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Unique range sets of 5 points for unbounded analytic functions inside an open disk

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Abstract

Let $\mathbb{K}$ be a complete algebraically closed p-adic field of characteristic $p \geq 0$ and let $\mathcal{A}_u(d(a, R^-))$ be the set of unbounded analytic functions inside the disk $d(a, R^-) = \{x \in \mathbb{K} \mid : |x - a| < R\}$. We recall the definition of urscm and the ultrametric Nevanlinna Theory on 3 small functions in order to find new urscm for $\mathcal{A}_u(d(a, R^-))$. Results depend on the characteristic. In characteristic 0, we can find urscm of 5 points. Some results on bi-urscm are given for meromorphic functions.

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1 Introduction and main result

We shall introduce URSCM for p-adic meromorphic functions. Many studies were made in the eighties and the nineties concerning URSCM for functions in $\mathbb{C}$, [3], [6], [16]. Studies were also made in the non-archimedean context by the late nineties and next [1], [2], [3], [4], [5], [8], [9], [10], [11], [13]. Here, we will only consider the situation in an ultrametric field.

Definitions and notation: Throughout the paper, E is an algebraically closed field of characteristic $p \geq 0$ without any assumption on the existence of an absolute value. A subset S of E is said to be *affinely rigid* if there is no similarity t on E other than the identity, such that $t(S) = S$.

We denote by $\mathbb{K}$ an algebraically closed field complete with respect to an ultrametric absolute value $|\ .\ |$ and of characteristic $p \geq 0$. We will denote by q the characteristic exponent of $\mathbb{K}$: if $p \neq 0$, then $q = p$ and if $p = 0$ then $q = 1$.

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Given $\alpha \in \mathbb{K}$ and $R \in \mathbb{R}_+^*$, we denote by $d(\alpha, R)$ the disk $\{x \in \mathbb{K} \mid |x-\alpha| \leq R\}$, by $d(\alpha, R^-)$ the disk $\{x \in \mathbb{K} \mid |x-\alpha| < R\}$, by $\mathcal{A}(\mathbb{K})$ the $\mathbb{K}$-algebra of analytic functions in $\mathbb{K}$ (i.e. the set of power series with an infinite radius of convergence) and by $\mathcal{M}(\mathbb{K})$ the field of meromorphic functions in $\mathbb{K}$ (i.e. the field of fractions of $\mathcal{A}(\mathbb{K})$).

In the same way, given $\alpha \in \mathbb{K}$, $R > 0$ we denote by $\mathcal{A}(d(\alpha, R^-))$ the $\mathbb{K}$-algebra of analytic functions in $d(\alpha, R^-)$ (i.e. the set of power series with an radius of convergence $\geq R$) and by $\mathcal{M}(d(\alpha, R^-))$ the field of fractions of $\mathcal{A}(d(\alpha, R^-))$. We then denote by $\mathcal{A}_b(d(\alpha, R^-))$ the $\mathbb{K}$-algebra of bounded analytic functions in $d(\alpha, r^-)$ and by $\mathcal{M}_b(d(\alpha, r^-))$ the field of fractions of $\mathcal{A}_b(d(\alpha, r^-))$. And we set $\mathcal{A}_u(d(\alpha, R^-)) = \mathcal{A}(d(\alpha, R^-)) \setminus \mathcal{A}_b(d(\alpha, R^-))$ and $\mathcal{M}_u(d(\alpha, R^-)) = \mathcal{M}(d(\alpha, R^-)) \setminus \mathcal{M}_b(d(\alpha, R^-))$.

Given a family of functions $\mathcal{F}$ defined in $\mathbb{K}$ or in a subset S of $\mathbb{K}$ (resp. in E or in a subset S of E), with values in $\mathbb{K}$ (resp. in E), S is called an *ursim for* $\mathcal{F}$ if for any two non-constant functions $f, g \in \mathcal{F}$ satisfying $f^{-1}(S) = g^{-1}(S)$, these functions are equal.

That definition particularly applies to $\mathcal{A}(\mathbb{K})$, $\mathcal{M}(\mathbb{K})$, $\mathcal{A}_u(d(a, R^-))$, $\mathcal{M}_u(d(a, R^-))$, $\mathbb{K}[x]$, $\mathbb{K}(x)$, $E[x]$, $E(x)$.

We will now recall the definition of URSCM. Given a subset S of E and $f \in E(x)$, we denote by $\mathcal{E}(f, S)$ the set in $E\times \mathbb{N}^*$:

$$\bigcup_{a\in S}\{(z,q) \in E\times \mathbb{N}^* \mid z \text{ is a zero of order } q \text{ of } f(x)-a\}.$$

Similarly, consider now meromorphic functions in the field $\mathbb{K}$. For a subset S of $\mathbb{K}$ and $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(a, R^-))$) we denote by $\mathcal{E}(f, S)$ the set in $\mathbb{K}\times \mathbb{N}^*$: $\bigcup_{a\in S}\{(z,q) \in \mathbb{K}\times \mathbb{N}^* \mid z$ is a zero of order q of $f(x)-a\}$.

Let $\mathcal{F}$ be a non-empty subset of $\mathcal{A}(\mathbb{K})$ (resp. of $\mathcal{M}(\mathbb{K})$, resp. of $\mathcal{A}(d(a, R^-))$, resp. of $\mathcal{M}(d(a, R^-))$). We say that two non-constant functions $f, g \in \mathcal{F}$ *share* S*, counting multiplicity* if $\mathcal{E}(f, S) = \mathcal{E}(g, S)$; and the set S is called *a unique range set counting multiplicity* (an *URSCM* in brief) *for* $\mathcal{F}$ if for any two non-constant $f, g \in \mathcal{F}$ sharing S counting multiplicity, one has $f = g$. Next, the set S will be called a *bi-URSCM* for $\mathcal{F}$ if for two non-constant functions $f,\ g \in \mathcal{M}_u(d(a, R^-))$ sharing S counting multiplicity and having the same poles, counting multiplicity, one has $f = g$ [8].

