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Nonuniform ordinary dichotomy for evolution families on the real line

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Abstract

This paper presents some Perron-type results for the nonuniform ordinary dichotomy of evolution families on the real line with nonuniform exponential growth. It it also mentioned the notion of the admissibility of the pair $(\mathcal{L}^1(X), \mathcal{L}^\infty(X))$ to an evolution family. This notion is used to obtain a result for the nonuniform ordinary dichotomy for an evolution family on the real line.

Key Words: Evolution family, admissibility, dichotomy on $\mathbb{R}$.
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1 Introduction

One of the most important asymptotic properties of a differential system is the exponential dichotomy, notion introduced by O. Perron in 1930 in [11].

J.L. Daleckij and M.G. Krein in [4], J.L. Massera and J.J. Schäffer, in [8, Chapter 8], have obtained some dichotomy results on $\mathbb{R}$ for differential equations, for the infinite dimensional case.

In 1974 M.G. Krein and J.L. Daleckij, in [4, Theorem 4.1, p. 81], shows that if $A \in \mathcal{B}(X)$ then $\sigma(A) \cap i\mathbb{R} = \emptyset$ if and only if the differential equation $\dot{x}(t) = Ax(t) + f(t)$ has an unique solution $x \in \mathcal{C}$, for all $f \in \mathcal{C}$, where $\mathcal{C}$ represents the Banach space of the continuous and bounded functions on $\mathbb{R}$ and $\sigma(A)$ represent the spectrum of the operator A.

Another important results in the study of the evolution equations were obtained by B.M. Levitan and V.V. Zhikov in [7] and A. Pazy in [10]. Some of the results were extended for the evolution families with nonuniform exponential growth by L. Barreira and C. Valls in [2] and [3].

In 1998 Y. Latushkin, T. Randolph and R. Schnaubelt, in [6], study the dichotomy on $\mathbb{R}$ for the evolution families with uniform exponential growth through the assigned evolution semigroup. The dichotomy on $\mathbb{R}_+$ was studied by A. Ben-Artzi and I. Gohberg in [1], N. van

Minh, F. Räbiger and R. Schnaubelt in [9] and N.T. Huy in [5]. Similar results for the dichotomy on the real line were obtained by A.L. Sasu and B. Sasu in [14] and A.L. Sasu in [15],[16].

In 2010, in [2], L. Barreira and C. Valls, using appropriate adapted norms (which can be seen as Lyapunov norms) mention the connection between a nonuniform exponentially stable evolution family and the admissibility of their associated L^p spaces, denoted by $\mathcal{L}^p(X)$, where $p > 1$. The authors, mentioned above, extend the previous results for the nonuniform exponential dichotomy in 2011, in [3]. They prove that if any $\mathcal{L}^p(X)$ space, with $p > 1$, is admissible to an evolution process with nonuniform exponential growth, then that evolution process is nonuniform exponentially dichotomic.

The results were extended in 2012, in [12], by C. Preda, P. Preda and A. Crăciunescu, which have obtained some R. Datko and Lyapunov-type characterizations for the nonuniform exponential stability and dichotomy of an evolution process with a nonuniform exponential growth. In the same year, in [13], C. Preda, P. Preda and C. Praţa presents some Perron-type results for the nonuniform exponential dichotomy of an evolution operator.

All the above results are obtained for all $t_0 \in \mathbb{R}_+$. In [14], [15] and [16] are considered systems described by evolution families with exponential growth on the real line. It is known that the exponential dichotomy is a generalization of the exponential stability, so it is expected that the above results in more stringent conditions should imply the exponential stability on $\mathbb{R}$.

The main purpose of this paper is to give a sufficient condition for the nonuniform dichotomy of an evolution family with nonuniform exponential growth on the real line, using the admissibility of the pair $(\mathcal{L}^1(X), \mathcal{L}^\infty(X))$.

