +\infty)$ excluding a Borel subset E of $(1, +\infty)$ with $\int\limits_E dr < +\infty$.

In 1997, Vojta [5] established a general form of the Second Main Theorem.

Latter, Ru [3] generalized the result of Vojta to the case where intesection multiplicities are truncated. He proved that.

Theorem A. *Let f be a linearly nondegenerate holomorphic mapping of $\mathbb{C}$ into $\mathbb{C}P^n$ and let $\{H_j\}_{j=1}^q$ be arbitrary hyperplanes in $\mathbb{C}P^n$. Let ψ and ϕ be increasing functions in $\mathbb{R}^+$ with $\int_e^\infty \frac{dr}{r\psi(r)} < \infty$, and $\int_e^\infty \frac{dr}{\phi(r)} = \infty$.*

Then the inequality

$$\int_0^{2\pi} \max_{K \in \mathcal{K}} \sum_{j \in K} \lambda_{H_j}(f(re^{i\theta}))\frac{d\theta}{2\pi} + N_{W(f)}(r)$$
$$\leq (n+1)T_f(r) + \frac{n(n+1)}{2} \log \frac{T_f(r)\psi(T_f(r))}{\phi(r)} + O(1),$$

holds for all r outside a set E with $\int_E \frac{dr}{\phi(r)} < \infty$. Here $\mathcal{K}$ is the set of all subsets $K \subset \{1, \dots, q\}$ such that the hyperplanes H_j, $j \in K$ are in general position, and $W(f)$ is the Wronskian of f.

We would like to emphasize here that in the above theorem of Ru, the hyperplanes $H_1, \dots, H_q$ are not assumed to be in general position.

Recently, the Second Main Theorem has been established for holomorphic maps in a projective variety intersecting hypersurfaces by Ru [4], Dethloff -Tan-Thai [1]. In 2009, Ru [4] proved that.

Theorem B. *Let $V \subset \mathbb{C}P^N$ be a smooth complex projective variety of dimension $n \geq 1$. Let f be an algebraically nondegenerate holomorphic mapping of $\mathbb{C}$ into V. Let $D_1, \dots, D_q$ be hypersurfaces in $\mathbb{C}P^N$ of degree d_j in general position in V. Then for every $\epsilon > 0$,*

$$\Big\| (q - n - 1 - \epsilon)T_f(r) \leq \sum_{j=1}^q \frac{1}{d_j} N(r, D_j).$$

We note that in Theorem B, the multiplicities of intersections are not truncated (all of them are taken in to the account of counting functions). Motivated by the case of hyperplanes, in this paper we generalize Theorem B to the case of arbitrary hypersurfaces, and the multiplicities of intersections are truncated (the multiplicities are taken in to the account do not exceed a common positive integer). In the case where hypersurfaces are in general position, from our below theorem we get a slight improvement of Theorem B that mulitiplicities in the counting functions are truncated by a positive integer. We will prove the following theorem.

Theorem 1. *Let $V \subset \mathbb{C}P^N$ be a smooth complex projective variety of dimension $n \geq 1$. Let f be an algebraically nondegenerate holomorphic mapping of $\mathbb{C}$ into V. Let $D_1, \ldots, D_q$ ($V \not\subset D_j$) be arbitrary hypersurfaces in $\mathbb{C}P^n$ of degree d_j. Then, for every $\epsilon > 0$, there exists a positive integer M depending on $\epsilon, d_j, q, n, \deg V$ such that*

$$\Big\| \int_0^{2\pi} \max_{K\in\mathcal{K}} \sum_{j\in K} \frac{1}{d_j}\lambda_{D_j}(f(re^{i\theta}))\frac{d\theta}{2\pi} + \int_1^r \frac{dt}{t} \max_{K\in\mathcal{K}} \sum_{j\in K, |z|<t} \frac{1}{d_j}\big(\nu_{D_j}(z) - \nu_{D_j}^{[M]}(z)\big)$$
$$\leq (n+1+\epsilon)T_f(r) \tag{1.1}$$

where $\mathcal{K}$ is the set of all subsets $K \subset \{1,\ldots,q\}$ such that the hypersurfaces $\{D_j,\ j \in K\}$ are in general position in V.