Particularly, if we consider a family $\mathcal{F} \subset \mathcal{A}(K)$ or $\mathcal{F} \subset \mathcal{A}_u(d(a, R^-))$ and a set $S = \{a_1, ..., a_t\} \subset \mathbb{K}$ (resp. a set $S = \{a_1, ..., a_t\} \subset E$) with $a_i \neq a_j\ \forall i \neq j$, we can set $P(X) = \prod_{j=1}^{t}(X-a_j)$ and then the set $S = \{a_1, ..., a_t\}$ is an URSCM for $\mathcal{F}$ if for any two functions $f,\ g \in \mathcal{F}$ such that $P \circ f$ and $P \circ g$ have the same zeros with the same multiplicity, then $f = g$.

Similarly, if we consider a family $\mathcal{F} \subset \mathcal{M}(K)$ or $\mathcal{F} \subset \mathcal{M}_u(d(a, R^-))$ and a set $S = \{a_1, ..., a_t\} \subset \mathbb{K}$ (resp. a set $S = \{a_1, ..., a_t\} \subset E$) with $a_i \neq a_j\ \forall i \neq j$, we can set $P(X) = \prod_{j=1}^{t}(X-a_j)$ and then the set $S = \{a_1, ..., a_t\}$ is a bi-URSCM for $\mathcal{F}$ if for any two functions $f,\ g \in \mathcal{F}$ having the same poles (counting multiplicity) such that $P \circ f$ and $P \circ g$ have the same zeros with the same multiplicity, then $f = g$.

Remark: An URSCM S for a family of functions $\mathcal{F} = \mathcal{M}(\mathbb{K}),\ \mathcal{A}(\mathbb{K})$, $\mathcal{M}_u(d(a,R^-)),\ \mathcal{A}_u(d(a,R^-))$ must obviously be affinely rigid. Indeed suppose that S is not affinely rigid and let t be a similarity of $\mathbb{K}$ such that $t(S) = S$. Then, if f belongs to $\mathcal{F}$, so does $f \circ t$ and therefore we can check that $\mathcal{E}(f,S) = \mathcal{E}(f \circ t, S)$. And it is a bi-URSCM if for any two functions $f,\ g \in \mathcal{F}$ such that $P \circ f$ and $P \circ g$ have the same zeros and the same poles, counting multiplicity, then $f = g$.

Similar definitions were given for meromorphic functions on $\mathbb{C}$ before these questions were examined on the field $\mathbb{K}$. URSCM of only 11 points for complex meromorphic functions in the whole field $\mathbb{C}$ where found in [16] and the same method showed the existence of URSCM of only 7 points for complex entire functions. So far, they are the smallest known in $\mathbb{C}$.

In the field $\mathbb{K}$, the same method lets us find URSCM of 11 points for $\mathcal{M}_u(d(a,R^-))$ and URSCM of 10 points for $\mathcal{M}(\mathbb{K})$.

In 1996, URSCM for polynomials on a field such as E were characterized: they are just the affinely rigid subsets of E [9]. Particularly, the smallest URSCM for polynomials are the affinely rigid sets of 3 points. Concerning entire functions on the field $\mathbb{K}$, URSCM of 3 points were found: they also are the affinely rigid sets of 3 points [9] and n points [19]. Next, URSCM of 7 points were found for unbounded analytic functions in a disk $d(a,R^-)$ [10]. Here we will show the existence of another family of URSCM for $\mathcal{A}_u(d(a,R^-))$, looking for sets of less than 7 points.

The notion of URSCM is closely linked to that of strong uniqueness polynomial.

Definition: A polynomial $P \in \mathbb{K}[x]$ is called *a strong uniqueness polynomial for a subset* $\mathcal{F} \subset E(x)$ (resp. $\mathcal{F} \subset \mathcal{M}(\mathbb{K})$, resp. $\mathcal{F} \subset \mathcal{M}(d(a,R^-))$) if, given $f,\ g \in \mathcal{F}$, the equality $P(f) = P(g)$ implies $f = g$.

The following basic result is immediate and useful to understand the role of URSCM:

Proposition A: *Let $S = \{a_1,...,a_n\} \subset E$, (resp. $S = \{a_1,...,a_n\} \subset \mathbb{K}$), let $a \in \mathbb{K}$, let $R \in \mathbb{R}_+^*$ and let $P(x) = \prod_{i=1}^{n}(x - a_i)$. Given any two functions $f,\ g \in E[x]$ (resp. $f,\ g \in \mathcal{A}(\mathbb{K})$, resp. $f,\ g \in \mathcal{A}(d(a,R^-))$) then $\mathcal{E}(f,S) = \mathcal{E}(g,S)$ if and only if $\dfrac{P(f)}{P(g)}$ is a constant in E^* (resp. is a constant in $\mathbb{K}^*$, resp. is an invertible function in $\mathcal{A}(d(a,R^-))$). Given any two functions $f,\ g \in E(x)$ (resp. $f,\ g \in \mathcal{M}(\mathbb{K})$, resp. $f,\ g \in \mathcal{M}(d(a,R^-))$) having the same poles counting multiplicity, then $\mathcal{E}(f,S) = \mathcal{E}(g,S)$ if and only if $\dfrac{P(f)}{P(g)}$ is a constant in E^* (resp. is a constant in $\mathbb{K}^*$, resp. is an invertible function in $\mathcal{A}(d(a,R^-))$).*

Corollary A1 *Let $S = \{a_1, ..., a_n\} \subset \mathbb{K}$ (resp. let $S = \{a_1, ..., a_n\} \subset E$) and let $P(x) = \prod_{i=1}^{n}(x - a_i)$. Then P is a polynomial of strong uniqueness for $\mathcal{A}(\mathbb{K})$ (resp. for $E[x]$) if and only if $S = \{a_1, ..., a_n\}$ is an URSCM for $\mathcal{A}(\mathbb{K})$ (resp. for $E[x]$).*

Remark: Let $P(x) = x^4 - 4x^3$ and let j be a primitive 3-rd root of 1. Clearly, $P(jf) = jP(f)\ \forall f \in \mathcal{M}(\mathbb{K})$, hence P is not a polynomial of strong uniqueness for $\mathcal{A}(\mathbb{K})$ or for $E[x]$.

