Unique range sets of 5 points for unbounded analytic functions inside an open disk

by

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Abstract

Let $\mathbb{K}$ be a complete algebraically closed p-adic field of characteristic $p \geq 0$ and let $A_u(d(a,R^-))$ be the set of unbounded analytic functions inside the disk $d(a,R^-) = \{ x \in \mathbb{K} \mid |x - a| < R \}$. We recall the definition of urscm and the ultrametric Nevanlinna Theory on 3 small functions in order to find new urscm for $A_u(d(a,R^-))$. Results depend on the characteristic. In characteristic 0, we can find urscm of 5 points. Some results on bi-urscm are given for meromorphic functions.

Key Words: p-adic analytic functions, URSCM, Nevanlinna, ultrametric, unicity, distribution of values.

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1 Introduction and main result

We shall introduce URSCM for p-adic meromorphic functions. Many studies were made in the eighties and the nineties concerning URSCM for functions in $\mathbb{C}$, [3], [6], [16]. Studies were also made in the non-archimedean context by the late nineties and next [1], [2], [3], [4], [5], [8], [9], [10], [11], [13]. Here, we will only consider the situation in an ultrametric field.

Definitions and notation: Throughout the paper, $E$ is an algebraically closed field of characteristic $p \geq 0$ without any assumption on the existence of an absolute value. A subset $S$ of $E$ is said to be affinely rigid if there is no similarity $t$ on $E$ other than the identity, such that $t(S) = S$.

We denote by $\mathbb{K}$ an algebraically closed field complete with respect to an ultrametric absolute value $| \cdot |$ and of characteristic $p \geq 0$. We will denote by $q$ the characteristic exponent of $\mathbb{K}$: if $p \neq 0$, then $q = p$ and if $p = 0$ then $q = 1$.

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Given \( \alpha \in \mathbb{K} \) and \( R \in \mathbb{R}^*_+ \), we denote by \( d(\alpha, R) \) the disk \( \{ x \in \mathbb{K} \mid |x - \alpha| \leq R \} \), by \( d(\alpha, R^-) \) the disk \( \{ x \in \mathbb{K} \mid |x - \alpha| < R \} \), by \( \mathcal{A}(\mathbb{K}) \) the \( \mathbb{K} \)-algebra of analytic functions in \( \mathbb{K} \) (i.e. the set of power series with an infinite radius of convergence) and by \( \mathcal{M}(\mathbb{K}) \) the field of meromorphic functions in \( \mathbb{K} \) (i.e. the field of fractions of \( \mathcal{A}(\mathbb{K}) \)).

In the same way, given \( \alpha \in \mathbb{K} \), \( R > 0 \) we denote by \( \mathcal{A}(d(\alpha, R^-)) \) the \( \mathbb{K} \)-algebra of analytic functions in \( d(\alpha, R^-) \) (i.e. the set of power series with a radius of convergence \( \geq R \)) and by \( \mathcal{M}(d(\alpha, R^-)) \) the field of fractions of \( \mathcal{A}(d(\alpha, R^-)) \). We then denote by \( \mathcal{A}_b(d(\alpha, R^-)) \) the \( \mathbb{K} \)-algebra of bounded analytic functions in \( d(\alpha, R^-) \) and by \( \mathcal{M}_b(d(\alpha, R^-)) \) the field of fractions of \( \mathcal{A}_b(d(\alpha, R^-)) \). And we set \( \mathcal{A}_u(d(\alpha, R^-)) = \mathcal{A}(d(\alpha, R^-)) \setminus \mathcal{A}_b(d(\alpha, R^-)) \) and \( \mathcal{M}_u(d(\alpha, R^-)) = \mathcal{M}(d(\alpha, R^-)) \setminus \mathcal{M}_b(d(\alpha, R^-)) \).

Given a family of functions \( \mathcal{F} \) defined in \( \mathbb{K} \) or in a subset \( S \) of \( \mathbb{K} \) (resp. in \( E \) or in a subset \( S \) of \( E \)), with values in \( \mathbb{K} \) (resp. in \( E \), \( S \) is called an **ursim** for \( \mathcal{F} \) if for any two non-constant functions \( f, g \in \mathcal{F} \) satisfying \( f^{-1}(S) = g^{-1}(S) \), these functions are equal.

That definition particularly applies to \( \mathcal{A}(\mathbb{K}) \), \( \mathcal{M}(\mathbb{K}) \), \( \mathcal{A}_u(d(\alpha, R^-)) \), \( \mathcal{M}_u(d(\alpha, R^-)) \), \( \mathbb{K}[x] \), \( \mathbb{K}(x) \), \( E[x] \), \( E(x) \).

We will now recall the definition of URSCM. Given a subset \( S \) of \( E \) and \( f \in E(x) \), we denote by \( \mathcal{E}(f, S) \) the set in \( E \times \mathbb{N}^* \):

\[
\bigcup_{a \in S} \{ (z, q) \in E \times \mathbb{N}^* \mid z \text{ is a zero of order } q \text{ of } f(x) - a \}.
\]

Similarly, consider now meromorphic functions in the field \( \mathbb{K} \). For a subset \( S \) of \( \mathbb{K} \) and \( f \in \mathcal{M}(\mathbb{K}) \) (resp. \( f \in \mathcal{M}(d(\alpha, R^-)) \)) we denote by \( \mathcal{E}_M(f, S) \) the set in \( \mathbb{K} \times \mathbb{N}^* \):

\[
\bigcup_{a \in S} \{ (z, q) \in \mathbb{K} \times \mathbb{N}^* \mid z \text{ is a zero of order } q \text{ of } f(x) - a \}.
\]

Let \( \mathcal{F} \) be a non-empty subset of \( \mathcal{A}(\mathbb{K}) \) (resp. of \( \mathcal{M}(\mathbb{K}) \), resp. of \( \mathcal{A}(d(\alpha, R^-)) \), resp. of \( \mathcal{M}(d(\alpha, R^-)) \)). We say that two non-constant functions \( f, g \in \mathcal{F} \) share \( S \), counting multiplicity if \( \mathcal{E}(f, S) = \mathcal{E}(g, S) \); and the set \( S \) is called a **unique range set counting multiplicity** (an URSCM in brief) for \( \mathcal{F} \) if for any two non-constant \( f, g \in \mathcal{F} \) sharing \( S \) counting multiplicity, one has \( f = g \). Next, the set \( S \) will be called a bi-URSCM for \( \mathcal{F} \) if for two non-constant functions \( f, g \in \mathcal{M}_u(d(\alpha, R^-)) \) sharing \( S \) counting multiplicity and having the same poles, counting multiplicity, one has \( f = g \) [8].

