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The Square-Cross Lemma

by
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Dedicated to Professor Constantin Năstăsescu on his 70th birthday

Abstract

We prove a homological result in additive exact categories, which we call the Square-Cross Lemma. Its applications include a property related to the Green formula from the theory of Ringel-Hall algebras as well as the Two-Square Lemma of Fay, Hardie and Hilton.

Key Words: Additive category, exact category, Two-Square Lemma.
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1 Introduction

Quillen exact categories [10] are suitable frameworks to develop homological algebra in additive categories more general than abelian categories. Moreover, one may need relative homological algebra, and choose an exact structure in an abelian category different than the usual one of all kernel-cokernel pairs. Their ubiquity and versatility made exact structures natural tools in algebra, algebraic geometry, algebraic K-theory, functional analysis etc. A recent exhaustive account on exact categories is presented by Bühler [2].

We use the setting of Quillen exact categories in order to establish a general homological result, and we present two applications. The first one is a bijective correspondence between certain sets of "squares" and "crosses" consisting of short exact sequences with respect to an exact structure, which extends a property related to the Green formula from the theory of Ringel-Hall algebras [6, 12, 13]. The second one is a generalization of the Two-Square Lemma of Fay, Hardie and Hilton [4], a useful result for obtaining a completely categorical construction of the connecting morphism from the Snake Lemma. For categorical terminology the reader is referred to [8, 9, 15].

2 Exact categories

We recall the concepts of exact category given by Quillen [10], as simplified by Keller [7], and exact chain complexes over exact categories (e.g., see [2]).

Definition 2.1. By an *exact category* we mean an additive category endowed with a distinguished class $\mathcal{E}$ of short exact sequences satisfying the axioms $[E0]$, $[E1]$, $[E2]$ and $[E2^{\mathrm{op}}]$ below. The short exact sequences in $\mathcal{E}$ are called *conflations*, while the kernels and cokernels appearing in such exact sequences are called *inflations* (denoted by $\rightarrowtail$) and *deflations* (denoted by $\twoheadrightarrow$) respectively.

$[E0]$ The identity morphism $1_0 : 0 \to 0$ is a deflation.
$[E1]$ The composition of two deflations is again a deflation.
$[E2]$ The pullback of a deflation along an arbitrary morphism exists and is again a deflation.
$[E2^{\mathrm{op}}]$ The pushout of an inflation along an arbitrary morphism exists and is again an inflation.

The duals of $[E0]$ and $[E1]$ on inflations also hold in any exact category [7].

Definition 2.2. Let $\mathcal{C}$ be an exact category. A chain complex $(A^\bullet, d_A^\bullet)$ over $\mathcal{C}$ is called *exact* (or *acyclic*) if each differential $d_A^{n-1} : A^{n-1} \to A^n$ factors as $A^{n-1} \overset{p^{n-1}}{\twoheadrightarrow} Z^n A \overset{i^{n-1}}{\rightarrowtail} A^n$ such that each sequence $Z^n A \overset{i^{n-1}}{\rightarrowtail} A^n \overset{p^n}{\twoheadrightarrow} Z^{n+1} A$ is a conflation.

The following types of additive categories will be useful for applications of our results (e.g., see [2]).

Definition 2.3. An additive category $\mathcal{C}$ is called:

1. *quasi-abelian* if it is pre-abelian (i.e., it has kernels and cokernels), any pushout of a kernel along an arbitrary morphism is a kernel, and any pullback of a cokernel along an arbitrary morphism is a cokernel.

2. *weakly idempotent complete* if every retraction has a kernel, or equivalently, every section has a cokernel.

Note that the present concept of pre-abelian category is different of that from [1]. There is the following hierarchy of additive categories, from particular to general: abelian category, quasi-abelian category, pre-abelian category and weakly idempotent complete category.

The following definition was given by Richman and Walker in a pre-abelian category [11], and extended to an arbitrary category in [3].