Section 2 is devoted to our preliminaries while Section 3 is dedicated to the main results. First we will specify the following terms: evolution family on $\mathbb{R}$ that has nonuniform exponential growth, family of projectors compatible with an evolution family on the real line, nonuniform ordinary dichotomic evolution family. In Definition 3.1 we describe the admissibility of the pair $(\mathcal{L}^1(X), \mathcal{L}^\infty(X))$ to an evolution family. It is worth mentioning that in this case the complement of the space $X_1(t_0)$ is unique, for all $t_0 \in \mathbb{R}$, unlike the case when $t_0 \in \mathbb{R}_+$ and the associated family of projectors has a similar behaviour to the case where the evolution family is generated by a differential system (see [4] and [8]).

We will use Theorem 3.1 in the demonstration of the most important result of this paper, namely Theorem 3.2. We have expanded the work done by L. Barreira and C. Valls in [2] and [3] for the nonuniform exponential dichotomy using the admissibility of the pair $(\mathcal{L}^1(X), \mathcal{L}^\infty(X))$. Using the same adapted norms and with personal methods, we will prove in Theorem 3.2 that if any pair $(\mathcal{L}^1(X), \mathcal{L}^\infty(X))$ is admissible to an evolution family with nonuniform exponential growth then that evolution family is nonuniform ordinary dichotomic.

2 Preliminaries

Let X be a Banach space and $\mathcal{B}(X)$ the Banach algebra of all linear and bounded operators acting on X. We will denote by $\|\cdot\|$ the norm on X and $\mathcal{B}(X)$ and $\Delta = \{(t, t_0) \in \mathbb{R}^2 : t \geq t_0\}$.

Definition 2.1. *An application* $\Phi : \Delta \to \mathcal{B}(X), \Phi = \{\Phi(t, t_0)\}_{t \geq t_0}$, *is called* an evolution family on $\mathbb{R}$ *if it satisfies the following properties:*

(i) $\Phi(t,t) = I$, *for all* $t \in \mathbb{R}$, *where* I *is the identity operator on* X;

(ii) $\Phi(t,\tau)\Phi(\tau,t_0) = \Phi(t,t_0)$, *for all* $t \geq \tau \geq t_0$;

(iii) $t_0 \mapsto \Phi(\cdot,t_0)x : [t_0,\infty) \to X$ *is continuous for all* $x \in X$ *and* $t \mapsto \Phi(t,\cdot)x : (-\infty,t] \to X$ *is continuous for all* $x \in X$.

Definition 2.2. *We say that the evolution family* Φ *is* with nonuniform exponential growth *if there exists* $M : \mathbb{R} \to \mathbb{R}_+^*$ *and* $\omega \in \mathbb{R}$ *such that*

$$\|\Phi(t,t_0)\| \leq M(t_0)e^{\omega(t-t_0)}, \text{ for all } t \geq t_0.$$

Following [2] and [3] we introduce the norm:

$$\|x\|_{t_0} = \sup_{\tau \geq t_0} e^{-\omega(\tau-t_0)}\|\Phi(\tau,t_0)x\|$$

and we obtain that

$$\|x\| \leq \|x\|_{t_0} \leq M(t_0)\|x\|, \text{ for all } t_0 \in \mathbb{R} \text{ and } x \in X$$

and

$$\|\Phi(t,t_0)x\|_t \leq e^{\omega(t-t_0)}\|x\|_{t_0}, \text{ for all } t \geq t_0 \text{ and } x \in X.$$

Further we mention the following notations:

$$L^1 = \{f : \mathbb{R} \to \mathbb{R} : \text{ f is measurable and } \|f\|_1 = \int_{-\infty}^{\infty} |f(t)|dt < \infty\},$$

$$L^\infty = \{f : \mathbb{R} \to \mathbb{R} : \text{ f is measurable and } \|f\|_\infty = ess\sup_{t\in\mathbb{R}} |f(t)| < \infty\},$$

$$\mathcal{L}^1(X) = \{f : \mathbb{R} \to X : \text{ f is Bochner measurable and } t \xrightarrow{g} \|f(t)\|_t \in L^1\}$$

and

$$\mathcal{L}^\infty(X) = \{f : \mathbb{R} \to X : \text{ f is Bochner measurable and } t \xrightarrow{g} \|f(t)\|_t \in L^\infty\}.$$

Remark 2.1. By [2] and [3] we obtain that $\mathcal{L}^1(X)$ and $\mathcal{L}^\infty(X)$ are Banach spaces with the norms $\|f\|_1^{'} = \|g\|_1$, respectively $\|f\|_\infty^{'} = \|g\|_\infty$, where $g : \mathbb{R} \to \mathbb{R}$, $g(t) = \|f(t)\|_t$.