As usual, if $p \neq 0$, given $a \in \mathbb{K}$ and $n \in \mathbb{N}$, we denote by $\sqrt[p^n]{a}$ the unique $b \in \mathbb{K}$ such that $b^{(p^n)} = a$.

Given $m,\ n \in \mathbb{N}$ we set $m \prec n$ if m divides n and $m \not\prec n$ if m does not divide n. When $p \neq 0$, we denote by $\mathcal{S}$ the $\mathbb{F}_p$-automorphism of $\mathbb{K}$ defined by $\mathcal{S}(x) = \sqrt[p]{x}$. More generally this mapping has continuation to a $\mathbb{K}$-algebra automorphism of $\mathbb{K}[X]$ as $\mathcal{S}(c\prod_{j=1}^{n}(X - a_j)) = \mathcal{S}(c)\prod_{j=1}^{n}(X - \mathcal{S}(a_j)),\ c \in \mathbb{K}$.

Proposition B: *Suppose $p \neq 0$. Let $r > 0$ and let $f \in \mathcal{M}(d(a, r^-))$. Then $\sqrt[p]{f}$ belongs to $\mathcal{M}(d(a, r^-))$ if and only if $f' = 0$. Moreover, there exists a unique $t \in \mathbb{N}$ such that $\sqrt[p^t]{f} \in \mathcal{M}(d(a, r^-))$ and $(\sqrt[p^t]{f})' \neq 0$.*

Proof: If f is of the form l^p with $l \in \mathcal{M}(d(a, r^-))$, then of course we have $f' = 0$. Now, suppose that $f' = 0$. If $f \in \mathcal{A}(d(a, r^-))$, then obviously all non-zero coefficients have an index multiple of p, hence f is of the form l^p, with $l \in \mathcal{A}(d(a, r^-))$. We now consider the general case when $f \in \mathcal{M}(d(a, r^-))$. Let $(b_n, t_n)_{n \in \mathbb{N}}$ be the sequence of poles of f inside $d(a, r^-)$ where t_n is the multiplicity order of b_n. By Theorem 25.5 [14] we can find $h \in \mathcal{A}(d(a, r^-))$ such that $\omega_{b_n}(h) \geq t_n\ \forall n \in \mathbb{N}$. Clearly fh^p belongs to $\mathcal{A}(d(a, r^-))$ and satisfies $(fh^p)' = 0$. Consqeuently, fh^p is of the form g^p, with $g \in \mathcal{A}(d(a, r^-))$, therefore $f = \left(\frac{g}{h}\right)^p$. On the other hand, the set of integers s such that $\sqrt[p^s]{f}$ belongs to $\mathcal{M}(d(a, r^-))$ is obviously bounded and therefore admits a biggest element, which ends the proof.

□

Definition and notation: Suppose $p \neq 0$. Given, $f \in \mathcal{M}(d(a, r^-))$, we will call *ramification index of* f the integer t such that $\sqrt[p^t]{f} \in \mathcal{M}(d(a, r^-))$ and $(\sqrt[p^t]{f})' \neq 0$.

In the same way, given an algebraically closed field B of characteristic $p \neq 0$ and $P(x) \in B[x]$, we call *ramification index of* P the unique integer t such that $\sqrt[p^t]{P} \in B[x]$ and $(\sqrt[p^t]{P})' \neq 0$. This ramification index will be denoted by $\text{ram}(f)$ for any $f \in \mathcal{M}(d(a, r^-))$ or $f \in \mathcal{M}(\mathbb{K})$ and similarly it will be denoted by $\text{ram}(P)$ for any $P \in B[x]$.

Henceforth, given $t \in \mathbb{N}^*$, we will denote by $\mathcal{A}_t(d(a, R^-))$ the subset of the functions $f \in \mathcal{A}(d(a, R^-))$ having a ramification index $\leq t$ and similarly, we put $\mathcal{A}_{u,t}(d(a, R^-)) = \mathcal{A}_t(d(a, R^-)) \cap \mathcal{A}_u(d(a, R^-))$.

Given $k \in \mathbb{K}^*$ and $n, m \in \mathbb{N}^*$ with $m < n$, we set $Q_{n,m,k}(x) = x^n - x^m + k$ and we denote by $Y_{n,m,k}$ the set of zeros of $Q_{n,m,k}$. In the same way, we set $Q_{n,k}(x) = x^n - x^{n-1} + k$ and we denote by $Y_{n,k}$ the set of zeros of $Q_{n,k}$.

Remark: Suppose $p \neq 0$ and let $f \in \mathcal{M}(d(a,r^-))$ have ramification index t as an element of $\mathcal{M}(d(a,r^-))$. For every $r' \in]0,r[$, f has the same ramification index as an element of $\mathcal{M}(d(a,r'^-))$ because of course, on one hand, $\sqrt[p^t]{f} \in \mathcal{M}(d(a,r'^-))$ and on the other hand, by properties of analytic functions, $(\sqrt[p^t]{f})'$ is not identically zero inside $d(a,r')$.

As recalled above, in [9] the smallest urscm for $\mathcal{A}_u(d(a,R^-))$ have 7 points. By Corollary 2.2 we can find a new family of urscm for $\mathcal{A}_u(d(a,R^-))$, with particularly urscm of 5 points.