Definition 2.4. Let $\mathcal{C}$ be a category. A kernel in $\mathcal{C}$ is called *semi-stable* if its pushout along an arbitrary morphism exists and is again a kernel. Dually, a cokernel in $\mathcal{C}$ is called *semi-stable* if its pullback along an arbitrary morphism exists and is again a cokernel. A kernel-cokernel pair $A \overset{i}{\to} B \overset{d}{\to} C$ in $\mathcal{C}$ is called *stable* if i is a semi-stable kernel and d is a semi-stable cokernel.

Remark 2.5. (1) Any additive category has a unique minimal exact structure, whose conflations are the split exact sequences (e.g., see [2]).

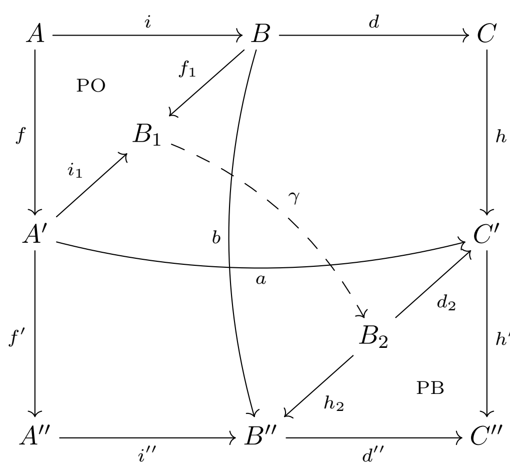
(2) Any quasi-abelian category has a unique maximal exact structure, whose conflations are the kernel-cokernel pairs [14].

(3) Any weakly idempotent complete category has a unique maximal exact structure, whose conflations are the stable exact sequences [3].

3 The Square-Cross Lemma

We are now in a position to establish the main result of the paper. In particular, our theorem holds for the maximal exact structure given by the kernel-cokernel pairs in a quasi-abelian category (and, in particular, abelian category), and for the maximal exact structure given by the stable exact sequences in a weakly idempotent complete additive category (see Remark 2.5). For a morphism $f : A \to B$ we denote by $\ker(f) : \mathrm{Ker}(f) \to A$ and $\mathrm{coker}(f) : B \to \mathrm{Coker}(f)$ its kernel and its cokernel respectively.

Theorem 3.1. *Consider the following commutative diagram in an additive category:*



where (B_1, i_1, f_1) *is the pushout of* i *and* f, *and* (B_2, d_2, h_2) *is the pullback of* d'' *and* h'. *Then:*

(i) There exists a unique morphism $\gamma : B_1 \to B_2$ *such that:*

$$\begin{bmatrix} h_2 \\ d_2 \end{bmatrix} \gamma \begin{bmatrix} i_1 & f_1 \end{bmatrix} = \begin{bmatrix} i''f' & b \\ a & hd \end{bmatrix}.$$

Assume further that the category is exact.

(ii) If the first row is a conflation, $a = 0$ *and* h *is a monomorphism, then* $i_1 = \ker(d_2\gamma)$. *If the last row is a conflation,* $a = 0$ *and* f' *is an epimorphism, then* $d_2 = \mathrm{coker}(\gamma i_1)$.

(iii) If the first row is a conflation, $a = 0$, $i''f' = 0$ *and* h, h' *are monomorphisms, then* $i_1 = \ker(\gamma)$. *If the last row is a conflation,* $a = 0$, $hd = 0$ *and* f, f' *are epimorphisms, then* $d_2 = \mathrm{coker}(\gamma)$.

(iv) If the rows and the columns are conflations, $a = 0$ *and* $b = 0$, *then the following sequence is exact:*

$$A \rightarrowtail^{f_1 i} B_1 \xrightarrow{\gamma} B_2 \overset{d''h_2}{\twoheadrightarrow} C''.$$