We set now $X_1(t_0) = \{x \in X : \delta_x \in \mathcal{L}^\infty(X)\}$, where $\delta_x(t) = \begin{cases} \Phi(t,t_0)x, & t \geq t_0 \\ 0, & t < t_0 \end{cases}$ and $X_2(t_0) = \{x \in X :$ there exists $\varphi_x \in \mathcal{L}^\infty(X)$ such that $\varphi_x(t) = \Phi(t,s)\varphi_x(s)$, for all $t_0 \geq t \geq s$; $\varphi_x(t_0) = x\}$.

If $X_1(t_0)$ is a closed and complemented subspace and $X_2(t_0)$ is a complement for $X_1(t_0)$, for all $t_0 \in \mathbb{R}$ then the family of projectors associated to the decomposition $X = X_1(t_0) \oplus X_2(t_0)$ are denoted by $\{P_i(t_0)\}_{t_0\in\mathbb{R}}$, for all $t_0 \in \mathbb{R}$ and $i = 1, 2$.

If $X = X_1 \oplus X_2$, we denote by

$$\gamma[X_1,X_2] = \inf_{x_i \in X_i - \{0\}} \left\| \frac{x_1}{\|x_1\|} - \frac{x_2}{\|x_2\|} \right\|, \ i = 1, 2.$$

Remark 2.2. By [8] we have that

$$\frac{1}{\|P_i\|} \leq \gamma[X_1, X_2] \leq \frac{2}{\|P_i\|},\ i = 1, 2,$$

where $P_i, i = 1, 2$ are the projectors associated to the decomposition $X = X_1 \oplus X_2$.

Remark 2.3. It can be seen that $\Phi(t, t_0)X_i(t_0) \subset X_i(t)$, for all $t \geq t_0$ and $i = 1, 2$.

Definition 2.3. *We say that a family of projectors $\{P(t)\}_{t\in\mathbb{R}}$ is* compatible *with the evolution family $\{\Phi(t, t_0)\}_{t\geq t_0}$ if the following properties are satisfied:*

(i) $P(t)\Phi(t, t_0) = \Phi(t, t_0)P(t_0)$, *for all* $t \geq t_0$*;*

(ii) The application $t \mapsto P(t)x : \mathbb{R} \to X$ *is bounded, for all* $x \in X$*;*

(iii) $\Phi(t, t_0) : KerP(t_0) \to KerP(t)$ *is an isomorphism, for all* $t \geq t_0$.

If we denote by $X_1(t_0) = P(t_0)X$, respectively $X_2(t_0) = (I - P(t_0))X$ we can formulate the following definition:

Definition 2.4. *We say that the evolution family $\{\Phi(t, t_0)\}_{t\geq t_0}$ is* nonuniform ordinary dichotomic *if there exist a family of projectors $\{P(t)\}_{t\in\mathbb{R}}$ compatible with $\{\Phi(t, t_0)\}_{t\geq t_0}$ and there exists $N_1, N_2 > 0$ such that:*

- $\|\Phi(t, t_0)x\|_t \leq N_1\|x\|_{t_0},$ *for all* $x \in X_1(t_0)$ *and* $t \geq t_0$.
- $\|\Phi(t, t_0)x\|_t \geq N_2\|x\|_{t_0},$ *for all* $x \in X_2(t_0)$ *and* $t \geq t_0$.

Remark 2.4. If the evolution family $\{\Phi(t, t_0)\}_{t\geq t_0}$ is nonuniform ordinary dichotomic then there exist a family of projectors $\{P(t)\}_{t\in\mathbb{R}}$ compatible with $\{\Phi(t, t_0)\}_{t\geq t_0}$ and there exists $N_1, N_2 > 0$ such that:

- $\|\Phi(t, t_0)x\| \leq N_1M(t_0)\|x\|,$ for all $x \in X_1(t_0)$ and $t \geq t_0$.
- $M(t)\|\Phi(t, t_0)x\| \geq N_2\|x\|,$ for all $x \in X_2(t_0)$ and $t \geq t_0$.