The proof of our theorem consists of two parts: In the first parts (section 3 until inequality (3.3)), by using the technique of Min Ru in [4], we approximate the first term of inequality (1.1) by an integration of a summation of linear forms. In the second part (from (3.3)), we apply the Hilbert weights to estimating the second term of (1.1) which gives a truncation for intersection multiplicities. This technique is completed different from the one used in the case of hyperplanes.

2 Some lemmas

Let $X \subset \mathbb{C}P^N$ be a projective variety of dimension n and degree $\triangle$. Let I_X be the prime ideal in $\mathbb{C}[x_0,\ldots,x_N]$ defining X. Denote by $\mathbb{C}[x_0,\ldots,x_N]_m$ the vector space of homogeneous polynomials in $\mathbb{C}[x_0,\ldots,x_N]$ of degree m (including 0). Put $I_X(m) := \mathbb{C}[x_0,\ldots,x_N]_m \cap I_X$.

The Hilbert function H_X of X is defined by

$$H_X(m) := \dim \mathbb{C}[x_0,\ldots,x_N]_m \big/ I_X(m).$$

For each tuple $c = (c_0,\ldots,c_N) \in \mathbb{R}_{\geq 0}^{N+1}$, and $m \in \mathbb{N}$, we define the m-th Hilbert weight $S_X(m,c)$ of X with respect to c by

$$S_X(m,c) := \max \sum_{i=1}^{H_X(m)} I_i \cdot c,$$

where $I_i = (I_{i0},\ldots,I_{iN}) \in \mathbb{N}_0^{N+1}$ and the maximum is taken over all sets $\{x^{I_i} = x_0^{I_{i0}} \cdots x_N^{I_{iN}}\}$ whose residue classes modulo $I_X(m)$ form a basis of the vector space $\mathbb{C}[x_0,\ldots,x_N]_m \big/ I_X(m)$.

Lemma 1 ([4], Lemma 3.2.). *Let $X \subset \mathbb{C}P^N$ be an algebraic variety of dimension n and degree $\triangle$. Let $m > \triangle$ be an integer and let $c = (c_0,\ldots,c_N) \in \mathbb{R}_{\geq 0}^{N+1}$. Let $\{i_0,\ldots,i_n\}$ be a subset of $\{0,\ldots,N\}$ such that $\{x = (x_0 : \cdots : x_N) \in \mathbb{C}P^N : x_{i_0} = \cdots = x_{i_n} = 0\} \cap X = \varnothing$. Then*

$$\frac{1}{mH_X(m)} S_X(m,c) \geq \frac{1}{(n+1)}(c_{i_0} + \cdots + c_{i_n}) - \frac{(2n+1)\triangle}{m} \cdot \max_{0\leq i\leq N} c_i.$$

Lemma 2 ([2], Lemma 3.2.13)**.** *Let f be a linearly nondegenerate holomorphic mapping of $\mathbb{C}$ into $\mathbb{C}P^N$ with the reduced representation $f = (f_0 : \cdots : f_N)$. Let $W(f) = W(f_0, \ldots, f_N)$ be the Wronskian of f. Then*

$$\nu_{\frac{f_0 \cdots f_N}{W(f)}} \le \sum_{i=0}^{N} \min\{\nu_{f_i}, N\}.$$

3 Proof of Theorem 1.

Let $Q_j,\ 1 \le j \le q$, be homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_N]$ of degree d_j defining D_j. Denote by $\mathcal{R}$ the set of all subsets $R \subset \{1, \ldots, q\}$ such that $\#R = n+1$ and $\cap_{j \in R} D_j \cap V = \varnothing$.