Theorem 1: *Let $t \in \mathbb{N}^*$ and let $f,\ g \in \mathcal{M}_u(d(a,R^-))$ be such that the function $\phi = \dfrac{f^n - f^m + k}{g^n - g^m + k}$ is invertible in $\mathcal{A}(d(a,R^-))$. Let t be the ramification index of $\dfrac{f^n - f^m - k(\phi - 1)}{f^n - f^m}$. If $2mq^t > n(2q^t - 1) + 3q^t$ then $f = g$.*

Corollary 1.1 : *Suppose $\mathbb{K}$ is of characteristic 0. If $2m > n+3$ then $Y(n,m,k)$ is a bi-urscm for $\mathcal{M}_u(d(a,R^-))$.*

Corollary 1.2 : *Suppose $\mathbb{K}$ is of characteristic 0. If $n \geq 6$, then $Y(n,k)$ is a bi-urscm for $\mathcal{M}_u(d(a,R^-))$.*

Theorem 2: *Let $t \in \mathbb{N}^*$ and let $f,\ g \in \mathcal{A}_u(d(a,R^-))$ be such that the function $\phi = \dfrac{f^n - f^m + k}{g^n - g^m + k}$ is invertible in $\mathcal{A}(d(a,R^-))$. Let t be the ramification index of $\dfrac{f^n - f^m - k(\phi - 1)}{f^n - f^m}$.*

If $2mq^t > n(2q^t - 1) + 2q^t$ then $f = g$.

Corollary 2.1 : *Suppose $\mathbb{K}$ is of characteristic 0. If $2m \geq n+3$, then $Y(n,m,k)$, is an urscm for $\mathcal{A}_u(d(a,R^-))$.*

Corollary 2.2 : *Suppose $\mathbb{K}$ is of characteristic 0. If $n \geq 5$, then $Y(n,k)$, is an urscm for $\mathcal{A}_u(d(a,R^-))$.*

Remark: We don't know whether there exists an urscm for $\mathcal{A}_u(d(a,R^-))$ of 4 points or 3 points.

2 The Proof

We must recall the definition of the counting functions in the Nevanlinna Theory.

Definitions and notation: Let $f \in \mathcal{M}(d(a,R^-))$ and let $\alpha \in d(a,R^-)$. If f admits α as a zero of order q, we set $\omega_\alpha(f) = q$; if f admits α as a pole of order q, we set $\omega_\alpha(f) = -q$; and if α is neither a zero nor a pole for f, we set $\omega_\alpha(f) = 0$.

We denote by $Z(r,f)$ the counting function of zeros of f in $d(0,r)$ in the following way:

Let (a_n), $1 \leq n \leq \sigma(r)$ be the finite sequence of zeros of f such that $0 < |a_n| \leq |a_{n+1}| \leq |a_{\sigma(r)}| \leq r$, of respective order s_n.

We set $Z(r,f) = \max(\omega_0(f), 0)\log r + \displaystyle\sum_{n=1}^{\sigma(r)} s_n(\log r - \log|a_n|)$.

Similarly, we set $N(r,f) = Z(r, \frac{1}{f})$.

In order to define the counting function of zeros of f without multiplicity, we put $\overline{\omega_0}(f) = 0$ if $\omega_0(f) \leq 0$ and $\overline{\omega_0}(f) = 1$ if $\omega_0(f) \geq 1$.

In the sequel, I will denote an interval of the form $[\rho, +\infty[$, with $\rho > 0$, and J will denote an interval of the form $[\rho, R[$.

Next, denoting by $E(r,f)$ the set $\{a \in d(0,r) \mid \omega_a(f) > 0,\ p^{\text{ram}(f)+1} \not\prec \omega_a(f)\}$,
if $0 \notin E(r,f)$ we set $\widetilde{Z}(r,f) = \sum_{\alpha \in E(r,f)} \log \frac{r}{|\alpha|}$
and if $0 \in E(r,f)$ we set $\widetilde{Z}(r,f) = \log r + \sum_{\alpha \in E(r,f), \alpha \neq 0} \log \frac{r}{|\alpha|}$.
Similarly we define $\widetilde{N}(r,f) = \widetilde{Z}(r, \frac{1}{f})$.

We can now define the Nevanlinna characteristic function of f: $T(r,f) = \max(Z(r,f), T(r,f))$.

Assume that f' is not identically 0.
Let $V(r,f) = \{a \in d(0,r) \mid \omega_a(f) < 0,\ p^{\text{ram}(f)+1} \prec \omega_a(f)\}$. We put
$N_0(r,f') = \sum_{\alpha \in V(r,f)} [\omega_\alpha(f') - \omega_\alpha(f)] \log \frac{r}{|\alpha|}$.

Given a finite subset S of $\mathbb{K}$, we put $\Lambda'(r,f,S) = \{a \in d(0,r) \mid f'(a) = 0,\ f(a) \notin S\}$ and $\Lambda''(r,f,S) = \{a \in d(0,r) \mid p^{\text{ram}(f)+1} \prec \omega_a(f - f(a)),\ f(a) \in S\}$. Then we can define
$Z_0^S(r,f') = \sum_{\alpha \in \Lambda'(r,f,S)} \omega_\alpha(f') \log \frac{r}{|\alpha|} + \sum_{\alpha \in \Lambda''(r,f,S)} [\omega_\alpha(f') - \omega_\alpha(f - f(\alpha))] \log \frac{r}{|\alpha|}$.

Remarks: 1) It is easily verified that all the above functions are positive.
2) If $p = 0$, we have $\overline{Z}(r,f) = \widetilde{Z}(r,f)$ and $\overline{N}(r,f) = \widetilde{N}(r,f)$.