3 Main Results

Let $\{\Phi(t, t_0)\}_{t\geq t_0}$ be an evolution family on $\mathbb{R}$ with nonuniform exponential growth.

Definition 3.1. *We say that the pair $(\mathcal{L}^1(X), \mathcal{L}^\infty(X))$ is* admissible *to $\{\Phi(t, t_0)\}_{t\geq t_0}$ if for all $f \in \mathcal{L}^1(X)$ it results that there is an unique $x_f \in \mathcal{L}^\infty(X)$ such that*

$$x_f(t) = \Phi(t, s)x_f(s) + \int_s^t \Phi(t, \tau)f(\tau)d\tau,\ \textit{for all}\ t \geq s.$$

Remark 3.1. If $x \in X_2(t_0),\ x \neq 0$ then $\Phi(t, t_0)x \neq 0$, for all $t \geq t_0$.

Theorem 3.1. *If the pair* $(\mathcal{L}^1(X), \mathcal{L}^\infty(X))$ *is admissible to* $\{\Phi(t,t_0)\}_{t\geq t_0}$ *then there is* $K>0$ *such that*

$$\|x_f\|_\infty^{'} \leq K\|f\|_1^{'}, \text{ for all } f \in \mathcal{L}^1(X).$$

Démonstration: Let $\mathcal{U} : \mathcal{L}^1(X) \to \mathcal{L}^\infty(X)$, defined by $\mathcal{U}f = x_f$.

It is easy to notice that $\mathcal{U}$ is a closed operator and by the Closed Graph Theorem we obtain that there is $K>0$ such that

$$||\mathcal{U}f||_\infty^{'} = ||x_f||_\infty^{'} \leq K||f||_1^{'}, \text{ for all } f \in \mathcal{L}^1(X).$$

□

Our main result is the following:

Theorem 3.2. *If the pair* $(\mathcal{L}^1(X), \mathcal{L}^\infty(X))$ *is admissible to* $\{\Phi(t,t_0)\}_{t\geq t_0}$ *then :*

(i) The subspaces $X_i(t_0)$ *are closed, for all* $t_0 \in \mathbb{R}$ *and* $i = 1, 2$;

(ii) $X = X_1(t_0) \oplus X_2(t_0)$, *for all* $t_0 \in \mathbb{R}$;

(iii) The evolution family $\{\Phi(t,t_0)\}_{t\geq t_0}$ *is nonuniform ordinary dichotomic;*

(iv) $\Phi(t_1,t_0)X_2(t_0) = X_2(t_1)$, *for all* $t_1 > t_0$;

(v) The application $\Phi(t_1,t_0) : X_2(t_0) \to X_2(t_1)$ *is invertible, for all* $t_1 > t_0$;

(vi) The family of projectors $\{P_i(t_0)\}_{t_0\in\mathbb{R}}$, $i = 1, 2$ *associated to the decomposition* $X = X_1(t_0) \oplus X_2(t_0)$, *for all* $t_0 \in \mathbb{R}$, *has the following property*

$$t \mapsto \|P_i(t_0)\|_{t_0} : \mathbb{R} \to \mathbb{R}_+ \text{ is bounded}, \ i = 1, 2.$$

Démonstration: (iii) Let $t_0 \in \mathbb{R}, \ \delta > 0, \ x \in X_1(t_0)$ with $\Phi(t,t_0)x \neq 0$, for all $t \geq t_0$ and

$$f : \mathbb{R} \to X, \ f(t) = \varphi_{[t_0,t_0+\delta]}(t)\frac{\Phi(t,t_0)x}{\|\Phi(t,t_0)x\|_t},$$

where $\varphi_{[t_0,t_0+\delta]}$ denotes the characteristic function of the interval $[t_0, t_0+\delta]$.

