On the existence of solutions for a Fredholm-type integral inclusion

by
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Abstract

We consider an integral inclusion of Fredholm type and we obtain several existence results by using suitable fixed point theorems.

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1   Introduction

This paper is concerned with the following integral inclusion

$$x(t) = \lambda(t) + \int_0^T G(t,s)u(s)ds, \tag{1.1}$$

$$u(t) \in F(t,x(t)), \quad a.e. \ (I), \tag{1.2}$$

where $I = [0,T]$, $\lambda(.) : I \to X$, $F(.,.) : I \times X \to \mathcal{P}(X)$, $G(.,.) : I \times I \to \mathbf{R}$, are given mappings and X is a separable Banach space.

In the last years we observe a remarkable amount of interest in the study of existence of solutions of several boundary value problems associated to differential inclusions of the form

$$\mathcal{D}x \in F(t,x), \tag{1.3}$$

where $\mathcal{D}$ is a differential operator and $F(.,.) : I \times X \to \mathcal{P}(X)$ is a set-valued map ([1-5,8,13] etc.). Most of these existence results are obtained using fixed point techniques and are based on an integral form of the right inverse to the operator $\mathcal{D}$. This means that for every f the unique solution y of the equation $\mathcal{D}y = f$ can be written in the form $y = \mathcal{R}f$, where the operator $\mathcal{R}$ posses nonnegative Green's function. Therefore, the existence of solutions to problem (1.3) together with boundary conditions reduces to the existence of solutions to problem (1.1)-(1.2).

The aim of this paper is to provide two existence results for problem (1.1)-(1.2), when the set-valued map $F(.,.)$ has convex or nonconvex values. Our results are essentially based on

a nonlinear alternative of Leray-Schauder type and on Bressan-Colombo selection theorem for lower semicontinuous set-valued maps with decomposable values. In this way we extend and unify some of the results mentioned above. More exactly, the main results in all papers cited above become consequences of our Theorems 3.2 and 3.4 below.

We note that another existence result for problem (1.1)-(1.2) in the case when $F(t,.)$ is Lipschitz with nonconvex values, is obtain in our previous paper [7] by the application of the set-valued contraction principle due to Covitz and Nadler.

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel and in Section 3 we prove our main results.

2 Preliminaries

In this section we sum up some basic facts that we are going to use later.

Let (X, d) be a metric space with the corresponding norm $|.|$ and let $I = [0, T]$. Denote by $\mathcal{L}(I)$ the σ-algebra of all Lebesgue measurable subsets of I, by $\mathcal{P}(X)$ the family of all nonempty subsets of X and by $\mathcal{B}(X)$ the family of all Borel subsets of X. If $A \subset I$ then $\chi_A(.) : I \to \{0, 1\}$ denotes the characteristic function of A. For any subset $A \subset X$ we denote by $\overline{A}$ the closure of A.

Recall that the Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by $d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\}$, $d^*(A, B) = \sup\{d(a, B); a \in A\}$, where $d(x, B) = \inf_{y \in B} d(x, y)$.

As usual, we denote by $C(I, X)$ the Banach space of all continuous functions $x(.) : I \to X$ endowed with the norm $|x(.)|_C = \sup_{t \in I} |x(t)|$ and by $L^1(I, X)$ the Banach space of all (Bochner) integrable functions $x(.) : I \to X$ endowed with the norm $|x(.)|_1 = \int_I |x(t)| \mathrm{d}t$.

A subset $D \subset L^1(I, X)$ is said to be *decomposable* if for any $u(\cdot), v(\cdot) \in D$ and any subset $A \in \mathcal{L}(I)$ one has $u\chi_A + v\chi_B \in D$, where $B = I \backslash A$.

Consider $T : X \to \mathcal{P}(X)$ a set-valued map. A point $x \in X$ is called a fixed point for $T(.)$ if $x \in T(x)$. $T(.)$ is said to be bounded on bounded sets if $T(B) := \cup_{x \in B} T(x)$ is a bounded subset of X for all bounded sets B in X. $T(.)$ is said to be compact if $\overline{T(B)}$ is relatively compact for any bounded sets B in X. $T(.)$ is said to be totally compact if $\overline{T(X)}$ is a compact subset of X. $T(.)$ is said to be upper semicontinuous if for any open set $D \subset X$, the set $\{x \in X; T(x) \subset D\}$ is open in X. $T(.)$ is called completely continuous if it is upper semicontinuous and totally bounded on X.

It is well known that a compact set-valued map $T(.)$ with nonempty compact values is upper semicontinuous if and only if $T(.)$ has a closed graph.