Particularly, if we consider a family \( \mathcal{F} \subset \mathcal{A}(\mathbb{K}) \) or \( \mathcal{F} \subset \mathcal{A}_u(d(\alpha, R^-)) \) and a set \( S = \{ a_1, ..., a_t \} \subset \mathbb{K} \) (resp. a set \( S = \{ a_1, ..., a_t \} \subset E \) with \( a_i \neq a_j \) \( \forall i \neq j \)), we can set \( P(X) = \prod_{j=1}^{t} (X - a_j) \) and then the set \( S = \{ a_1, ..., a_t \} \) is an URSCM for \( \mathcal{F} \) if for any two functions \( f, g \in \mathcal{F} \) such that \( P \circ f \) and \( P \circ g \) have the same zeros with the same multiplicity, then \( f = g \).

Similarly, if we consider a family \( \mathcal{F} \subset \mathcal{M}(\mathbb{K}) \) or \( \mathcal{F} \subset \mathcal{M}_u(d(\alpha, R^-)) \) and a set \( S = \{ a_1, ..., a_t \} \subset \mathbb{K} \) (resp. a set \( S = \{ a_1, ..., a_t \} \subset E \) with \( a_i \neq a_j \) \( \forall i \neq j \)), we can set \( P(X) = \prod_{j=1}^{t} (X - a_j) \) and then the set \( S = \{ a_1, ..., a_t \} \) is a bi-URSCM for \( \mathcal{F} \) if for any two
functions \( f, g \in \mathcal{F} \) having the same poles (counting multiplicity) such that \( P \circ f \) and \( P \circ g \) have the same zeros with the same multiplicity, then \( f = g \).

**Remark:** An URSCM \( S \) for a family of functions \( \mathcal{F} = \mathcal{M}(\mathbb{K}), \mathcal{A}(\mathbb{K}), \mathcal{M}_u(d(a, R^-)), \mathcal{A}_u(d(a, R^-)) \) must obviously be affinely rigid. Indeed suppose that \( S \) is not affinely rigid and let \( t \) be a similarity of \( \mathbb{K} \) such that \( t(S) = S \). Then, if \( f \) belongs to \( \mathcal{F} \), so does \( f \circ t \) and therefore we can check that \( E(f, S) = E(f \circ t, S) \). And it is a bi-URSCM if for any two functions \( f, g \in \mathcal{F} \) such that \( P \circ f \) and \( P \circ g \) have the same zeros and the same poles, counting multiplicity, then \( f = g \).

Similar definitions were given for meromorphic functions on \( \mathbb{C} \) before these questions were examined on the field \( \mathbb{K} \). URSCM of only 11 points for complex meromorphic functions in the whole field \( \mathbb{C} \) was found in [16] and the same method showed the existence of URSCM of only 7 points for complex entire functions. So far, they are the smallest known in \( \mathbb{C} \).

In the field \( \mathbb{K} \), the same method lets us find URSCM of 11 points for \( \mathcal{M}_u(d(a, R^-)) \) and URSCM of 10 points for \( \mathcal{A}_u(d(a, R^-)) \). In 1996, URSCM for polynomials on a field such as \( \mathbb{E} \) were characterized: they are just the affinely rigid subsets of \( \mathbb{E} \) [9]. Particularly, the smallest URSCM for polynomials are the affinely rigid sets of 3 points. Concerning entire functions on the field \( \mathbb{K} \), URSCM of 3 points were found: they also are the affinely rigid sets of 3 points [9] and \( n \) points [19]. Next, URSCM of 7 points were found for unbounded analytic functions in a disk \( d(a, R^-) \) [10]. Here we will show the existence of another family of URSCM for \( \mathcal{A}_u(d(a, R^-)) \), looking for sets of less than 7 points.

The notion of URSCM is closely linked to that of strong uniqueness polynomial.

**Definition:** A polynomial \( P \in \mathbb{K}[x] \) is called a strong uniqueness polynomial for a subset \( \mathcal{F} \subset E(x) \) (resp. \( \mathcal{F} \subset \mathcal{M}(\mathbb{K}), \mathcal{F} \subset \mathcal{M}(d(a, R^-)) \)) if, given \( f, g \in \mathcal{F} \), the equality \( P(f) = P(g) \) implies \( f = g \).

The following basic result is immediate and useful to understand the role of URSCM:

**Proposition A:** Let \( S = \{a_1, \ldots, a_n\} \subset E \), (resp. \( S = \{a_1, \ldots, a_n\} \subset \mathbb{K} \)), let \( a \in \mathbb{K} \), let \( R \in \mathbb{R}^+ \) and let \( P(x) = \prod_{i=1}^{n}(x - a_i) \). Given any two functions \( f, g \in E[x] \) (resp. \( f, g \in \mathcal{A}(\mathbb{K}) \), resp. \( f, g \in \mathcal{A}(d(a, R^-)) \)) then \( E(f, S) = E(g, S) \) if and only if \( \frac{P(f)}{P(g)} \) is a constant in \( E^* \) (resp. is a constant in \( \mathbb{K}^* \), resp. is an invertible function in \( \mathcal{A}(d(a, R^-)) \)). Given any two functions \( f, g \in E(x) \) (resp. \( f, g \in \mathcal{M}(\mathbb{K}), \mathcal{M}(d(a, R^-)) \) having the same poles counting multiplicity, then \( E(f, S) = E(g, S) \) if and only if \( \frac{P(f)}{P(g)} \) is a constant in \( E^* \) (resp. is a constant in \( \mathbb{K}^* \), resp. is an invertible function in \( \mathcal{A}(d(a, R^-)) \)).
Corollary A1 Let \( S = \{a_1, ..., a_n\} \subseteq \mathbb{K} \) (resp. let \( S = \{a_1, ..., a_n\} \subseteq E \)) and let \( P(x) = \prod_{i=1}^{n} (x - a_i) \). Then \( P \) is a polynomial of strong uniqueness for \( \mathcal{A}(\mathbb{K}) \) (resp. for \( E[x] \)) if and only if \( S = \{a_1, ..., a_n\} \) is an URSCM for \( \mathcal{A}(\mathbb{K}) \) (resp. for \( E[x] \)).