Lemma 1: *Let $f \in \mathcal{M}(d(0,R^-))$, let $t = r(f)$ and let $g = \sqrt[q^t]{f}$. Then $\widetilde{Z}(r,f) = \widetilde{Z}(r,g)$ and $\widetilde{N}(r,f) = \widetilde{N}(r,g)$.*

Proof: Let a be a zero of f and let $s = \omega_a(f)$. Then s is of the form nt with $n \in \mathbb{N}^*$. If $n = 1$, then a belongs to both $E(r,f)$ and $E(r,g)$; and if $n > 1$, then $a \notin E(r,f)$. But then a is a zero of order n of g and hence, a does not belong to $E(r,g)$. □

The following Lemmas 2 and 3 are easily checked [12]:

Lemma 2: *Let $\alpha_1, \cdots, \alpha_n \in \mathbb{K}$ be pairwise distinct, let $P(u) = \prod_{i=1}^{n}(u - \alpha_i)$ and let $f \in \mathcal{M}(d(0,R^-))$. Then $Z(r,P(f)) = \sum_{i=1}^{n} Z(r, f - \alpha_i)$ and $\widetilde{Z}(r,P(f)) = \sum_{i=1}^{n} \widetilde{Z}(r, f - \alpha_i)$.*

Lemma 3: *Let $f \in \mathcal{M}(d(0,R^-))$ be such that f' is not identically zero and let $\alpha \in d(0,R^-)$. We have $\omega_\alpha(f') = \omega_\alpha(f) - 1$ if $p \not\prec \omega_\alpha(f)$ and $\omega_\alpha(f') \geq \omega_\alpha(f)$ if $p \prec \omega_\alpha(f)$.*

Lemmas 4 and 5 are consequences of Lemma 3.

Lemma 4: *Let $f \in \mathcal{M}(d(0,R^-))$ be such that $f' \neq 0$ and let S be a finite subset of $\mathbb{K}$. Then:*

$$\sum_{b\in S}\Big(Z(r,f-b)-\widetilde{Z}(r,f-b)\Big) = Z(r,f') - Z_0^S(r,f').$$

Lemma 5: *Let $f \in \mathcal{M}(d(0,R^-))$ be such that $f' \neq 0$ and let $0<r<R$. Then $N(r,f') = N(r,f) + \widetilde{N}(r,f) - N_0(r,f')$.*

Lemma 6: *Let $f \in \mathcal{M}(d(0,R^-))$ be such that $f' \neq 0$ and let $0<r<R$. Then:*

$$Z(r,f') \leq Z(r,f) + \widetilde{N}(r,f) - N_0(r,f') - \log r + O(1), (r \in J).$$

Proof: Without loss of generality, up to change of variable, we can assume that both f and f' have no zero and no pole at 0. Let $|f|(r)$ denote the circular value of f defined as $|f|(r) = \lim_{|x|\to r,\ |x|\neq r} |f(x)|$

By classical results such as Theorem 23.13 [14], we have $Z(r,f) - N(r,f) = \log(|f|(r)) - \log(|(f(0)|)$, and $Z(r,f') - N(r,f') = \log(|f'|(r)) - \log(|f'|(0))$. But, it is well-known that $|f'|(r) \leq \dfrac{|f|(r)}{r}$ (Theorem 1.5.10 [15]); hence we obtain

$$Z(r,f') \leq N(r,f') - N(r,f) + Z(r,f) - \log r + O(1).$$

Moreover, by Lemma 4 we have $N(r,f') - N(r,f) = \widetilde{N}(r,f) - N_0(r,f')$, which completes the proof.

□

We know Proposition C [9].

Proposition C : *Let $f \in \mathcal{M}(d(0,R^-))$. Then f belongs to $\mathcal{M}_b(d(0,R^-))$ if and only if $T(r,f)$ is bounded when r tends to R.*

Corollary C1: *Let $f \in \mathcal{M}(d(0,R^-))$. Then $\mathcal{M}_b(d(0,R^-))$ is a subset of $\mathcal{M}_f(d(0,R^-))$ and $\mathcal{A}_b(d(0,R^-))$ is a subset of $\mathcal{A}_f(d(0,R^-))$.*

Remark: Particularly, an invertible function $f \in \mathcal{A}(d(a,R^-))$ has a constant absolute value and therefore lies in $\mathcal{A}_b(d(a,R^-))$.

The following Theorem D1 is known as Second Main Theorem on Three Small Functions [17]. It holds in p-adic analysis as well as in complex analysis, where it was shown first [17].

Notice that this theorem was generalized to any finite set of small functions by Yamanoy in complex analysis [18], through methods that have no equivalent on a p-adic field.

Remark: Let $f \in \mathcal{M}_u(d(0, R^-))$ and let $w \in \mathcal{M}_b(d(0, R^-))$. Then of course, $w \in \mathcal{M}_f(d(0, R^-))$.

The previous results enable us to prove the ultrametric Nevanlinna Main Theorem in a basic form:

Theorem D1: *Let $\alpha_1, ..., \alpha_n \in \mathbb{K}$, with $n \geq 2$, and let $f \in \mathcal{M}(d(0, R^-))$ (resp. $f \in \mathcal{M}(\mathbb{K})$) of ramification index t. Let $S = \{ \sqrt[q^t]{\alpha_1}, ..., \sqrt[q^t]{\alpha_n} \}$. Then we have:*

$$\frac{(n-1)T(r,f)}{q^t} \leq \sum_{i=1}^{n} \widetilde{Z}(r, f - \alpha_i) + Z(r, (\sqrt[q^t]{f})') - Z_0^S(r, (\sqrt[q^t]{f})') + O(1) \quad \forall r \in J$$

(resp. $\forall r \in I$).

Moreover, if f belongs to $\mathcal{A}(d(0, R^-))$ (resp. $f \in \mathcal{A}(\mathbb{K})$), then

$$\frac{nT(r,f)}{q^t} \leq \sum_{i=1}^{n} \widetilde{Z}(r, f - \alpha_i) + Z(r, (\sqrt[q^t]{f})') - Z_0^S(r, (\sqrt[q^t]{f})') + O(1) \; \forall r \in J$$

(resp. $\forall r \in I$).

Now, following the same method as in Theorem 2.5.9 [15], we can obtain that classical form of the Nevalinna inequality where $\overline{Z}$ and $\overline{N}$ are replaced by $\widetilde{Z}$ and $\widetilde{N}$.