Claim: If $\mathcal{R} \ne \varnothing$ and $d_1 = \cdots = d_q := d$, then for every $\epsilon > 0$, there exists a positive integer M depending on $\epsilon, d, q, n, \deg V$, such that

$$\Big\| \int_0^{2\pi} \max_{R \in \mathcal{R}} \sum_{j \in R} \frac{1}{d} \lambda_{D_j}(f(re^{i\theta})) \frac{d\theta}{2\pi} + \int_1^r \frac{dt}{t} \max_{R \in \mathcal{R}} \sum_{j \in R, |z| < t} \frac{1}{d} \big(\nu_{D_j}(z) - \nu_{D_j}^{[M]}(z)\big)$$
$$\le (n+1+\epsilon) T_f(r). \tag{3.1}$$

Since $\mathcal{R} \ne \varnothing$, we have that $\cap_{j=1}^q D_j \cap V = \varnothing$. We define a map $\Phi : V \longrightarrow \mathbb{C}P^{q-1}$ by $\Phi(x) = (Q_1(x) : \cdots : Q_q(x))$. Then Φ is a finite morphism and $Y := im\Phi$ is a complex projective subvariety of $\mathbb{C}P^{q-1}$ and $\dim Y = n$ and $\triangle := \deg Y \le d^n \cdot \deg V$.

For a positive integer m, denote by $\{I_1, \ldots, I_{q_m}\}$ the set of all $I_i := (I_{i1}, \ldots, I_{iq}) \mathbb{N}_0^q$ with $I_{i1} + \cdots + I_{iq} = m$. We have $q_m := \binom{q+m-1}{m}$.

Let F be a holomorphic mapping of $\mathbb{C}$ into $\mathbb{C}P^{q_m - 1}$ with the reduced representation $F = \big(Q_1^{I_{11}}(f) \cdots Q_q^{I_{1q}}(f) : \cdots : Q_1^{I_{q_m 1}}(f) \cdots Q_q^{I_{q_m q}}(f)\big)$. Let P be the smallest sub-space in $\mathbb{C}P^{q_m-1}$ containing ImF. Then by an argument as in [4], page 261, we have $\dim P = H_Y(m) - 1$. We define hyperplanes H_j $(j = 1, \ldots, q_m)$ in the complex projective space P by $H_j := \{(z_1 : \cdots : z_{q_m}) \in \mathbb{C}P^{q_m - 1} :\ z_j = 0\} \cap P$.

Denote by $\mathcal{L}$ the set of all subsets J of $\{1, \ldots, q_m\}$ such that $\#J = H_Y(m)$ and the hyperplanes $H_j, j \in J$, are in general position in P. Then $\mathcal{L}$ is also the set of all subsets J of $\{1, \ldots, q_m\}$ such that $\{y^{I_j},\ j \in J\}$ $(y = (y_1, \ldots, y_q))$ is a basis of $\mathbb{C}[y_1, \ldots, y_q] \diagup I_Y(m)$.

Similarly to (3.19) in [4], for every $z \in \mathbb{C}$, we have

$$\max_{R \in \mathcal{R}} \frac{1}{(n+1)} \sum_{j \in R} \lambda_{D_j}(f(z)) \le \frac{1}{mH_Y(m)} \max_{L \in \mathcal{L}} \sum_{i \in L} \lambda_{H_i}(F(z)) + d \log \|f(z)\|$$
$$- \frac{1}{m} \log \|F(z)\| + \frac{(2n+1)\triangle}{m} \sum_{1 \le j \le q} \lambda_{D_j}(f(z)) + O(1).$$

Therefore, by Theorem A (for F and $\epsilon = 1$), we have

$$\begin{aligned}
\Big\| \int_0^{2\pi} \max_{R\in\mathcal{R}} \sum_{j\in R} \lambda_{D_j}(f(re^{i\theta}))\frac{d\theta}{2\pi} &\le \frac{n+1}{mH_Y(m)}\int_0^{2\pi} \max_{L\in\mathcal{L}} \sum_{i\in L} \lambda_{H_i}(F(re^{i\theta}))\frac{d\theta}{2\pi} \\
&\quad + d(n+1)T_f(r) - \frac{n+1}{m}T_F(r) \\
&\quad + \frac{(2n+1)(n+1)\triangle}{m}\sum_{1\le j\le q}\big(d\cdot T_f(r) - N_f(r,D_j)\big) + O(1) \\
&\le \frac{(n+1)(H_Y(m)+1)}{mH_Y(m)}T_F(r) - \frac{n+1}{mH_Y(m)}N_{W(F)}(r) \\
&\quad + d(n+1)T_f(r) - \frac{n+1}{m}T_F(r) \\
&\quad + \frac{(2n+1)(n+1)dq\triangle}{m}T_f(r) + O(1) \\
&\le \frac{n+1}{mH_Y(m)}T_F(r) - \frac{n+1}{mH_Y(m)}N_{W(F)}(r) \\
&\quad + d(n+1)T_f(r) + \frac{(2n+1)(n+1)dq\triangle}{m}T_f(r) + O(1).
\end{aligned} \tag{3.2}$$