Remark: Let \( \in A \) be a primitive \( p \)-th root of \( 1 \). Clearly, \( P(jf) = jP(f) \forall f \in \mathcal{M}(\mathbb{K}) \), hence \( P \) is not a polynomial of strong uniqueness for \( \mathcal{A}(\mathbb{K}) \) or for \( E[x] \).

As usual, if \( p \neq 0 \), given \( a \in \mathbb{K} \) and \( n \in \mathbb{N} \), we denote by \( r^n \sqrt[p]{a} \) the unique \( b \in \mathbb{K} \) such that \( b^p = a \).

Given \( m, n \in \mathbb{N} \) we set \( m < n \) if \( m \) divides \( n \) and \( m \not\sim n \) if \( m \) does not divide \( n \). When \( p \neq 0 \), we denote by \( S \) the \( \mathbb{F}_p \)-automorphism of \( \mathbb{K} \) defined by \( S(x) = \sqrt[p]{x} \).

More generally this mapping has continuation to a \( \mathbb{K} \)-algebra automorphism of \( \mathbb{K}[X] \) as \( S(c)\prod_{j=1}^{n} (X - a_j) = S(c)\prod_{j=1}^{n} (X - S(a_j)), \ c \in \mathbb{K} \).

Proposition B: Suppose \( p \neq 0 \). Let \( r > 0 \) and let \( f \in \mathcal{M}(d(a, r^{-})) \). Then \( \sqrt[p]{f} \) belongs to \( \mathcal{M}(d(a, r^{-})) \) if and only if \( f' = 0 \). Moreover, there exists a unique \( t \in \mathbb{N} \) such that \( \sqrt[p]{f} \in \mathcal{M}(d(a, r^{-})) \) and \( (\sqrt[p]{f})' \neq 0 \).

Proof: If \( f \) is of the form \( l^p \) with \( l \in \mathcal{M}(d(a, r^{-})) \), then of course we have \( f' = 0 \). Now, suppose that \( f' = 0 \). If \( f \in \mathcal{A}(d(a, r^{-})) \), then obviously all non-zero coefficients have an index multiple of \( p \), hence \( f \) is of the form \( l^p \), with \( l \in \mathcal{A}(d(a, r^{-})) \). We now consider the general case when \( f \in \mathcal{M}(d(a, r^{-})) \). Let \( (b_n, t_n)_{n \in \mathbb{N}} \) be the sequence of poles of \( f \) inside \( d(a, r^{-}) \) where \( t_n \) is the multiplicity order of \( b_n \). By Theorem 25.5 [14] we can find \( h \in \mathcal{A}(d(a, r^{-})) \) such that \( \omega_{b_n}(h) \geq t_n \forall n \in \mathbb{N} \). Clearly \( fh^p \) belongs to \( \mathcal{A}(d(a, r^{-})) \) and satisfies \( (fh^p)' = 0 \). Consequently, \( fh^p \) is of the form \( g^p \), with \( g \in \mathcal{A}(d(a, r^{-})) \), therefore \( f = \left( \frac{g}{h} \right)^p \). On the other hand, the set of integers \( s \) such that \( \sqrt[p]{f} \) belongs to \( \mathcal{M}(d(a, r^{-})) \) is obviously bounded and therefore admits a biggest element, which ends the proof.

Definition and notation: Suppose \( p \neq 0 \). Given, \( f \in \mathcal{M}(d(a, r^{-})) \), we will call ramification index of \( f \) the integer \( t \) such that \( \sqrt[p]{f} \in \mathcal{M}(d(a, r^{-})) \) and \( (\sqrt[p]{f})' \neq 0 \).

In the same way, given an algebraically closed field \( B \) of characteristic \( p \neq 0 \) and \( P(x) \in B[x] \), we call ramification index of \( P \) the unique integer \( t \) such that \( \sqrt[p]{P} \in B[x] \) and \( (\sqrt[p]{P})' \neq 0 \). This ramification index will be denoted by \( \text{ram}(f) \) for any \( f \in \mathcal{M}(d(a, r^{-})) \) or \( f \in \mathcal{M}(\mathbb{K}) \) and similarly it will be denoted by \( \text{ram}(P) \) for any \( P \in B[x] \).

Henceforth, given \( t \in \mathbb{N}^* \), we will denote by \( \mathcal{A}_{t}(d(a, R^{-})) \) the subset of the functions \( f \in \mathcal{A}(d(a, R^{-})) \) having a ramification index \( \leq t \) and similarly, we put \( \mathcal{A}_{t}(d(R^{-})) = \mathcal{A}_{t}(d(a, R^{-})) \cap \mathcal{A}_{a}(d(a, R^{-})) \).

Given \( k \in \mathbb{K}^* \) and \( n, m \in \mathbb{N}^* \) with \( m < n \), we set \( Q_{n,m,k}(x) = x^n - x^m + k \) and we denote by \( Y_{n,m,k} \) the set of zeros of \( Q_{n,m,k} \). In the same way, we set \( Q_{n,k}(x) = x^n - x^{n-1} + k \) and we denote by \( Y_{n,k} \) the set of zeros of \( Q_{n,k} \).
Remark: Suppose \( p \neq 0 \) and let \( f \in \mathcal{M}(d(a,r^−)) \) have ramification index \( t \) as an element of \( \mathcal{M}(d(a,r^−)) \). For every \( r' \in ]0,r[ \), \( f \) has the same ramification index as an element of \( \mathcal{M}(d(a,r'^−)) \) because of course, on one hand, \( \sqrt{f} \in \mathcal{M}(d(a,r'^−)) \) and on the other hand, by properties of analytic functions, \( (\sqrt{f})' \) is not identically zero inside \( d(a,r') \).