Proof: (i) We have $i''f'f = bi$ and $af = hdi$, whence the pushout property of $ABA'B_1$ implies that there exist a unique morphism $g_1 : B_1 \to B''$ such that $g_1 i_1 = i''f'$ and $g_1 f_1 = b$, and a unique morphism $g_1' : B_1 \to C'$ such that $a = g_1' i_1$ and $g_1' f_1 = hd$. Then $d''g_1 \begin{bmatrix} i_1 & f_1 \end{bmatrix} = \begin{bmatrix} d''i''f' & d''b \end{bmatrix} = \begin{bmatrix} h'a & h'hd \end{bmatrix} = h'g_1' \begin{bmatrix} i_1 & f_1 \end{bmatrix}$. The pushout property of $ABA'B_1$ ensures that the morphism $\begin{bmatrix} i_1 & f_1 \end{bmatrix} : A' \oplus B \to B_1$ is a cokernel and, in particular, an epimorphism. Then we must have $d''g_1 = h'g_1'$. Now by the pullback property of $B_2C'B''C''$ there exists a unique morphism $\gamma : B_1 \to B_2$ such that $h_2\gamma = g_1$ and $d_2\gamma = g_1'$. Then we have:

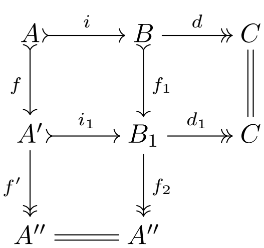
$$\begin{bmatrix} h_2 \\ d_2 \end{bmatrix} \gamma \begin{bmatrix} i_1 & f_1 \end{bmatrix} = \begin{bmatrix} h_2\gamma i_1 & h_2\gamma f_1 \\ d_2\gamma i_1 & d_2\gamma_1 f_1 \end{bmatrix} = \begin{bmatrix} g_1 i_1 & g_1 f_1 \\ g_1' i_1 & g_1' f_1 \end{bmatrix} = \begin{bmatrix} i''f' & b \\ a & hd \end{bmatrix}.$$

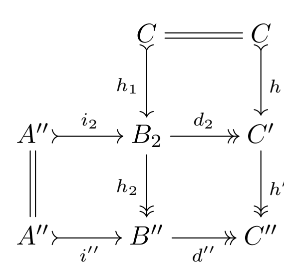
Since $\begin{bmatrix} i_1 & f_1 \end{bmatrix} : A' \oplus B \to B_1$ is an epimorphism, and $\begin{bmatrix} h_2 \\ d_2 \end{bmatrix} : B_2 \to B'' \oplus C'$ is a monomorphism by the pullback property of $B_2C'B''C''$, the morphism $\gamma : B_1 \to B_2$ with the required property is uniquely determined.

(ii) Assume that the first row is a conflation, $a = 0$ and h is a monomorphism. Then the pushout $ABA'B_1$ implies the existence of a conflation $A' \overset{i_1}{\to} B_1 \overset{d_1}{\to} C$. Since h is a monomorphism, it follows that $i_1 = \ker(d_1) = \ker(hd_1) = \ker(d_2\gamma)$.

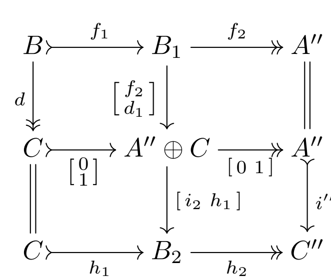
(iii) Assume that the first row is a conflation, $a = 0$, $i''f' = 0$ and h, h' are monomorphisms. Since pullbacks preserve monomorphisms, h_2 is a monomorphism. We have $h_2\gamma i_1 = i''f' = 0$, whence $\gamma i_1 = 0$. Now let $k : K \to B_1$ be a morphism such that $\gamma k = 0$. Then $d_2\gamma k = 0$, and since $i_1 = \ker(d_2\gamma)$, there is a unique morphism $k' : K \to A'$ such that $k = i_1 k'$. Hence $i_1 = \ker(\gamma)$.

(iv) First note that $f_1 i$ is an inflation and $d''h_2$ is a deflation by the axioms of an exact category. Also, f_1, i_1 are inflations, and d_2, h_2 are deflations, hence we may consider $f_2 = \operatorname{coker}(f_1)$, $d_1 = \operatorname{coker}(i_1)$, $i_2 = \ker(d_2)$ and $h_1 = \ker(h_2)$. The pushout $ABA'B_1$ and the pullback $B_2C'B''C''$ yield the following commutative diagrams in which the rows and the columns are conflations:





Then the following diagram is commutative:



The Short Five Lemma [2, Corollary 3.2] implies that $\begin{bmatrix} f_2 \\ d_1 \end{bmatrix}$ is a deflation and $\begin{bmatrix} i_2 & h_1 \end{bmatrix}$ is an inflation. Since the upper-left and the lower-right squares are both pullback-pushout squares [2, Proposition 2.12], it follows that $\ker\left(\begin{bmatrix} f_2 \\ d_1 \end{bmatrix}\right) = f_1 i$ and $\operatorname{coker}(\begin{bmatrix} i_2 & h_1 \end{bmatrix}) = d'' h_2$. We also have:

$$\begin{bmatrix} h_2 \\ d_2 \end{bmatrix}\begin{bmatrix} i_2 & h_1 \end{bmatrix}\begin{bmatrix} f_2 \\ d_1 \end{bmatrix}\begin{bmatrix} i_1 & f_1 \end{bmatrix} = \begin{bmatrix} h_2 i_2 & h_2 h_1 \\ d_2 i_2 & d_2 h_1 \end{bmatrix}\begin{bmatrix} f_2 i_1 & f_2 f_1 \\ d_1 i_2 & d_1 f_1 \end{bmatrix} = \begin{bmatrix} i'' & 0 \\ 0 & h \end{bmatrix}\begin{bmatrix} f' & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} i'' f' & 0 \\ 0 & hd \end{bmatrix}.$$

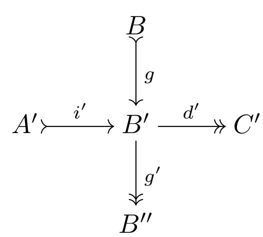
Now the uniqueness of γ implies that $\gamma = \begin{bmatrix} i_2 & h_1 \end{bmatrix}\begin{bmatrix} f_2 \\ d_1 \end{bmatrix}$. Finally, since $\begin{bmatrix} i_2 & h_1 \end{bmatrix}$ is a monomorphism and $\begin{bmatrix} f_2 \\ d_1 \end{bmatrix}$ is an epimorphism, it follows that:

$$\ker(\gamma) = \ker\left(\begin{bmatrix} f_2 \\ d_1 \end{bmatrix}\right) = f_1 i, \quad \operatorname{coker}(\gamma) = \operatorname{coker}(\begin{bmatrix} i_2 & h_1 \end{bmatrix}) = d'' h_2.$$

Hence the required sequence is exact. □

4 A bijective correspondence

Let $\mathcal{C}$ be an abelian category with the exact structure given by the kernel-cokernel pairs. For an object B', denote by $\mathcal{O}(B')$ the set of all diagrams from Theorem 3.1 with all rows and columns being kernel-cokernel pairs such that γ factors through B' as $\gamma = b_2 b_1$ for some kernel b_1 and cokernel b_2, and by $\mathcal{Q}(B')$ the set of all diagrams



where the horizontal and the vertical sequences are kernel-cokernel pairs.

Remark 4.1. Let k be a finite field, Λ a finitary (i.e., the Ext^1-groups of finite Λ-modules are finite) hereditary k-algebra and $\mathcal{P}$ the set of isomorphism classes of finite Λ-modules. Define a multiplication on the $\mathbb{Q}$-space with basis $\mathcal{P}$ by counting the number of submodules K of a given module M with prescribed isomorphism classes both of K and M/K. Thus one constructs the *Ringel-Hall algebra* $\mathcal{H}$ with coefficients in $\mathbb{Q}$ [12]. Green defined a comultiplication $\rho : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ such that $\mathcal{H}$ becomes a bialgebra, where ρ is an algebra homomorphism with respect to some twisted multiplication $\mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ [6]. In the proof of the compatibility for multiplication and comultiplication on $\mathcal{H}$, it appears the so-called *Green formula*, which is essentially based on a bijective correspondence between sets of the form $\mathcal{O}(B')$ and $\mathcal{Q}(B')$ (see [6, 13] for details). We show that such a homological result, which may be of independent interest, still holds in a more general categorical setting.