In Theorem 9 in [11], O' Regan proved a nonlinear alternative of Leray-Schauder type. Namely, if D and $\overline{D}$ are open and closed subsets in a normed linear space X such that $0 \in D$ and if $T : \overline{D} \to \mathcal{P}(X)$ is a completely continuous set-valued map with compact convex values, then either the inclusion $x \in T(x)$ has a solution, or there exists $x \in \partial D$ (the boundary of D) such that $\lambda x \in T(x)$ for some $\lambda > 1$.

Obviously, the next two statements are consequences of this result.

Corollary 2.1 *Let $B_r(0)$ and $\overline{B_r(0)}$ be the open and closed balls in a normed linear space X centered at the origin and of radius r and let $T : \overline{B_r(0)} \to \mathcal{P}(X)$ be a completely continuous*

set-valued map with compact convex values. Then either

    i) the inclusion $x \in T(x)$ has a solution, or

    ii) there exists $x \in X$ with $|x| = r$ and $\lambda x \in T(x)$ for some $\lambda > 1$.

Corollary 2.2 *Let $B_r(0)$ and $\overline{B_r(0)}$ be the open and closed balls in a normed linear space X centered at the origin and of radius r and let $T : \overline{B_r(0)} \to X$ be a completely continuous single valued map with compact convex values. Then either*

    i) the equation $x = T(x)$ has a solution, or

    ii) there exists $x \in X$ with $|x| = r$ and $x = \lambda T(x)$ for some $\lambda < 1$.

We recall that a multifunction $T(.) : X \to \mathcal{P}(X)$ is said to be lower semicontinuous if for any closed subset $C \subset X$, the subset $\{s \in X; G(s) \subset C\}$ is closed. If $F(.,.) : I \times X \to \mathcal{P}(X)$ is a set-valued map with compact values and $x(.) \in C(I, X)$ we define

$$S_F(x) := \{f \in L^1(I, X); \quad f(t) \in F(t, x(t)) \quad a.e. \ (I)\}.$$

We say that $F(.,.)$ is of *lower semicontinuous type* if $S_F(.)$ is lower semicontinuous with closed and decomposable values. A set-valued map $G : I \to \mathcal{P}(X)$ with nonempty compact convex values is said to be *measurable* if for any $x \in X$ the function $t \to d(x, G(t))$ is measurable. A set-valued map $F(.,.) : I \times X \to \mathcal{P}(X)$ is said to be *Carathéodory* if $t \to F(t, x)$ is measurable for any $x \in X$ and $x \to F(t, x)$ is upper semicontinuous for almost all $t \in I$. $F(.,.)$ is said to be L^1-*Carathéodory* if for any $l > 0$ there exists $h_l(.) \in L^1(I, \mathbf{R})$ such that $\sup\{|v|; v \in F(t, x)\} \leq h_l(t)$ a.e. (I), $\forall x \in \overline{B_l(0)}$.

Finally, by a solution of problem (1.1)-(1.2) we mean a function $x(.) \in C(I, X)$ for which there exists $u(.) \in L^1(I, X)$ such that (1.1)-(1.2) are satisfied.

3   The main results

We consider first the case when $F(.,.)$ is convex valued.

Hypothesis 3.1 i) $F(.,.) : I \times X \to \mathcal{P}(X)$ *has nonempty compact convex values and is Carathéodory.*

    ii) *There exist $\varphi(.) \in L^1(I, \mathbf{R})$ with $\varphi(t) > 0$ a.e. (I) and there exists a nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ such that*

$$\sup\{|v|; \quad v \in F(t, x)\} \leq \varphi(t)\psi(|x|) \quad a.e. \ (I), \quad \forall x \in X.$$

    iii) *The mappings $\lambda(.) : I \to X$ and $G(.,.) : I \times I \to \mathbf{R}$ are continuous.*

Denote $G_0 := \sup_{t,s \in I} |G(t, s)|$.

Theorem 3.2 *Assume that Hypothesis 3.1 is satisfied and there exists $r > 0$ such that*

$$r > G_0|\varphi|_1\psi(r). \tag{3.1}$$

Then problem (1.1)-(1.2) has at least one solution $x(.)$ such that $|x(.)|_C < r$.

Proof: Let $r > 0$ be as in (3.1) and consider the set-valued map $T : \overline{B_r(0)} \to \mathcal{P}(C(I, X))$ defined by

$$T(x) := \{v(.) \in C(I, X); \ v(t) := \lambda(t) + \int_0^T G(t, s)f(s)ds, \ f \in \overline{S_F(x)}\}. \qquad (3.2)$$

We show that $T(.)$ satisfies the hypotheses of Corollary 2.1.

First, we show that $T(x) \subset C(I, X)$ is convex for any $x \in C(I, X)$.