As recalled above, in [9] the smallest urscm for \( \mathcal{A}_u(d(a,R^−)) \) have 7 points. By Corollary 2.2 we can find a new family of urscm for \( \mathcal{A}_u(d(a,R^−)) \), with particularly urscm of 5 points.

**Theorem 1:** Let \( t \in \mathbb{N}^+ \) and let \( f, g \in \mathcal{M}_u(d(a,R^−)) \) be such that the function \( \phi = \frac{f^n - f^m + k}{g^n - g^m + k} \) is invertible in \( \mathcal{A}(d(a,R^−)) \). Let \( t \) be the ramification index of \( \frac{f^n - f^m - k(\phi - 1)}{f^n - f^m} \). If \( 2mq^t > n(2q^t - 1) + 3q^t \) then \( f = g \).

**Corollary 1.1:** Suppose \( K \) is of characteristic 0. If \( 2m > n+3 \) then \( Y(n,m,k) \) is a bi-urscm for \( \mathcal{M}_u(d(a,R^−)) \).

**Corollary 1.2:** Suppose \( K \) is of characteristic 0. If \( n \geq 6 \), then \( Y(n,k) \) is a bi-urscm for \( \mathcal{M}_u(d(a,R^−)) \).

**Theorem 2:** Let \( t \in \mathbb{N}^+ \) and let \( f, g \in \mathcal{A}_u(d(a,R^−)) \) be such that the function \( \phi = \frac{f^n - f^m + k}{g^n - g^m + k} \) is invertible in \( \mathcal{A}(d(a,R^−)) \). Let \( t \) be the ramification index of \( \frac{f^n - f^m - k(\phi - 1)}{f^n - f^m} \). If \( 2mq^t > n(2q^t - 1) + 2q^t \) then \( f = g \).

**Corollary 2.1:** Suppose \( K \) is of characteristic 0. If \( 2m \geq n+3 \), then \( Y(n,m,k) \), is an urscm for \( \mathcal{A}_u(d(a,R^−)) \).

**Corollary 2.2:** Suppose \( K \) is of characteristic 0. If \( n \geq 5 \), then \( Y(n,k) \), is an urscm for \( \mathcal{A}_u(d(a,R^−)) \).

Remark: We don’t know whether there exists an urscm for \( \mathcal{A}_u(d(a,R^−)) \) of 4 points or 3 points.

2 The Proof

We must recall the definition of the counting functions in the Nevanlinna Theory.

**Definitions and notation:** Let \( f \in \mathcal{M}(d(a,R^−)) \) and let \( \alpha \in d(a,R^−) \). If \( f \) admits \( \alpha \) as a zero of order \( q \), we set \( \omega_\alpha(f) = q; \) if \( f \) admits \( \alpha \) as a pole of order \( q \), we set \( \omega_\alpha(f) = -q; \) and if \( \alpha \) is neither a zero nor a pole for \( f \), we set \( \omega_\alpha(f) = 0 \).

We denote by \( Z(r,f) \) the counting function of zeros of \( f \) in \( d(0,r) \) in the following way:

Let \( (a_n), 1 \leq n \leq \sigma(r) \) be the finite sequence of zeros of \( f \) such that \( 0 < |a_n| \leq |a_{n+1}| \leq |a_{\sigma(r)}| \leq r \), of respective order \( s_n \).

We set \( Z(r,f) = \max(\omega_0(f),0) \log r + \sum_{n=1}^{\sigma(r)} s_n(\log r - \log |a_n|) \).
Similarly, we set \( N(r, f) = Z(r, \frac{1}{f}) \).

In order to define the counting function of zeros of \( f \) without multiplicity, we put \( \omega_0(f) = 0 \) if \( \omega_0(f) \leq 0 \) and \( \omega_0(f) = 1 \) if \( \omega_0(f) \geq 1 \).

In the sequel, \( I \) will denote an interval of the form \( [\rho, +\infty] \), with \( \rho > 0 \), and \( J \) will denote an interval of the form \( [\rho, R] \).

Next, denoting by \( E(r, f) \) the set \( \{ a \in d(0, r) \mid \omega_a(f) > 0, \ p^{\text{ram}(f)+1} \neq \omega_a(f) \} \), if \( 0 \not\in E(r, f) \) we set \( \tilde{Z}(r, f) = \sum_{\alpha \in E(r, f)} \log\frac{r}{|\alpha|} \)

and if \( 0 \in E(r, f) \) we set \( \tilde{Z}(r, f) = \log r + \sum_{\alpha \in E(r, f), \alpha \neq 0} \log\frac{r}{|\alpha|} \).

Similarly we define \( \tilde{N}(r, f) = \tilde{Z}(r, \frac{1}{f}) \).

We can now define the Nevanlinna characteristic function of \( f \): \( T(r, f) = \max(Z(r, f), T(r, f)) \).

Assume that \( f' \) is not identically 0.

Let \( V(r, f) = \{ a \in d(0, r) \mid \omega_a(f) < 0, \ p^{\text{ram}(f)+1} < \omega_a(f) \} \). We put
\[
N_0(r, f') = \sum_{a \in V(r, f)} [\omega_a(f') - \omega_a(f)] \log\frac{r}{|\alpha|}.
\]

Given a finite subset \( S \) of \( \mathbb{K} \), we put \( \mathcal{N}'(r, f, S) = \{ a \in d(0, r) \mid f'(a) = 0, \ f(a) \not\in S \} \) and \( \mathcal{N}'(r, f, S) = \{ a \in d(0, r) \mid p^{\text{ram}(f)+1} \leq \omega_a(f - f(a)), \ f(a) \in S \} \). Then we can define
\[
Z_0^S(r, f') = \sum_{a \in \mathcal{N}'(r, f, S)} \omega_a(f') \log\frac{r}{|\alpha|} + \sum_{a \in \mathcal{N}'(r, f, S)} [\omega_a(f') - \omega_a(f - f(a))] \log\frac{r}{|\alpha|}.
\]

Remarks: 1) It is easily verified that all the above functions are positive.
2) If \( p = 0 \), we have \( \tilde{Z}(r, f) = \tilde{Z}(r, f) \) and \( \tilde{N}(r, f) = \tilde{N}(r, f) \).