We notice that $f \in \mathcal{L}^1(X)$ and $\|f\|_1^{'} = \delta$. Now we consider

$$y : \mathbb{R} \to X, \ y(t) = \begin{cases} \int\limits_{-\infty}^{t} \varphi_{[t_0,t_0+\delta]}(\tau)\dfrac{d\tau}{\|\Phi(\tau,t_0)x\|_\tau}\Phi(t,t_0)x, \ t \geq t_0 \\ 0, \ t < t_0. \end{cases}$$

As $y \in \mathcal{L}^\infty(X)$ and $y(t) = \Phi(t,s)y(s) + \int\limits_s^t \Phi(t,\tau)f(\tau)d\tau$, for all $t \geq s$ it results that $y(t) = x_f(t)$, for all $t \in \mathbb{R}$. By Theorem 3.1 it follows that there is $K>0$ such that

$$\|y(t)\|_t \leq K\delta \ \text{ a.e.}$$

If $t \geq t_0 + \delta$ then

$$y(t) = \int_{t_0}^{t_0+\delta} \frac{d\tau}{\|\Phi(\tau, t_0)x\|_\tau} \Phi(t, t_0)x,$$

thus

$$\frac{1}{\delta} \int_{t_0}^{t_0+\delta} \frac{d\tau}{\|\Phi(\tau, t_0)x\|_\tau} \|\Phi(t, t_0)x\|_t \leq K \quad \text{a.e.} \quad t \geq t_0 + \delta.$$

From the above relation for $\delta \to 0$ it results that

$$\|\Phi(t, t_0)x\|_t \leq K\|x\|_{t_0}, \text{ for all } t \geq t_0 \text{ and } x \in X_1(t_0) \text{ with } \Phi(t, t_0)x \neq 0. \tag{3.1}$$

Let $x \in X$ with the property that there is $t_1 > t_0$ such that $\Phi(t_1, t_0)x = 0$. Then

$$\Phi(t, t_0)x = \Phi(t, t_1)\Phi(t_1, t_0)x = 0, \text{ for all } t \geq t_1.$$

We denote by $\sigma = \inf\{t \in \mathbb{R} : t \geq t_0, \Phi(t, t_0)x = 0\}$. It follows that $\Phi(\sigma, t_0)x = = 0$, hence $\Phi(t, t_0)x \neq 0$, for all $t \in [t_0, \sigma)$. On account of (3.1) we obtain that

$$\|\Phi(t, t_0)x\|_t \leq K\|x\|_{t_0}, \text{ for all } t \in [t_0, \sigma).$$

Thus

$$\|\Phi(t, t_0)x\|_t \leq K\|x\|_{t_0}, \text{ for all } t \geq t_0 \text{ and } x \in X_1(t_0). \tag{3.2}$$

We consider $x \in X_2(t_0),\ x \neq 0,\ t_0 \in \mathbb{R},\ \delta > 0$ and

$$g : \mathbb{R} \to X,\ g(t) = \varphi_{[t_0, t_0+\delta]}(t) \frac{\Phi(t, t_0)x}{\|\Phi(t, t_0)x\|_t}.$$

We can easily notice that $g \in \mathcal{L}^1(X)$ and $\|g\|_1^{'} = \delta$. Let

$$z : \mathbb{R} \to X,\ z(t) = \begin{cases} -\int_t^\infty \varphi_{[t_0, t_0+\delta]}(\tau) \frac{d\tau}{\|\Phi(\tau, t_0)x\|_\tau} \Phi(t, t_0)x, & t \geq t_0 \\ -\int_{t_0}^{t_0+\delta} \frac{d\tau}{\|\Phi(\tau, t_0)x\|_\tau} \varphi_x(t), & t < t_0. \end{cases}$$

As $z \in \mathcal{L}^\infty(X)$ and $z(t) = \Phi(t, s)z(s) + \int_s^t \Phi(t, \tau)g(\tau)d\tau$, for all $t \geq s$ it follows that $z(t) = x_g(t)$, for all $t \in \mathbb{R}$. By Theorem 3.1 it follows that there is $K > 0$ such that

$$\|z(t)\|_t \leq K\delta \quad \text{a.e.}$$

If $t \leq t_0$ then

$$\frac{1}{\delta} \int_{t_0}^{t_0+\delta} \frac{d\tau}{\|\Phi(\tau, t_0)x\|_\tau} \|\varphi_x(t)\|_t \leq K \quad \text{a.e. } t \leq t_0.$$