If $v_1, v_2 \in T(x)$ then there exist $f_1, f_2 \in S_F(x)$ such that for any $t \in I$ one has

$$v_i(t) = \lambda(t) + \int_0^T G(t, s)f_i(s)ds, \quad i = 1, 2.$$

Let $0 \leq \alpha \leq 1$. Then for any $t \in I$ we have

$$(\alpha v_1 + (1 - \alpha)v_2)(t) = \lambda(t) + \int_0^T G(t, s)[\alpha f_1(s) + (1 - \alpha)f_2(s)]ds.$$

The values of $F(.,.)$ are convex, thus $S_F(x)$ is a convex set and hence $\alpha h_1 + (1-\alpha)h_2 \in T(x)$.

Secondly, we show that $T(.)$ is bounded on bounded sets of $C(I, X)$.

Let $B \subset C(I, X)$ be a bounded set. Then there exist $m > 0$ such that $|x|_C \leq m \ \forall x \in B$.

If $v \in T(x)$ there exists $f \in S_F(x)$ such that $v(t) = \lambda(t) + \int_0^T G(t, s)f(s)$ ds. One may write for any $t \in I$

$$|v(t)| \leq |\lambda(t)| + \int_0^T |G(t, s)|.|f(s)|ds \leq |\lambda(t)| + \int_0^T |G(t, s)|\varphi(s)\psi(|x(t)|)ds$$

and therefore

$$|v|_C \leq |\lambda|_C + G_0|\varphi|_1\psi(m) \quad \forall v \in T(x),$$

i.e., $T(B)$ is bounded.

We show next that $T(.)$ maps bounded sets into equi-continuous sets.

Let $B \subset C(I, X)$ be a bounded set as before and $v \in T(x)$ for some $x \in B$. There exists $f \in S_F(x)$ such that $v(t) = \lambda(t) + \int_0^T G(t, s)f(s)ds$. Then for any $t, \tau \in I$ we have

$$|v(t) - v(\tau)| \leq |\int_0^T G(t, s)f(s)ds - \int_0^T G(\tau, s)f(s)ds| \leq$$

$$\leq \sup_{s \in I} |G(t, s) - G(\tau, s)|.|\varphi|_1\psi(m).$$

Let us note that

$$\lim_{t \to tau} \max_{s \in I} |G(t, s) - G(\tau, s)| = 0 \qquad (3.3)$$

Assume by contrary that there exists $\varepsilon_0 > 0$, $t_m \to \tau$, $s_m \in I$ such that $\varepsilon_0 < |G(t_m, s_m) - G(\tau, s_m)|$. Since I is compact we may assume that $s_m \to s_0 \in I$ as $m \to \infty$. Therefore,

$$\varepsilon_0 < |G(t_m, s_m) - G(\tau, s_m)| \leq |G(t_m, s_m) - G(\tau, s_0)| + |G(\tau, s_0) - G(\tau, s_m)|.$$

We take $m \to \infty$ and by the continuity of $G(.,.)$ we get a contradiction.

It follows that $|v(t) - v(\tau)| \to 0$ as $t \to \tau$. Therefore, $T(B)$ is an equi-continuous set in $C(I, X)$.

We apply now Arzela-Ascoli's theorem we deduce that $T(.)$ is completely continuous on $C(I, X)$.

In the next step of the proof we prove that $T(.)$ has a closed graph.

Let $x_n \in C(I, X)$ be a sequence such that $x_n \to x^*$ and $v_n \in T(x_n)$ $\forall n \in \mathbf{N}$ such that $v_n \to v^*$. We prove that $v^* \in T(x^*)$.

Since $v_n \in T(x_n)$, there exists $f_n \in S_F(x_n)$ such that $v_n(t) = \lambda(t) + \int_0^T G(t, s) f_n(s) ds$.

Define $\Gamma : L^1(I, X) \to C(I, X)$ by $(\Gamma(f))(t) := \lambda(t) + \int_0^T G(t, s) f(s) ds$. One has $\max_{t \in I} |v_n(t) - v^*(t)| = |v_n(.) - v^*(.)|_C \to 0$ as $n \to \infty$

It follows (e.g., [10]) that $\Gamma \circ S_F$ has closed graph and from the definition of Γ we get $v_n \in \Gamma \circ S_F(x_n)$. Since $x_n \to x^*$, $v_n \to v^*$ it follows the existence of $f^* \in S_F(x^*)$ such that $v^*(t) = \lambda(t) + \int_0^T G(t, s) f^*(s) ds$.

Therefore, $T(.)$ is upper semicontinuous and compact on $\overline{B_r(0)}$. We apply Corollary 2.1 to deduce that either i) the inclusion $x \in T(x)$ has a solution in $\overline{B_r(0)}$, or ii) there exists $x \in X$ with $|x|_C = r$ and $\lambda x \in T(x)$ for some $\lambda > 1$.