Lemma 1: Let \( f \in \mathcal{M}(d(0, R^-)) \), let \( t = r(f) \) and let \( g = \sqrt{T} \). Then \( \tilde{Z}(r, f) = \tilde{Z}(r, g) \) and \( \tilde{N}(r, f) = \tilde{N}(r, g) \).

Proof: Let \( a \) be a zero of \( f \) and let \( s = \omega_a(f) \). Then \( s \) is of the form \( nt \) with \( n \in \mathbb{N}^* \). If \( n = 1 \), then \( a \) belongs to both \( E(r, f) \) and \( E(r, g) \); and if \( n > 1 \), then \( a \notin E(r, f) \). But then \( a \) is a zero of order \( n \) of \( g \) and hence, \( a \) does not belong to \( E(r, g) \).

The following Lemmas 2 and 3 are easily checked [12]:

Lemma 2: Let \( \alpha_1, \ldots, \alpha_n \in \mathbb{K} \) be pairwise distinct, let \( P(u) = \prod_{i=1}^{n} (u - \alpha_i) \) and let \( f \in \mathcal{M}(d(0, R^-)) \). Then \( Z(r, P(f)) = \sum_{i=1}^{n} Z(r, f - \alpha_i) \) and \( \tilde{Z}(r, P(f)) = \sum_{i=1}^{n} \tilde{Z}(r, f - \alpha_i) \).

Lemma 3: Let \( f \in \mathcal{M}(d(0, R^-)) \) be such that \( f' \) is not identically zero and let \( \alpha \in d(0, R^-) \). We have \( \omega_\alpha(f') = \omega_\alpha(f) - 1 \) if \( p \neq \omega_\alpha(f) \) and \( \omega_\alpha(f') \geq \omega_\alpha(f) \) if \( p = \omega_\alpha(f) \).
Lemmas 4 and 5 are consequences of Lemma 3.

**Lemma 4:** Let \( f \in \mathcal{M}(d(0, R^-)) \) be such that \( f' \neq 0 \) and let \( S \) be a finite subset of \( \mathbb{K} \). Then:
\[
\sum_{b \in S} \left( Z(r, f - b) - \tilde{Z}(r, f - b) \right) = Z(r, f') - Z_0^S(r, f').
\]

**Lemma 5:** Let \( f \in \mathcal{M}(d(0, R^-)) \) be such that \( f' \neq 0 \) and let \( 0 < r < R \). Then:
\[
N(r, f') = N(r, f) + \tilde{N}(r, f) - N_0(r, f').
\]

**Lemma 6:** Let \( f \in \mathcal{M}(d(0, R^-)) \) be such that \( f' \neq 0 \) and let \( 0 < r < R \). Then:
\[
Z(r, f') \leq Z(r, f) + \tilde{N}(r, f) - N_0(r, f') - \log r + O(1),(r \in J).
\]

**Proof:** Without loss of generality, up to change of variable, we can assume that both \( f \) and \( f' \) have no zero and no pole at 0. Let \( |f|(r) \) denote the circular value of \( f \) defined as \( |f|(r) = \lim_{|x| \to r, |x| \neq r} |f(x)| \).

By classical results such as Theorem 23.13 [14], we have \( Z(r, f) - N(r, f) = \log(|f|(r)) - \log(||f(0)||) \), and \( Z(r, f') - N(r, f') = \log(|f'(r)|) - \log(||f'(0)||) \). But, it is well-known that \( |f'(r)| \leq \frac{|f|(r)}{r} \) (Theorem 1.5.10 [15]); hence we obtain
\[
Z(r, f') \leq N(r, f') - N(r, f) + Z(r, f) - \log r + O(1).
\]

Moreover, by Lemma 4 we have \( N(r, f') - N(r, f) = \tilde{N}(r, f) - N_0(r, f') \), which completes the proof.

We know Proposition C [9].

**Proposition C :** Let \( f \in \mathcal{M}(d(0, R^-)) \). Then \( f \) belongs to \( \mathcal{M}_b(d(0, R^-)) \) if and only if \( T(r, f) \) is bounded when \( r \) tends to \( R \).

**Corollary C1:** Let \( f \in \mathcal{M}(d(0, R^-)) \). Then \( \mathcal{M}_b(d(0, R^-)) \) is a subset of \( \mathcal{M}_f(d(0, R^-)) \) and \( \mathcal{A}_b(d(0, R^-)) \) is a subset of \( \mathcal{A}_f(d(0, R^-)) \).

**Remark:** Particularly, an invertible function \( f \in \mathcal{A}(d(a, R^-)) \) has a constant absolute value and therefore lies in \( \mathcal{A}_b(d(a, R^-)) \).

The following Theorem D1 is known as Second Main Theorem on Three Small Functions [17]. It holds in \( p \)-adic analysis as well as in complex analysis, where it was shown first [17].
Notice that this theorem was generalized to any finite set of small functions by Yamanoy in complex analysis [18], through methods that have no equivalent on a p-adic field.

**Remark:** Let \( f \in \mathcal{M}_d(d(0, R^-)) \) and let \( w \in \mathcal{M}_b(d(0, R^-)) \). Then of course, \( w \in \mathcal{M}_f(d(0, R^-)) \).

The previous results enable us to prove the ultrametric Nevanlinna Main Theorem in a basic form:

**Theorem D1:** Let \( \alpha_1, ..., \alpha_n \in \mathbb{K} \), with \( n \geq 2 \), and let \( f \in \mathcal{M}(d(0, R^-)) \) (resp. \( f \in \mathcal{M}(\mathbb{K}) \)) of ramification index \( t \). Let \( S = \{ \sqrt[n]{\alpha_1}, ..., \sqrt[n]{\alpha_n} \} \). Then we have:
\[
\frac{(n-1)T(r, f)}{q^t} \leq \sum_{i=1}^{n} Z(r, f - \alpha_i) + Z(r, (\sqrt[n]{f})') - Z_0^S(r, (\sqrt[n]{f})') + O(1) \quad \forall r \in J
\]
(resp. \( \forall r \in I \)).