For $\delta \to 0$, from the above relation we obtain that

$$\|\varphi_x(t)\|_t \leq K\|x\|_{t_0} \quad \text{a.e.} \quad t \leq t_0.$$

But $\varphi_x(t) = \Phi(t,s)\varphi_x(s)$, for all $t_0 \geq t \geq s$. For $t = t_0$ it follows that

$$\varphi_x(t_0) = x = \Phi(t_0,t)\varphi_x(t), \text{ for all } t_0 \geq t.$$

We have that

$$\|\varphi_x(t)\|_t \leq K\|\Phi(t_0,t)\varphi_x(t)\|_{t_0} \quad \text{a.e.} \quad t \leq t_0.$$

We obtain that

$$\|\varphi_x(t_0)\|_{t_0} \leq K\|\Phi(t,t_0)\varphi_x(t_0)\|_t \quad \text{a.e.} \quad t \geq t_0,$$

thus

$$\|x\|_{t_0} \leq K\|\Phi(t,t_0)x\|_t \quad \text{a.e.} \quad t \geq t_0.$$

So

$$\|\Phi(t,t_0)x\|_t \geq \frac{1}{K}\|x\|_{t_0}, \text{ for all } t \geq t_0 \text{ and } x \in X_2(t_0).$$

(i) On account to (3.2) we get that $X_1(t_0)$ is a closed subspace, for all $t_0 \in \mathbb{R}$.
Further we will prove that $X_2(t_0)$ is a closed subspace, for all $t_0 \in \mathbb{R}$.
Let $x \in \overline{X_2(t_0)}$, which means that there is $x_n \in X_2(t_0)$ such that $x_n \to x$.
We consider $f_n : \mathbb{R} \to X,\ f_n(t) = -\varphi_{[t_0,t_0+1]}(t)\Phi(t,t_0)x_n$, for all $n \in \mathbb{N}$. It results that $f_n \in \mathcal{L}^1(X)$ and $\|f_n\|_1^{'} \leq e^{\omega}\|x_n\|_{t_0}$. We set now

$$y_n : \mathbb{R} \to X,\ y_n(t) = \begin{cases} \int\limits_t^{\infty} \varphi_{[t_0,t_0+1]}(\tau)d\tau\Phi(t,t_0)x_n, & t \geq t_0 \\ \varphi_{x_n}(t), & t < t_0, \end{cases} \qquad \text{for all } n \in \mathbb{N}.$$

As $y_n \in \mathcal{L}^{\infty}(X)$ and $y_n(t) = \Phi(t,s)y_n(s) + \int\limits_s^t \Phi(t,\tau)f_n(\tau)d\tau$, for all $t \geq s$ it results that $y_n(t) = x_{f_n}(t)$, for all $t \in \mathbb{R}$ and $n \in \mathbb{N}$. If we apply Theorem 3.1 we get that

$$\|y_n\|_{\infty}^{'} \leq K\|f_n\|_1{}' \leq Ke^{\omega}\|x_n\|_{t_0},$$

which implies that

$$\|y_n - y_m\|_{\infty}^{'} \leq Ke^{\omega}\|x_n - x_m\|_{t_0},$$

thus

$$\|y_n(t) - y_m(t)\|_t \leq K\|x_n - x_m\|_{t_0} \xrightarrow[n,m\to\infty]{} 0 \quad \text{a.e.}$$

We obtain that $\lim\limits_{n\to\infty} y_n(t) = y(t)$ a.e. But $y_n(t_0) = \varphi_{x_n}(t_0) = x_n \xrightarrow[n\to\infty]{} y(t_0)$, so $y(t_0) = x$.
We have that

$$y_n(t) = \Phi(t,s)y_n(s), \text{ for all } s \leq t \leq t_0$$

and

$$y(t) = \Phi(t,s)y(s), \text{ for all } s \leq t \leq t_0.$$

It follows that $x \in X_2(t_0)$, thus $X_2(t_0)$ is a closed subspace, for all $t_0 \in \mathbb{R}$.