Theorem D2: *Let $\alpha_1, ..., \alpha_n \in \mathbb{K}$, with $n \geq 2$, and let $f \in \mathcal{M}(d(0, R^-))$ (resp. $f \in \mathcal{M}(\mathbb{K})$) of ramification index t. Let $S = \{ \sqrt[q^t]{\alpha_1}, ..., \sqrt[q^t]{\alpha_n} \}$. Then we have:*

$$\frac{(n-1)T(r,f)}{q^t} \leq \sum_{i=1}^{n} \widetilde{Z}(r, f - \alpha_i) + \widetilde{N}(r, f) - Z_0^S(r, (\sqrt[q^t]{f})') - N_0(r, (\sqrt[q^t]{f})') - \log r + O(1) \quad \forall r \in J$$

(resp. $\forall r \in I$).

Proof of Theorems D1 and D2: The proof of Theorems D1 and D2 was given in [12]. We will recall it. For convenience, we put $g = \sqrt[q^t]{f}$, and $\beta_i = \sqrt[q^t]{\alpha_i}$ for every $i = 1, ..., n$. So $S = \{\beta_1, ..., \beta_n\}$.

Let $f \in \mathcal{M}(K)$ (resp. $f \in \mathcal{M}(d(a, R^-))$) and let $(a_n, s_n)_{n \in \mathbb{N}}$ be the set of zeros of f in $\mathbb{K}$ (resp. in $d(a, R^-)$) with $|a_n| \leq |a_{n+1}|$ whereas s_n is the order of multiplicity of a_n. We denote by $\mathcal{D}(f)$ the sequence $(a_n, s_n)_{n \in \mathbb{N}}$.

By Theorem 25.5 [14] there exist $\phi, \; \psi \in \mathcal{A}(d(0, R^-))$ such that $g = \dfrac{\phi}{\psi}$, and

(1) $Z(r, \phi) \leq Z(r, g) + 1$,

(2) $Z(r, \psi) \leq N(r, g) + 1$.

By Lemma 2.5.5 [15], there exists $A \in \mathbb{R}$ and for any $r \in J$ (resp. $r \in I$), there exists $l(r) \in \{1, ..., n\}$ such that $Z(r, \phi - \beta_j \psi) \geq \max(Z(r, \phi), Z(r, \psi)) + A \quad \forall j \neq l(r)$, therefore there exists $B \in \mathbb{R}$ such that

(3) $Z(r, \phi - \beta_i \psi) \geq T(r, g) + B \quad \forall i \neq l(r), \quad \forall r \in J$ (resp. $\forall r \in I$).

We check that $\mathcal{D}(\phi)-\mathcal{D}(\frac{\phi}{\psi})=\mathcal{D}(\psi)-\mathcal{D}(\frac{\psi}{\phi})$, therefore
$\mathcal{D}(\phi-\beta_i\psi)=\mathcal{D}(g-\beta_i)+\mathcal{D}(\psi)-\mathcal{D}(\frac{1}{g-\beta_i})=\mathcal{D}(g-\beta_i)+\mathcal{D}(\psi)-\mathcal{D}(\frac{1}{g})$.

Then, applying counting functions, we have $Z(r,\phi-\beta_i\psi)=Z(r,g-\beta_i)+Z(r,\psi)-N(r,g)$, and therefore, by (2), we obtain
(4) $Z(r,\phi-\beta_i\psi)\leq Z(r,g-\beta_i)+1$.
Then, by (3) and (4) we obtain $(n-1)\Big(T(r,g)+B\Big)$
$\leq \sum_{\substack{1\leq i\leq n,\\ i\neq l(r)}} Z(r,\phi-\beta_i\psi)\leq \sum_{\substack{1\leq i\leq n,\\ i\neq l(r)}} Z(r,g-\beta_i)+n-1 \quad \forall r\in J$ (resp. $\forall r\in I$).
Putting $M=(n-1)(1-B)$, we obtain:
(5) $(n-1)T(r,g)\leq \sum_{i=1}^{n} Z(r,g-\beta_i)+M-Z(r,g-\beta_{l(r)}) \quad \forall r\in J$ (resp. $\forall r\in I$).

By Lemma 4, we have
$\sum_{i=1}^{n} Z(r,g-\beta_i)=\sum_{i=1}^{n}\widetilde{Z}(r,g-\beta_i)+Z(r,g')-Z_0^S(r,g')$, hence by (5) we obtain,

(6) $(n-1)T(r,g)\leq \sum_{i=1}^{n}\widetilde{Z}(r,g-\beta_i)+Z(r,g')-Z_0^S(r,g')-Z(r,g-\beta_{l(r)})+O(1)\ \forall r\in J$ (resp. $\forall r\in I$).

Now, since $T(r,g)=\dfrac{T(r,f)}{q^t}$ and since $\widetilde{Z}(r,g-\beta_i)=\widetilde{Z}(r,f-\alpha_j)\ \forall j=i,...,n$, we obtain

$$\frac{(n-1)T(r,f)}{q^t}\leq \sum_{i=1}^{n}\widetilde{Z}(r,f-\alpha_i)+Z(r,(\sqrt[q^t]{f})')-Z_0^S(r,(\sqrt[q^t]{f})')+O(1)\ \forall r\in J$$

(resp. $\forall r\in I$).