Moreover, if \( f \) belongs to \( \mathcal{A}(d(0, R^-)) \) (resp. \( f \in \mathcal{A}(\mathbb{K}) \)), then
\[
\frac{nT(r, f)}{q^t} \leq \sum_{i=1}^{n} Z(r, f - \alpha_i) + Z(r, (\sqrt[n]{f})') - Z_0^S(r, (\sqrt[n]{f})') + O(1) \quad \forall r \in J
\]
(resp. \( \forall r \in I \)).

Now, following the same method as in Theorem 2.5.9 [15], we can obtain that classical form of the Nevanlinna inequality where \( Z \) and \( N \) are replaced by \( \tilde{Z} \) and \( \tilde{N} \).

**Theorem D2:** Let \( \alpha_1, ..., \alpha_n \in \mathbb{K} \), with \( n \geq 2 \), and let \( f \in \mathcal{M}(d(0, R^-)) \) (resp. \( f \in \mathcal{M}(\mathbb{K}) \)) of ramification index \( t \). Let \( S = \{ \sqrt[n]{\alpha_1}, ..., \sqrt[n]{\alpha_n} \} \). Then we have:
\[
\frac{(n-1)T(r, f)}{q^t} \leq \sum_{i=1}^{n} \tilde{Z}(r, f - \alpha_i) + \tilde{N}(r, f) - Z_0^S(r, (\sqrt[n]{f})') - N_0(r, (\sqrt[n]{f})') - \log r + O(1) \quad \forall r \in J
\]
(resp. \( \forall r \in I \)).

**Proof of Theorems D1 and D2:** The proof of Theorems D1 and D2 was given in [12]. We will recall it. For convenience, we put \( g = \sqrt[n]{f} \) and \( \beta_i = \sqrt[n]{\alpha_i} \) for every \( i = 1, ..., n \). So \( S = \{ \beta_1, ..., \beta_n \} \).

Let \( f \in \mathcal{M}(K) \) (resp. \( f \in \mathcal{M}(d(a, R^-)) \)) and let \( (a_n, s_n)_{n \in \mathbb{N}} \) be the set of zeros of \( f \) in \( \mathbb{K} \) (resp. in \( d(a, R^-) \)) with \( |a_n| \leq |a_{n+1}| \) whereas \( s_n \) is the order of multiplicity of \( a_n \). We denote by \( D(f) \) the sequence \( (a_n, s_n)_{n \in \mathbb{N}} \).

By Theorem 25.5 [14] there exist \( \phi, \psi \in \mathcal{A}(d(0, R^-)) \) such that \( g = \frac{\phi}{\psi} \), and
\[
(1) \quad Z(r, \phi) \leq Z(r, g) + 1,
(2) \quad Z(r, \psi) \leq N(r, g) + 1.
\]

By Lemma 2.5.5 [15], there exists \( A \in \mathbb{R} \) and for any \( r \in J \) (resp. \( r \in I \)), there exists \( l(r) \in \{1, ..., n\} \) such that \( Z(r, \phi - \beta_j \psi) \geq \max(Z(r, \phi), Z(r, \psi)) + A \quad \forall j \neq l(r), \) therefore there exists \( B \in \mathbb{R} \) such that
\[
(3) \quad Z(r, \phi - \beta_i \psi) \geq T(r, g) + B \quad \forall i \neq l(r), \quad \forall r \in J \quad (\text{resp.} \quad \forall r \in I).
\]
We check that $\mathcal{D}(\phi) - \mathcal{D}(\frac{\phi}{\psi}) = \mathcal{D}(\psi) - \mathcal{D}(\frac{\psi}{\phi})$, therefore

\[ \mathcal{D}(\phi - \beta_i \psi) = \mathcal{D}(g - \beta_i) + \mathcal{D}(\psi) - \mathcal{D}(\frac{1}{g - \beta_i}). \]

Then, applying counting functions, we have $Z(r, \phi - \beta_i \psi) = Z(r, g - \beta_i) + Z(r, \psi) - N(r, g)$, and therefore, by (2), we obtain

(4) $Z(r, \phi - \beta_i \psi) \leq Z(r, g - \beta_i) + 1$.

Then, by (3) and (4) we obtain $(n - 1) \left( T(r, g) + B \right)$

\[ \leq \sum_{1 \leq i \leq n} Z(r, \phi - \beta_i \psi) \leq \sum_{1 \leq i \leq n} Z(r, g - \beta_i) + n - 1 \quad \forall r \in J \text{ (resp. } \forall r \in I). \]

Putting $M = (n - 1)(1 - B), we obtain:

(5) $(n - 1)T(r, g) \leq \sum_{i=1}^{n} Z(r, g - \beta_i) + M - Z(r, g - \beta_{l(r)}) \quad \forall r \in J \text{ (resp. } \forall r \in I).$

By Lemma 4, we have

\[ \sum_{i=1}^{n} Z(r, g - \beta_i) = \sum_{i=1}^{n} \tilde{Z}(r, g - \beta_i) + Z(r, g') - Z_0^S(r, g'), \]

hence by (5) we obtain,

(6) $(n - 1)T(r, g) \leq \sum_{i=1}^{n} \tilde{Z}(r, g - \beta_i) + Z(r, g') - Z_0^S(r, g') - Z(r, g - \beta_{l(r)}) + O(1) \quad \forall r \in J \text{ (resp. } \forall r \in I).$

Now, since $T(r, g) = \frac{T(r, f)}{q^t}$ and since $\tilde{Z}(r, g - \beta_i) = \tilde{Z}(r, f - \alpha_i) \quad \forall j = i, ..., n$, we obtain

\[ \frac{(n - 1)T(r, f)}{q^t} \leq \sum_{i=1}^{n} \tilde{Z}(r, f - \alpha_i) + Z(r, (\sqrt{T})') - Z_0^S(r, (\sqrt{T})') + O(1) \quad \forall r \in J \text{ (resp. } \forall r \in I). \]