For an arbitrary $\epsilon > 0$, we choose m such that $\frac{(2n+1)(n+1)dq\triangle}{m} < \frac{\epsilon}{4}$ and $\frac{(n+1)d}{H_Y(m)} < \frac{\epsilon}{4}$. Then, by (3.2) we get

$$\Big\| \int_0^{2\pi} \max_{R\in\mathcal{R}} \sum_{j\in R} \lambda_{D_j}(f(re^{i\theta}))\frac{d\theta}{2\pi} \le \big((n+1)d + \frac{\epsilon}{2}\big)T_f(r) - \frac{n+1}{mH_Y(m)}N_{W(F)}(r). \tag{3.3}$$

For each $J := \{j_1,\dots,j_{H_Y(m)}\} \in \mathcal{L}$, then there exists a constant $c_J \neq 0$ such that

$$W(F) = c_J \cdot W(Q_1^{I_{j_1 1}}(f)\cdots Q_q^{I_{j_1 q}}(f),\dots,Q_1^{I_{j_{H_Y(m)}1}}(f)\cdots Q_q^{I_{j_{H_Y(m)}q}}(f)).$$

On the other hand, by Lemma 2,

$$\nu_{\frac{Q_1^{I_{j_1 1}}(f)\cdots Q_q^{I_{j_1 q}}(f)\cdots Q_1^{I_{j_{H_Y(m)}1}}(f)\cdots Q_q^{I_{j_{H_Y(m)}q}}(f)}{W\left(Q_1^{I_{j_1 1}}(f)\cdots Q_q^{I_{j_1 q}}(f),\dots,Q_1^{I_{j_{H_Y(m)}1}}(f)\cdots Q_q^{I_{j_{H_Y(m)}q}}(f)\right)}} \le \sum_{1\le i\le H_Y(m)} \nu^{[H_Y(m)-1]}_{Q_1^{I_{j_i 1}}(f)\cdots Q_q^{I_{j_i q}}(f)}.$$

Hence, for all $J \in \mathcal{L}$, we have

$$\begin{aligned}
\nu_{W(F)} &\ge \nu_{Q_1^{I_{j_1 1}}(f)\cdots Q_q^{I_{j_1 q}}(f)\cdots Q_1^{I_{j_{H_Y(m)}1}}(f)\cdots Q_q^{I_{j_{H_Y(m)}q}}(f)} - \sum_{1\le i\le H_Y(m)} \nu^{[H_Y(m)-1]}_{Q_1^{I_{j_i 1}}(f)\cdots Q_q^{I_{j_i q}}(f)} \\
&\ge \sum_{1\le j\le q}\sum_{i\in J} I_{ij}\big(\nu_{Q_j(f)} - \nu^{[H_Y(m)-1]}_{Q_j(f)}\big).
\end{aligned} \tag{3.4}$$

For every $z \in \mathbb{C}$, let $c_z := (c_{1,z}, \ldots, c_{q,z})$ where $c_{j,z} := \nu_{Q_j(f)}(z) - \nu_{Q_j(f)}^{[H_Y(m)-1]}(z)$. Then, by definition of the Hilbert weight, there exists $J_z \in \mathcal{L}$ such that

$$S_Y(m, c_z) = \sum_{i \in J_z} I_i \cdot c_z = \sum_{1 \le j \le q} \sum_{i \in J_z} I_{ij} \big(\nu_{Q_j(f)}(z) - \nu_{Q_j(f)}^{[H_Y(m)-1]}(z)\big).$$