Suppose now that f belongs to $\mathcal{A}(d(a,R^-))$ or to $\mathcal{A}(\mathbb{K})$. Then so does g. By Lemma 2.5.5 [15] we have $Z(r,g-\beta_{l(r)})=T(r,g)+O(1)\ \forall r\in J$ (*resp.* $\forall r\in I$) so, by (6) we obtain
$nT(r,g)\ \leq\ \sum_{i=1}^{n}\widetilde{Z}(r,g-\beta_i)+Z(r,g')-Z_0^S(r,g')+O(1)\ \forall r\in J$ (resp. $\forall r\in I$), and consequently,
$\dfrac{nT(r,f)}{q^t}\leq \sum_{i=1}^{n}\widetilde{Z}(r,f-\alpha_i)+Z(r,(\sqrt[q^t]{f})')-Z_0^S(r,(\sqrt[q^t]{f})')+O(1)\ \forall r\in J$ (resp. $\forall r\in I$).
Now, returning to the general case, we have $g'=(g-\beta_{l(r)})'$ and $\widetilde{N}(r,g)=\widetilde{N}(r,g-\beta_{l(r)})$. So, by Lemma 6, we have:
(7) $Z(r,g')-Z(r,g-\beta_{l(r)})\leq \widetilde{N}(r,g)-N_0(r,g')-\log r+O(1)$.
Finally, by (6), (7) we obtain
$\dfrac{(n-1)T(r,f)}{q^t}\leq \sum_{i=1}^{n}\widetilde{Z}(r,f-\alpha_i)+\widetilde{N}(r,f)-Z_0^S(r,(\sqrt[q^t]{f})')-N_0(r,(\sqrt[q^t]{f})')-\log r\ \forall r\in J$ (resp. $\forall r\in I$).

That completes the proof.

Theorem D3: *Let* $f \in \mathcal{M}(\mathbb{K})$ *(resp.* $f \in \mathcal{M}_u(d(0,R^-)))$ *and let* $u_1,\ u_2, u_3 \in \mathcal{M}_f(\mathbb{K})$ *(resp.* $u_1, u_2, u_3 \in \mathcal{M}_f(d(0,R^-)))$ *be pairwise distinct. Let*
$\phi(x) = \dfrac{(f(x) - u_1(x))(u_2(x) - u_3(x))}{(f(x) - u_3(x))(u_2(x) - u_1(x))}$ *and let* t *be the ramifucation index of* ϕ.

Then $\dfrac{T(r,f)}{q^t} \leq \sum_{j=1}^{3} \widetilde{Z}(r, f - u_j) + o(T(r,f))$.

Proof: By Theorem D2, we have
(1) $\dfrac{T(r,\phi)}{q^t} \leq \widetilde{Z}(r,\phi) + \widetilde{Z}(r,\phi - 1) + \widetilde{N}(r,\phi) + O(1)$.

Next, we have $T(r,f) \leq T(r, f - u_j) + T(r, u_j)$ $(j = 1,2,3)$, hence $T(r,f) \leq T(r, \dfrac{u_3 - u_1}{f - u_3}) + o(T(r,f))$, thereby $T(r,f) \leq T(r, \dfrac{u_3 - u_1}{f - u_3} + 1) + o(T(r,f)) = T(r, \dfrac{f - u_1}{f - u_3}) + o(T(r,f))$.

Now, $T(r, \dfrac{u_2 - u_1}{u_2 - u_3}) = o(T(r,f)$. Consequently, by writing $\dfrac{f - u_1}{f - u_3} = \phi\Big(\dfrac{u_2 - u_1}{u_2 - u_3}\Big)$ we have $T(r, \dfrac{f - u_1}{f - u_3}) \leq T(r,\phi) + T(r, \dfrac{u_2 - u_1}{u_2 - u_3}) \leq T(r,\phi) + o(T(r,f))$ and finally $T(r,f) \leq T(r,\phi) + o(T(r,f))$. Thus, by (1) we obtain
(2) $\dfrac{T(r,f)}{q^t} \leq \widetilde{Z}(r,\phi) + \widetilde{Z}(r,\phi - 1) + \widetilde{N}(r,\phi) + o(T(r,f))$.

Now, we can check that
$\widetilde{Z}(r,\phi) + \widetilde{Z}(r,\phi-1) + \widetilde{N}(r,\phi) \leq \sum_{j=1}^{3} \widetilde{Z}(r, f - u_j) + \sum_{1 \leq j < k \leq 3} \widetilde{Z}(r, u_k - u_j) \leq \sum_{j=1}^{3} \widetilde{Z}(r, f - u_j) + o(T(r,f))$
which, by (2), completes the proof. □

We are now ready to state and prove Theorem D4.

Theorem D4: *Let* $f \in \mathcal{M}(\mathbb{K})$ *(resp.* $f \in \mathcal{M}_u(d(0,R^-)))$*, let* $u_1,\ u_2 \in \mathcal{M}_f(\mathbb{K})$ *(resp.* $u_1, u_2 \in \mathcal{M}_f(d(0,R^-)))$ *be distinct and let* t *be the ramification index of* $\dfrac{f(x) - u_1(x)}{f(x) - u_2(x)}$*. Then*

$$\frac{T(r,f)}{q^t} \leq \widetilde{Z}(r, f - u_1) + \widetilde{Z}(r, f - u_2) + \widetilde{N}(r,f) + o(T(r,f)).$$

Proof: Let $g = \dfrac{1}{f},\ w_j = \dfrac{1}{u_j},\ j = 1,2,\ w_3 = 0$. Clearly, $T(r,g) = T(r,f) + O(1),\ T(r,w_j) = T(r,u_j),\ j = 1,2$, so we can apply Theorem D3 to $g,\ w_1,\ w_2,\ w_3$. On the other hand,

$$\frac{(g(x) - w_1(x))w_2(x)}{(g(x) - w_2(x))w_1(x)} = \frac{f(x) - u_1(x)}{f(x) - u_2(x)}.$$

Thus by Theorem D3 we have: $\dfrac{T(r,g)}{q^t} \le \widetilde{Z}(r,g-w_1)+\widetilde{Z}(r,g-w_2)+\widetilde{Z}(r,g)+o(T(r,g))$.