Suppose now that $f$ belongs to $\mathcal{A}(d(a, R^-))$ or to $\mathcal{A}(IK)$. Then so does $g$. By Lemma 2.5.5 [15] we have $Z(r, g - \beta_{l(r)}) = T(r, g) + O(1) \quad \forall r \in J \text{ (resp. } \forall r \in I)$, so by (6) we obtain

\[ nT(r, g) \leq \sum_{i=1}^{n} \tilde{Z}(r, g - \beta_i) + Z(r, g') - Z_0^S(r, g') + O(1) \quad \forall r \in J \text{ (resp. } \forall r \in I), \]

and consequently,

\[ \frac{nT(r, f)}{q^t} \leq \sum_{i=1}^{n} \tilde{Z}(r, f - \alpha_i) + Z(r, (\sqrt{T})') - Z_0^S(r, (\sqrt{T})') + O(1) \quad \forall r \in J \text{ (resp. } \forall r \in I). \]

Now, returning to the general case, we have $g' = (g - \beta_{l(r)})'$ and $\tilde{N}(r, g) = \tilde{N}(r, g - \beta_{l(r)}).$

So, by Lemma 6, we have:

(7) $Z(r, g') - Z(r, g - \beta_{l(r)}) \leq \tilde{N}(r, g) - N_0(r, g') - \log r + O(1)$.

Finally, by (6), (7) we obtain

\[ \frac{(n - 1)T(r, f)}{q^t} \leq \sum_{i=1}^{n} \tilde{Z}(r, f - \alpha_i) + \tilde{N}(r, f) - Z_0^S(r, (\sqrt{T})') - N_0(r, (\sqrt{T})') - \log r \quad \forall r \in J \text{ (resp. } \forall r \in I). \]
That completes the proof.

**Theorem D3:** Let \( f \in \mathcal{M}(\mathbb{K}) \) (resp. \( f \in \mathcal{M}_u(d(0,R^-)) \)) and let \( u_1, u_2, u_3 \in \mathcal{M}_f(\mathbb{K}) \) (resp. \( u_1, u_2, u_3 \in \mathcal{M}_f(d(0,R^-)) \)) be pairwise distinct. Let

\[
\phi(x) = \frac{(f(x) - u_1(x))(u_2(x) - u_3(x))}{(f(x) - u_3(x))(u_2(x) - u_1(x))}
\]

and let \( t \) be the ramification index of \( \phi \).

Then

\[
\frac{T(r,f)}{q^t} \leq \sum_{j=1}^{3} \bar{Z}(r, f-u_j) + o(T(r,f)).
\]

**Proof:** By Theorem D2, we have

1. \[
\frac{T(r,\phi)}{q^t} \leq \bar{Z}(r,\phi) + \bar{Z}(r,\phi-1) + \bar{N}(r,\phi) + O(1).
\]

Next, we have \( T(r,f) \leq T(r,f-u_j) + T(r,u_j) \) (\( j = 1, 2, 3 \)), hence \( T(r,f) \leq T(r,\frac{u_3-u_1}{f-u_3}) + o(T(r,f)) \), whereby \( T(r,f) \leq T(r,\frac{u_3-u_1}{f-u_3} + 1) + o(T(r,f)) = T(r,\frac{f-u_1}{f-u_3}) + o(T(r,f)) \).

Now, \( T(r,\frac{u_2-u_1}{u_2-u_3}) = o(T(r,f)) \). Consequently, by writing \( \frac{f-u_1}{f-u_3} = \phi(\frac{u_2-u_1}{u_2-u_3}) \) we have

\[
T(r,\frac{f-u_1}{u_2-u_3}) \leq T(r,\phi) + T(r,\frac{u_2-u_1}{u_2-u_3}) \leq T(r,\phi) + o(T(r,f)) \]

and finally \( T(r,f) \leq T(r,\phi) + o(T(r,f)) \). Thus, by (1) we obtain

2. \[
\frac{T(r,f)}{q^t} \leq \bar{Z}(r,\phi) + \bar{Z}(r,\phi-1) + \bar{N}(r,\phi) + o(T(r,f)).
\]

Now, we can check that

\[
\bar{Z}(r,\phi) + \bar{Z}(r,\phi-1) + \bar{N}(r,\phi) \leq \sum_{j=1}^{3} \bar{Z}(r, f-u_j) + \sum_{1 \leq j < k \leq 3} \bar{Z}(r, u_k-u_j) \leq \sum_{j=1}^{3} \bar{Z}(r, f-u_j) + o(T(r,f))
\]

which, by (2), completes the proof. \( \square \)

We are now ready to state and prove Theorem D4.

**Theorem D4:** Let \( f \in \mathcal{M}(\mathbb{K}) \) (resp. \( f \in \mathcal{M}_u(d(0,R^-)) \)), let \( u_1, u_2 \in \mathcal{M}_f(\mathbb{K}) \) (resp. \( u_1, u_2 \in \mathcal{M}_f(d(0,R^-)) \)) be distinct and let \( t \) be the ramification index of \( \frac{f(x) - u_1(x)}{f(x) - u_2(x)} \). Then

\[
\frac{T(r,f)}{q^t} \leq \bar{Z}(r, f-u_1) + \bar{Z}(r, f-u_2) + \bar{N}(r, f) + o(T(r,f)).
\]

**Proof:** Let \( g = \frac{1}{f}, \ w_j = \frac{1}{u_j}, \ j = 1, 2, \ w_3 = 0 \). Clearly, \( T(r,g) = T(r,f) + O(1), \ T(r,w_j) = T(r,u_j), \ j = 1, 2 \), so we can apply Theorem D3 to \( g, \ w_1, \ w_2, \ w_3 \). On the other hand,

\[
\frac{(g(x) - w_1(x))w_2(x)}{(g(x) - w_2(x))w_1(x)} = \frac{f(x) - u_1(x)}{f(x) - u_2(x)}
\]
Thus by Theorem D3 we have: \[ \frac{T(r,g)}{q^t} \leq \tilde{Z}(\tau, g - w_1) + \tilde{Z}(\tau, g - w_2) + \tilde{Z}(\tau, g) + o(T(\tau, g)). \]

But we notice that \( \tilde{Z}(\tau, g - w_j) = \tilde{Z}(\tau, f - u_j) \) for \( j = 1, 2 \) and \( \tilde{Z}(\tau, g) = \tilde{N}(\tau, f) \). Moreover, we know that \( o(T(\tau, g)) = o(T(\tau, f)) \). Consequently, the claim is proven when \( u_1u_2 \) is not identically zero.