(ii) To prove that $X = X_1(t_0) \oplus X_2(t_0)$, for all $t_0 \in \mathbb{R}$, we consider $x \in X$, $t_0 \in \mathbb{R}$ and

$$f : \mathbb{R} \to X, \; f(t) = -\varphi_{[t_0,t_0+1]}(t)\Phi(t,t_0)x.$$

It results that $f \in \mathcal{L}^1(X)$. From the hypothesis we obtain that there is an unique function $u \in \mathcal{L}^\infty(X)$ such that

$$u(t) = \Phi(t,s)u(s) + \int_s^t \Phi(t,\tau)f(\tau)d\tau, \text{ for all } t \geq s.$$

If $t \geq t_0 + 1$ then $u(t) = \Phi(t,t_0)(u(t_0) - x)$. It results that $u(t_0) - x \in X_1(t_0)$.

We notice that $u(t) = \Phi(t,s)u(s)$, for all $s \leq t \leq t_0$. Because $u \in \mathcal{L}^\infty(X)$ it follows that $u(t_0) \in X_2(t_0)$. Therefore

$$X = X_1(t_0) + X_2(t_0), \text{ for all } t_0 \in \mathbb{R}.$$

Let now $z \in X_1(t_0) \cap X_2(t_0)$ and $v : \mathbb{R} \to X, \; v(t) = \begin{cases} \Phi(t,t_0)z, \; t \geq t_0 \\ \varphi_z(t), \; t < t_0. \end{cases}$

Since $v \in \mathcal{L}^\infty(X)$ and $v(t) = \Phi(t,s)v(s)$, for all $t \geq s$ it results that $v = 0$ in $\mathcal{L}^\infty(X)$ and being continuous on the right in t_0 it follows that $v(t_0) = 0$, thus $z = 0$. Hence

$$X = X_1(t_0) \oplus X_2(t_0), \text{ for all } t_0 \in \mathbb{R}.$$

(iv) Further we will show that $\Phi(t_1,t_0)X_2(t_0) = X_2(t_1)$, for all $t_1 > t_0$.

Set now $x \in X_2(t_0), \; t_1 > t_0$ and

$$u : \mathbb{R} \to X, \; u(t) = \begin{cases} \Phi(t,t_0)x, \; t_0 \leq t \leq t_1 \\ \varphi_x(t), \; t < t_0 \\ 0, \; t > t_1. \end{cases}$$

It results that $u \in \mathcal{L}^\infty(X)$ and we can easily prove that $u(t) = \Phi(t,s)u(s)$, for all $t \geq s$.

If we make the substitutions $t = t_1$ and $s = t_0$ in the above relation we get that

$$u(t_1) = \Phi(t_1,t_0)u(t_0) = \Phi(t_1,t_0)x \in X_2(t_1),$$

hence

$$\Phi(t_1,t_0)X_2(t_0) \subset X_2(t_1), \text{ for all } t_1 > t_0. \tag{3.3}$$

Let $y \in X_2(t_1)$. Then there is $\varphi_y \in \mathcal{L}^\infty(X)$ such that $\varphi_y(t) = \Phi(t,s)\varphi_y(s)$, for all $t_1 \geq t \geq s$ and $\varphi_y(t_1) = y$. Since $y = \varphi_y(t_1) = \Phi(t_1,t_0)\varphi_y(t_0)$ it results that

$$X_2(t_1) \subset \Phi(t_1,t_0)X_2(t_0), \text{ for all } t_1 > t_0. \tag{3.4}$$