But we notice that $\widetilde{Z}(r,g-w_j)=\widetilde{Z}(r,f-u_j)$ for $j=1,2$ and $\widetilde{Z}(r,g)=\widetilde{N}(r,f)$. Moreover, we know that $o(T(r,g)) = o(T(r,f))$. Consequently, the claim is proven when u_1u_2 is not identically zero. □

Next, by setting $g=f-u_1$ and $u=u_2-u_1$, we obtain Corollary D5:

Corollary D5: *Let* $g\in\mathcal{M}(\mathbb{K})$ *(resp.* $g\in\mathcal{M}_u(d(0,R^-))$*), let* $u\in\mathcal{M}_g(\mathbb{K})$ *(resp.* $u\in\mathcal{M}_g(d(0,R^-))$*) and let* t *be the ramification index of* $\dfrac{g-u}{g}$.

Then $\dfrac{T(r,g)}{q^t}\le \widetilde{Z}(r,g)+\widetilde{Z}(r,g-u)+\widetilde{N}(r,g)+o(T(r,g))$.

Corollary D6: *Let* $f\in\mathcal{A}(\mathbb{K})$ *(resp.* $f\in\mathcal{A}_u(d(0,R^-))$*) and let* $u_1,\ u_2\in\mathcal{A}_f(\mathbb{K})$ *(resp.* $u_1,u_2\in\mathcal{A}_f(d(0,R^-))$*) be distinct and let* t *be the ramification index of* $\dfrac{f-u_1}{f-u_2}$*. Then*

$$\frac{T(r,f)}{q^t}\le \widetilde{Z}(r,f-u_1)+\widetilde{Z}(r,f-u_2)+o(T(r,f)).$$

Corollary D7: *Let* $f\in\mathcal{A}(\mathbb{K})$ *(resp.* $f\in\mathcal{A}_u(d(0,R^-))$*) and let* $u\in\mathcal{A}_f(\mathbb{K})$ *(resp.* $u\in\mathcal{A}_f(d(0,R^-))$*) be non-identically zero and let* t *be the ramification index of* $\dfrac{f-u}{f}$*. Then*

$$\frac{T(r,f)}{q^t}\le \widetilde{Z}(r,f)+\widetilde{Z}(r,f-u)+o(T(r,f)).$$

In the proof of Theorems 1 and 2 we will need the following lemma:

Lemma 7: *Let* $f\in\mathcal{A}(\mathbb{K})$ *(resp.* $f\in\mathcal{A}_u(d(0,R^-))$*) and let* t *be the ramification index of* f*. Let* $m,\ n\in\mathbb{N}^*$*,* $m<n$*, be prime to* p*. Then the ramification index of* f^n-f^m *is also equal to* t*.*

Proof: Since the lemma is trivial when $p=0$, we suppose $p\ne 0$, hence $p=q$. Set $h=p^t$ and $F=f^n-f^m$. By hypothesis, since both $m,\ n$ are prime to p, the ramification index of both $f^m,\ f^n$ is equal to t and hence so are those of f^{n-m} and $f^{n-m}-1$. Let $g=\sqrt[h]{f}$. Then g belongs to $\mathcal{A}(\mathbb{K})$ (resp. to $\mathcal{A}_u(d(0,R^-))$) and so does $\sqrt[h]{F}$. Let $G=\sqrt[h]{F}$. Then we can check that $G'=g'g^{m-1}(ng^{n-m}-m)$ hence G' is not identically 0. Consequently, the ramification index of F is t. □

Proof of Theorem 1 and Theorem 2: We can obviously suppose $a = 0$. Suppose that $f,\ g$ are two distinct functions. Let $F = f^n - f^m$. By Corollary D5, we can obtain

$$\frac{T(r,F)}{q^t} \leq \widetilde{Z}(r,F) + \widetilde{Z}(r,F - k(\phi - 1)) + \widetilde{N}(r,F) + o(T(r,f).$$

Now, clearly, $\widetilde{N}(r,F) \leq T(r,f)$ and $\widetilde{Z}(r,F) \leq \widetilde{Z}(r,f) + \widetilde{Z}(r,f^{n-m} - 1) + O(1)$ and $\widetilde{Z}(r,f^{n-m} - 1) \leq (n-m)T(r,f)$; hence

(1) $\widetilde{Z}(r,F) \leq (n-m+1)T(r,f) + o(T(r,f))$.

Similarly,

(2) $\widetilde{Z}(r,F - k(\phi - 1)) = \widetilde{Z}(r,g^n - g^m) \leq \widetilde{Z}(r,g) + \widetilde{Z}(r,g^{n-m} - 1)$.

Of course, since ϕ is bounded, by Proposition C we have $T(r,f) = T(r,g) + O(1)$; hence , by (1) and (2), we obtain $T(r,F) \leq q^t(2n - 2m + 3)T(r,f) + o(T(r,f))$.

On the other hand, by 2.4.15 [15], we have $T(r,F) = nT(r,f) + O(1)$; hence $nT(r,f) \leq q^t(2n - 2m + 3)T(r,f) + o(T(r,f))$. That yields $2mq^t \leq n(2q^t - 1) + 3q^t$, a contradiction to the hypothesis of Theorem 1.

Next, in the hypotheses of Theorem 2, we have $N(r,f) = N(r,g) = 0$; hence we can get $T(r,F) \leq q^t(2n-2m+2)T(r,f) + o(T(r,f))$ and hence $2mq^t \leq n(2q^t - 1) + 2q^t$, a contradiction to the hypotheses of Theorem 2. That ends the proofs of Theorems 1 and 2.

Proof of Corollary 1.1: Suppose $Y(n,m,k)$ is not a bi-urscm for $\mathcal{M}_u(d(a,R^-))$ and let $f,\ g \in \mathcal{M}_u(d(a,R^-))$ be such that $\mathcal{E}(f,Y(n,m,k)) = \mathcal{E}(g,Y(n,m,k))$. By Proposition A, the function $\phi = \dfrac{f^n - f^m + k}{g^n - g^m + k}$ is an invertible element of $\mathcal{A}(d(a,R^-))$. And since $\mathbb{K}$ has characteristic zero, we have $nT(r,f) > 2(n-m+1)T(r,f) + o(T(r,f))$, hence by Theorem 1, $f = g$.

The proof of Corollary 2.1 is similar by applying Theorem 2.

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