Next, by setting \( g = f - u_1 \) and \( u = u_2 - u_1 \), we obtain Corollary D5:

**Corollary D5:** Let \( g \in \mathcal{M}(\mathbb{K}) \) (resp. \( g \in \mathcal{M}_u(d(0, R^{-})) \)), let \( u \in \mathcal{M}_g(\mathbb{K}) \) (resp. \( u \in \mathcal{M}_g(d(0, R^{-})) \)) and let \( t \) be the ramification index of \( \frac{g - u}{g} \).

Then \( \frac{T(r,g)}{q^t} \leq \tilde{Z}(\tau, g) + \tilde{Z}(\tau, g - u) + \tilde{N}(\tau, g) + o(T(\tau, g)). \)

**Corollary D6:** Let \( f \in \mathcal{A}(\mathbb{K}) \) (resp. \( f \in \mathcal{A}_u(d(0, R^{-})) \)) and let \( u_1, u_2 \in \mathcal{A}_f(\mathbb{K}) \) (resp. \( u_1, u_2 \in \mathcal{A}_f(d(0, R^{-})) \)) be distinct and let \( t \) be the ramification index of \( \frac{f - u_1}{f - u_2} \). Then

\[ \frac{T(r,f)}{q^t} \leq \tilde{Z}(\tau, f - u_1) + \tilde{Z}(\tau, f - u_2) + o(T(\tau, f)). \]

**Corollary D7:** Let \( f \in \mathcal{A}(\mathbb{K}) \) (resp. \( f \in \mathcal{A}_u(d(0, R^{-})) \)) and let \( u \in \mathcal{A}_f(\mathbb{K}) \) (resp. \( u \in \mathcal{A}_f(d(0, R^{-})) \)) be non-identically zero and let \( t \) be the ramification index of \( \frac{f - u}{f} \). Then

\[ \frac{T(r,f)}{q^t} \leq \tilde{Z}(\tau, f) + \tilde{Z}(\tau, f - u) + o(T(\tau, f)). \]

In the proof of Theorems 1 and 2 we will need the following lemma:

**Lemma 7:** Let \( f \in \mathcal{A}(\mathbb{K}) \) (resp. \( f \in \mathcal{A}_u(d(0, R^{-})) \)) and let \( t \) be the ramification index of \( f \). Let \( m, n \in \mathbb{N}^*, m < n, \) be prime to \( p \). Then the ramification index of \( f^n - f^m \) is also equal to \( t \).

**Proof:** Since the lemma is trivial when \( p = 0 \), we suppose \( p \neq 0 \), hence \( p = q \). Set \( h = p^t \) and \( F = f^n - f^m \). By hypothesis, since both \( m, n \) are prime to \( p \), the ramification index of both \( f^m, f^n \) is equal to \( t \) and hence so are those of \( f^{n-m} \) and \( f^{n-m} - 1 \). Let \( g = \sqrt[2]{f} \). Then \( g \) belongs to \( \mathcal{A}(\mathbb{K}) \) (resp. to \( \mathcal{A}_u(d(0, R^{-})) \)) and so does \( \sqrt[2]{F} \). Let \( G = \sqrt[2]{F} \). Then we can check that \( G' = g'g^{m-1}(ng^{e-m} - m) \) hence \( G' \) is not identically 0. Consequently, the ramification index of \( F \) is \( t \). \( \square \)
Proof of Theorem 1 and Theorem 2: We can obviously suppose $a = 0$. Suppose that $f$, $g$ are two distinct functions. Let $F = f^n - f^m$. By Corollary D5, we can obtain

$$\frac{T(r,F)}{q^t} \leq \bar{Z}(r,F) + \bar{Z}(r,F - k(\phi - 1)) + \bar{N}(r,F) + o(T(r,f)).$$

Now, clearly, $\bar{N}(r,F) \leq T(r,f)$ and $\bar{Z}(r,F) \leq \bar{Z}(r,f) + \bar{Z}(r,F^{m-n} - 1) + O(1)$ and $\bar{Z}(r,f^{n-m} - 1) \leq (n - m)T(r,f)$; hence

(1) $\bar{Z}(r,F) \leq (n - m + 1)T(r,f) + o(T(r,f)).$

Similarly,

(2) $\bar{Z}(r,F - k(\phi - 1)) = \bar{Z}(r,g^n - g^m) \leq \bar{Z}(r,g) + \bar{Z}(r,g^{n-m} - 1).$

Of course, since $\phi$ is bounded, by Proposition C we have $T(r,f) = T(r,g) + O(1)$; hence, by (1) and (2), we obtain $T(r,F) \leq q^t(2n - 2m + 3)T(r,f) + o(T(r,f)).$

On the other hand, by 2.4.15 [15], we have $T(r,F) = nT(r,f) + O(1)$; hence

$$nT(r,f) \leq q^t(2n - 2m + 3)T(r,f) + o(T(r,f)).$$

That yields $2mq^t \leq n(2q^t - 1) + 3q^t$, a contradiction to the hypothesis of Theorem 1.

Next, in the hypotheses of Theorem 2, we have $N(r,f) = N(r,g) = 0$; hence we can get $T(r,F) \leq q^t(2n - 2m + 2)T(r,f) + o(T(r,f))$ and hence $2mq^t \leq n(2q^t - 1) + 2q^t$, a contradiction to the hypotheses of Theorem 2. That ends the proofs of Theorems 1 and 2.

Proof of Corollary 1.1: Suppose $Y(n,m,k)$ is not a bi-urscm for $M_n(d(a,R^-))$ and let $f$, $g \in M_n(d(a,R^-))$ be such that $E(f,Y(n,m,k)) = E(g,Y(n,m,k))$. By Proposition A, the function $\phi = \frac{f^n - f^m + k}{g^n - g^m + k}$ is an invertible element of $A(d(a,R^-))$. And since $\mathbb{K}$ has characteristic zero, we have $nT(r,f) > 2(n - m + 1)T(r,f) + o(T(r,f))$, hence by Theorem 1, $f = g$.

The proof of Corollary 2.1 is similar by applying Theorem 2.

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URSCM of 5 points for $A_u(d(a, R^-))$