By the relations (3.3) and (3.4) we obtain that

$$\Phi(t_1,t_0)X_2(t_0)=X_2(t_1), \text{ for all } t_1>t_0. \tag{3.5}$$

(v) To prove that $\Phi(t_1,t_0):X_2(t_0)\to X_2(t_1)$ is invertible, for all $t_1>t_0$, we consider $z\in X_2(t_0)$ such that $\Phi(t_1,t_0)z=0$. It results that

$$\|\Phi(t_1,t_0)z\|_{t_1}=0\geq N_2\|z\|_{t_0},$$

so $\|z\|_{t_0}=0$, thus $z=0$. As $z=0$ and by the relation (3.5) we get that

$$\Phi(t_1,t_0):X_2(t_0)\to X_2(t_1) \text{ is invertible, for all } t_1>t_0.$$

(vi) We must show now that

$$t_0\to\|P_i(t_0)\|_{t_0}:\mathbb{R}\to\mathbb{R}_+ \text{ is bounded, } i=1,2,$$

where $\{P_i(t_0)\}_{t_0\in\mathbb{R}}$ is the family of projectors associated to the decomposition $X=X_1(t_0)\oplus X_2(t_0)$, for all $t_0\in\mathbb{R}$.

Let $t_0\in\mathbb{R},\ \delta>0,\ x_1\in X_1(t_0)$ with $\|x_1\|_{t_0}=1$, $x_2\in X_2(t_0)$ with $\|x_2\|_{t_0}=1$ and the functions

$$f_1:\mathbb{R}\to X,\ f_1(t)=\begin{cases}\Phi(t,t_0)x_1,\ t\geq t_0\\ 0,\ t<t_0,\end{cases}$$

$$f_2:\mathbb{R}\to X,\ f_2(t)=\begin{cases}\Phi(t,t_0)x_2,\ t\geq t_0\\ \varphi_{x_2}(t),\ t<t_0.\end{cases}$$

We notice that $f_1(t)\in X_1(t)$ and $f_2(t)\in X_2(t)$, for all $t\geq t_0$.

We consider $g:\mathbb{R}\to X,\ g(t)=f_1(t)+f_2(t)$.

Let $h:\mathbb{R}\to X,\ h(t)=\varphi_{[t_0,t_0+\delta]}(t)\dfrac{g(t)}{\|g(t)\|_t}$. We notice that $h\in\mathcal{L}^1(X)$ and $\|h\|_1^{'}=\delta$.

Now we consider $y:\mathbb{R}\to X$,

$$y(t)=\int_{-\infty}^{t}\varphi_{[t_0,t_0+\delta]}(\tau)\frac{d\tau}{\|g(\tau)\|_\tau}f_1(t)-\int_{t}^{\infty}\varphi_{[t_0,t_0+\delta]}(\tau)\frac{d\tau}{\|g(\tau)\|_\tau}f_2(t)$$

$$=\begin{cases}\int\limits_{-\infty}^{t}\varphi_{[t_0,t_0+\delta]}(\tau)\dfrac{d\tau}{\|g(\tau)\|_\tau}\Phi(t,t_0)x_1-\int\limits_{t}^{\infty}\varphi_{[t_0,t_0+\delta]}(\tau)\dfrac{d\tau}{\|g(\tau)\|_\tau}\Phi(t,t_0)x_2,\ t\geq t_0\\ -\int\limits_{t}^{\infty}\varphi_{[t_0,t_0+\delta]}(\tau)\dfrac{d\tau}{\|g(\tau)\|_\tau}\varphi_{x_2}(t),\ t<t_0.\end{cases}$$

We notice that $y\in\mathcal{L}^\infty(X)$ and $y(t)=\Phi(t,s)y(s)+\int\limits_s^t\Phi(t,\tau)h(\tau)d\tau$, for all $t\geq s$ we have that $y(t)=x_h(t)$, for all $t\in\mathbb{R}$. By Theorem 3.1 it results that

$$\|y(t)\|_t\leq K\delta.$$

Hence

$$\frac{1}{\delta}\int_{t_0}^{t_0+\delta}\frac{d\tau}{\|g(\tau)\|_\tau}\le K.$$

If $\delta\to 0$, from the above relation we get that $\|g(t_0)\|_{t_0}\ge\dfrac{1}{K}$, for all $t_0\in\mathbb{R}$, which means that

$$\|P_i(t_0)\|_{t_0}\le 2K,\ \text{for all } t_0\in\mathbb{R},\ i=1,2.$$

□

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