Then, by Lemma 1, for very $R \in \mathcal{R}$ we have

$$\begin{aligned}
\frac{1}{mH_Y(m)} & \sum_{1 \le j \le q} \sum_{i \in J_z} I_{ij} \big(\nu_{Q_j(f)}(z) - \nu_{Q_j(f)}^{[H_Y(m)-1]}(z)\big) \\
& \ge \frac{1}{n+1} \sum_{j \in R} \big(\nu_{Q_j(f)}(z) - \nu_{Q_j(f)}^{[H_Y(m)-1]}(z)\big) \\
& \quad - \frac{(2n+1)\triangle}{m} \max_{1 \le j \le q} \big(\nu_{Q_j(f)}(z) - \nu_{Q_j(f)}^{[H_Y(m)-1]}(z)\big) \\
& \ge \frac{1}{n+1} \sum_{j \in R} \big(\nu_{Q_j(f)}(z) - \nu_{Q_j(f)}^{[H_Y(m)-1]}(z)\big) - \frac{(2n+1)\triangle}{m} \sum_{1 \le j \le q} \nu_{Q_j(f)}(z).
\end{aligned}$$

Combining with (3.4), for every $R \in \mathcal{R}$ and $z \in \mathbb{C}$, we have

$$\begin{aligned}
\frac{1}{mH_Y(m)} \nu_{W(F)(z)} \ge \frac{1}{n+1} & \sum_{j \in R} \big(\nu_{Q_j(f)}(z) - \nu_{Q_j(f)}^{[H_Y(m)-1]}(z)\big) \\
& - \frac{(2n+1)\triangle}{m} \sum_{1 \le j \le q} \nu_{Q_j(f)}(z).
\end{aligned}$$

This implies that

$$\begin{aligned}
\frac{n+1}{mH_Y(m)} \nu_{W(F)} \ge \max_{R \in \mathcal{R}} & \sum_{j \in R} \big(\nu_{Q_j(f)} - \nu_{Q_j(f)}^{[H_Y(m)-1]}\big) \\
& - \frac{(n+1)(2n+1)\triangle}{m} \sum_{1 \le j \le q} \nu_{Q_j(f)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{n+1}{mH_Y(m)} N_{W(F)}(r) \ge \int_1^r \frac{dt}{t} \max_{R \in \mathcal{R}} & \sum_{j \in R, |z| < t} \big(\nu_{Q_j(f)}(z) - \nu_{Q_j(f)}^{[H_Y(m)-1]}(z)\big) \\
& - \frac{(n+1)(2n+1)\triangle}{m} \sum_{1 \le j \le q} N_f(r, D_j)
\end{aligned}$$

$$\geq \int_1^r \frac{dt}{t} \max_{R\in\mathcal{R}} \sum_{j\in R,|z|<t} \big(\nu_{Q_j(f)}(z) - \nu^{[H_Y(m)-1]}_{Q_j(f)}(z)\big)$$
$$- \frac{(n+1)(2n+1)dq\triangle}{m} \sum_{1\leq j\leq q} T_f(r) - O(1)$$
$$\geq \int_1^r \frac{dt}{t} \max_{R\in\mathcal{R}} \sum_{j\in R,|z|<t} \big(\nu_{D_j}(z) - \nu^{[H_Y(m)-1]}_{D_j}(z)\big) - \frac{\epsilon}{4}T_f(r).$$

Combining with (3.3) we get

$$\Big\| \int_0^{2\pi} \max_{R\in\mathcal{R}} \sum_{j\in R} \lambda_{D_j}(f(re^{i\theta}))\frac{d\theta}{2\pi} + \int_1^r \frac{dt}{t} \max_{R\in\mathcal{R}} \sum_{j\in R,|z|<t} \big(\nu_{D_j}(z) - \nu^{[H_Y(m)-1]}_{D_j}(z)\big)$$
$$\leq \big((n+1)d+\epsilon\big)T_f(r).$$

So, we get (3.1).

We now prove the theorem for the general case. Denote by $\mathcal{K}'$ the set of all $K \in \mathcal{K}$ such that K does not contain any other set in $\mathcal{K}$. For each $K \in \mathcal{K}'$, we choose hyperplanes $H_{(K,1)}, \dots H_{(K,n+1-\#K)}$ in $\mathbb{C}P^N$ such that

$$(\cap_{D\in K} D) \cap H_{(K,1)} \cap \cdots \cap H_{(K,n+1-\#L)} \cap V = \varnothing.$$