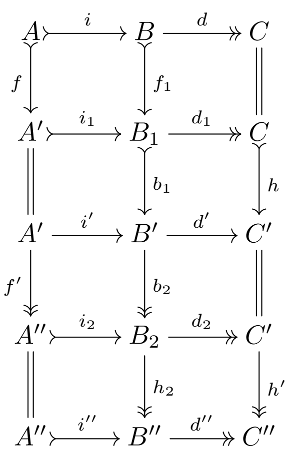
Theorem 4.2. *With the above notation,* $|\mathcal{O}(B')| = |\mathcal{Q}(B')|$.

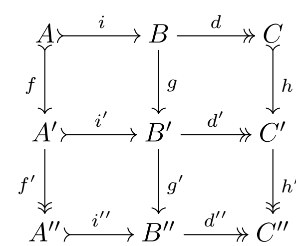
Proof: First, start with a diagram in $\mathcal{O}(B')$ and use the same notation as in the proof of Theorem 3.1. Denote $i' = b_1 i_1$, $d' = d_2 b_2$, $g = b_1 f_1$ and $g' = h_2 b_2$. Then

$$d' b_1 \begin{bmatrix} i_1 & f_1 \end{bmatrix} = \begin{bmatrix} d_2\gamma i_1 & d_2\gamma f_1 \end{bmatrix} = \begin{bmatrix} g_2 i_1 & g_2 f_1 \end{bmatrix} = \begin{bmatrix} 0 & hd \end{bmatrix} = h d_1 \begin{bmatrix} i_1 & f_1 \end{bmatrix},$$

$$\begin{bmatrix} h_2 \\ d_2 \end{bmatrix} b_2 i' = \begin{bmatrix} h_2\gamma i_1 \\ d_2\gamma i_1 \end{bmatrix} = \begin{bmatrix} g_1 i_1 \\ g_2 i_1 \end{bmatrix} = \begin{bmatrix} i'' f' \\ 0 \end{bmatrix} = \begin{bmatrix} h_2 \\ d_2 \end{bmatrix} i_2 f'.$$

Since $\begin{bmatrix} i_1 & f_1 \end{bmatrix}$ is an epimorphism and $\begin{bmatrix} h_2 \\ d_2 \end{bmatrix}$ is a monomorphism, $d' b_1 = h d_1$ and $b_2 i' = i_2 f'$. Then we have the following left-hand side commutative diagram:



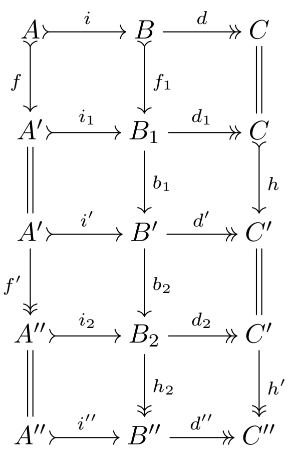


where the first two and the last two rows are kernel-cokernel pairs. Since $A'B'A''B_2$ is a pullback, $i' = \ker(d')$ by [8, Chapter I, Proposition 13.2], hence the middle row is also a kernel-cokernel pair. Since $g'g = 0$, the 3×3 Lemma [2, Corollary 3.6] applied to the above right-hand side commutative diagram yields that its middle column is a kernel-cokernel pair. Therefore, we have the required diagram in $\mathcal{Q}(B')$.

Now start with a diagram in $\mathcal{Q}(B')$. Let (A, i, f) be the pullback of the morphisms i' and g, and let (C'', d'', h') be the pushout of the morphisms d' and g'. Since $ABA'B'$ is a pullback square and $B'C'B''C''$ is a pushout square, it follows that $i = \ker(d'g)$ and $h' = \operatorname{coker}(d'g)$ [8, Chapter I, Proposition 13.2]. Since the category is abelian, we have

$$\operatorname{Coker}(i) = \operatorname{Coker}(\ker(d'g)) \cong \operatorname{Ker}(\operatorname{coker}(d'g)) = \operatorname{Ker}(h'),$$

and we denote it by C. Also, denote $d = \operatorname{coker}(i) : B \to C$ and $h = \ker(h') : C \to C'$. Similarly, let $f' = \operatorname{coker}(f) : A' \to A''$ and $i'' = \ker(d'') : A'' \to B''$. Considering the pushout (B_1, i_1, f_1) of the morphisms i and f, and the pullback (B_2, d_2, h_2) of the morphisms d'' and h', we obtain the following commutative diagram with rows being kernel-cokernel pairs:



where $g = b_1 f_1$ and $g' = h_2 b_2$. Clearly, b_1 is a kernel and b_2 is a cokernel. It follows that:

$$\begin{bmatrix} h_2 \\ d_2 \end{bmatrix} b_2 b_1 \begin{bmatrix} i_1 & f_1 \end{bmatrix} = \begin{bmatrix} h_2 b_2 \\ d_2 b_2 \end{bmatrix} \begin{bmatrix} b_1 i_1 & b_1 f_1 \end{bmatrix} = \begin{bmatrix} g' \\ d' \end{bmatrix} \begin{bmatrix} i' & g \end{bmatrix} = \begin{bmatrix} i'' f' & 0 \\ 0 & hd \end{bmatrix}.$$

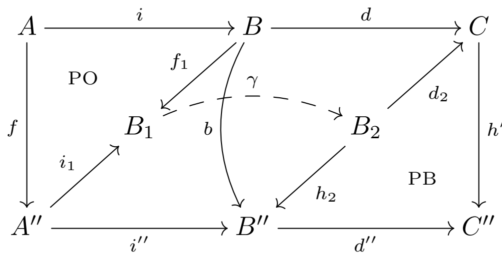
By Theorem 3.1, we must have $\gamma = b_2 b_1$. Therefore, we obtain the required diagram in $\mathcal{O}(B')$.

Finally, in order to prove that the above correspondences define a bijection, start with a diagram in $\mathcal{O}(B')$ and construct the first diagram from the proof of this theorem. Then we are done if we check that $ABA'B'$ is a pullback and $B'C'B''C''$ is a pushout. We only show the first part, the other one being dual. Let $u : D \to A'$ and $v : D \to B$ be morphisms such that $i'u = gv$. We have $hdv = d'gv = d'i'u = 0$, whence $dv = 0$ because h is a monomorphism. Then there exists a unique morphism $w : D \to A$ such that $iw = v$. We have $i'fw = giw = gv = i'u$, whence $fw = u$ because i' is a monomorphism. The uniqueness of w with the required properties follows easily by similar arguments. Hence $ABA'B'$ is a pullback. □

5 The Two-Square Lemma

In this section we use the Square-Cross Lemma in order to immediately deduce a generalized version of the Two-Square Lemma of Fay, Hardie and Hilton [4, Lemma 3]. This result was essential in [4] for providing an easy completely categorical construction of the connecting morphism from the Snake Lemma. For pre-abelian categories such a result was established by Generalov [5]. In particular, our theorem holds for the maximal exact structure given by the stable exact sequences in a weakly idempotent complete additive category (see Remark 2.5), extending the above results.

Theorem 5.1 (Two-Square Lemma)**.** *Consider the following commutative diagram in an additive category:*



where $di = 0$, $d''i'' = 0$, (B_1, i_1, f_1) *is the pushout of* i *and* f, *and* (B_2, d_2, h_2) *is the pullback of* d'' *and* h'. *Then:*

(i) There exists a unique morphism $\gamma : B_1 \to B_2$ *such that:*

$$\begin{bmatrix} h_2 \\ d_2 \end{bmatrix} \gamma \begin{bmatrix} i_1 & f_1 \end{bmatrix} = \begin{bmatrix} i'' & b \\ 0 & d \end{bmatrix}.$$

Assume further that the category is exact.

(ii) If the first row is a conflation, then $i_1 = \ker(d_2\gamma)$. *If the last row is a conflation, then* $d_2 = \operatorname{coker}(\gamma i_1)$.

(iii) If the first row is a conflation, $i'' = 0$ *and* h' *is a monomorphism, then* $i_1 = \ker(\gamma)$. *If the last row is a conflation,* $d = 0$ *and* f *is an epimorphism, then* $d_2 = \operatorname{coker}(\gamma)$.

Proof: (i) In Theorem 3.1 take $A' = A''$, $C' = C$, $f' = 1''_A$, $h = 1_C$ and $a = 0$.

(ii), (iii) These are immediate from Theorem 3.1, using (i). □

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