Assume that ii) is true. With the same arguments as in the second step of our proof we get $r = |x(.)|_C \leq G_0 |\varphi|_1 \psi(r)$ which contradicts (3.1). Hence only i) is valid and theorem is proved.
□

We consider now the case when $F(.,.)$ is not necessarily convex valued. Our existence result in this case is based on the Leray-Schauder alternative for single valued maps and on Bressan Colombo selection theorem.

Hypothesis 3.3 i) $F(.,.) : I \times X \to \mathcal{P}(X)$ *has compact values,* $F(.,.)$ *is* $\mathcal{L}(I) \otimes \mathcal{B}(X)$ *measurable and* $x \to F(t, x)$ *is lower semicontinuous for almost all* $t \in I$.

ii) *There exist* $\varphi(.) \in L^1(I, \mathbf{R})$ *with* $\varphi(t) > 0$ *a.e.* (I) *and there exists a nondecreasing function* $\psi : [0, \infty) \to (0, \infty)$ *such that*

$$\sup\{|v|; \quad v \in F(t, x)\} \leq \varphi(t) \psi(|x|) \quad a.e. \ (I), \quad \forall x \in X.$$

iii) *The mappings* $\lambda(.) : I \to X$ *and* $G(.,.) : I \times I \to \mathbf{R}$ *are continuous.*

Theorem 3.4 *Assume that Hypothesis 3.3 is satisfied and there exists* $r > 0$ *such that condition (3.1) is satisfied.*

Then problem (1.1)-(1.2) has at least one solution on I.

Proof: We note first that if Hypothesis 3.3 is satisfied then $F(.,.)$ is of lower semicontinuous type (e.g., [9]). Therefore, with Bressan-Colombo selection theorem ([6]) we deduce that there exists $f(.) : C(I, X) \to L^1(I, X)$ such that $f(x) \in S_F(x)$ $\forall x \in C(I, X)$.

We consider the corresponding problem

$$x(t) = \lambda(t) + \int_0^T G(t,s)f(x(s))ds, \quad t \in I \tag{3.4}$$

in the space $C(I, X)$.

Let $r > 0$ that satisfies condition (3.1) and define the set-valued map $T : \overline{B_r(0)} \to \mathcal{P}(C(I, X))$ by

$$(T(x))(t) := \lambda(t) + \int_0^T G(t,s)f(x(s))ds.$$

Obviously, the integral equation (3.4) is equivalent with the operator equation

$$x(t) = (T(x))(t), \quad t \in I. \tag{3.5}$$

It remains to show that $T(.)$ satisfies the hypotheses of Corollary 2.2.

We show that $T(.)$ is continuous on $\overline{B_r(0)}$. From Hypotheses 3.3. ii) we have

$$|f(x(t))| \le \varphi(t)\psi(|x(t)|) \quad a.e. \ (I)$$

for all $x(.) \in C(I, X)$. Let $x_n, x \in \overline{B_r(0)}$ such that $x_n \to x$. Then

$$|f(x_n(t))| \le \varphi(t)\psi(r) \quad a.e. \ (I).$$

From Lebesgue's dominated convergence theorem and the continuity of $f(.)$ we obtain, for all $t \in I$

$$\lim_{n\to\infty}(T(x_n))(t) = \lim_{n\to\infty}(\lambda(t) + \int_0^T G(t,s)f(x_n(s))ds) =$$

$$= \lambda(t) + \int_0^T G(t,s)f(x(s))ds = (T(x))(t),$$

i.e., $T(.)$ is continuous on $\overline{B_r(0)}$.

Repeating the arguments in the proof of Theorem 3.2 with corresponding modifications it follows that $T(.)$ is compact on $\overline{B_r(0)}$. We apply Corollary 2.2 and we find that either i) the equation $x = T(x)$ has a solution in $\overline{B_r(0)}$, or ii) there exists $x \in X$ with $|x|_C = r$ and $x = \lambda T(x)$ for some $\lambda < 1$.

As in the proof of Theorem 3.2 if the statement ii) holds true, then we obtain a contradiction to (3.1). Thus only the statement i) is true and problem (1.1) has a solution $x(.) \in C(I, X)$ with $|x(.)|_C < r$. □

Remark 3.5 Similar results to the ones in Theorems 3.2 and 3.4 may be obtained if instead of Fredholm type integral inclusions we consider Volterra type integral inclusions of the form

$$x(t) = \lambda(t) + \int_0^t G(t,s)u(s)ds, \quad u(t) \in F(t, x(t)), \quad a.e. \ (I).$$

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