Set $\mathcal{Q} := \{D_1,\dots,D_q\} \cup \{H_{(K,i)},\ K\in\mathcal{K}', 1\leq i\leq n+1-\#K\}$. Denote by d the least common multiple of $d_1,\dots,d_q$ and put $d_j^* = \frac{d}{d_j}$. Set $\{\widetilde{D_1},\dots,\widetilde{D_p}\} := \{D_1^{d_j^*},\dots,D_q^{d_j^*}\} \cup \{H^d_{(K,i)},\ K\in\mathcal{K}', 1\leq i\leq n+1-\#K\}$. Denote by $\widetilde{\mathcal{R}}$ the set of all subsets $\widetilde{R}\subset\{1,\dots,p\}$ such that $\#\widetilde{R} = n+1$ and $\cap_{j\in\widetilde{R}}\widetilde{D_j}\cap V = \varnothing$. It is clear that for each $K\in\mathcal{K}$ there exists $\widetilde{R}_K\in\widetilde{\mathcal{R}}$ such that $\{D_j^{d_j^*},\ j\in K\}\subset\{\widetilde{D_i},\ i\in\widetilde{R_K}\}$.

We now apply (3.1) for hypersurfaces $\widetilde{D_1},\dots,\widetilde{D_p}$, and get that

$$\Big\| \int_0^{2\pi} \max_{K\in\mathcal{K}} \sum_{j\in K} \frac{1}{d_j}\lambda_{D_j}(f(re^{i\theta}))\frac{d\theta}{2\pi} + \int_1^r \frac{dt}{t} \max_{K\in\mathcal{K}} \sum_{j\in K,|z|<t} \frac{1}{d_j}\big(\nu_{D_j}(z) - \nu^{[M]}_{D_j}(z)\big)$$
$$\leq \int_0^{2\pi} \max_{K\in\mathcal{K}} \sum_{j\in K} \frac{1}{d}\lambda_{D_j^{d_j^*}}(f(re^{i\theta}))\frac{d\theta}{2\pi} + \int_1^r \frac{dt}{t} \max_{K\in\mathcal{K}} \sum_{j\in K,|z|<t} \frac{1}{d}\big(\nu_{D_j^{d_j^*}}(z) - \nu^{[M]}_{D_j^{d_j^*}}(z)\big)$$
$$\leq \int_0^{2\pi} \max_{\widetilde{R}\in\widetilde{\mathcal{R}}} \sum_{j\in\widetilde{R}} \frac{1}{d}\lambda_{\widetilde{D_j}}(f(re^{i\theta}))\frac{d\theta}{2\pi} + \int_1^r \frac{dt}{t} \max_{\widetilde{R}\in\widetilde{\mathcal{R}}} \sum_{j\in R,|z|<t} \frac{1}{d}\big(\nu_{\widetilde{D_j}}(z) - \nu^{[M]}_{\widetilde{D_j}}(z)\big)$$
$$\overset{(3.1)}{\leq} (n+1+\epsilon)T_f(r),$$

where the positive integer M depends on $\epsilon, d, q, n, \deg V$. This completes the proof of Theorem 1. ∎

Acknowledgement. This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.02-2013.13. The first named author was partially supported by Vietnam Institute for Advanced Study in Mathematics, and by the Abdus Salam International Centre for Theoretical Physics, Italy. The second named author would like to thank Professor Do Duc Thai for scientific encouragement.

References

[1] G. Dethloff, T. V. Tan and D. D. Thai, An extension of the Cartan-Nochka second main theorem for hypersurfaces, Internat. J. Math. **22** (2011), 863-885.

[2] H. Fujimoto, *Value didtribution theory of the Gauss map of minimal surfaces in* $\mathbb{R}^m$, Braunschweig, Vieweg -Sohn, 1993.

[3] M. Ru, On a general form of the Second Main Theorem, Trans. AMS **349** (1997), 5093-5105.

[4] M. Ru, Holomorphic curves into algebraic varieties, Ann. of Math., **169** (2009), 255-267.

[5] P. Vojta, On Cartan's theorem and Cartan's conjecture, Amer. J. Math. **119** (1997), 1-17.

Received: 15.01.2014
Accepted: 30.10.2014

Department of Mathematics,
Hanoi National University of Education
E-mail: [1]`tranvantanhn@yahoo.com`,
[2]`vvtruong.dnb@moet.edu.vn`