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The circular Morse-Smale characteristic of closed surfaces

by
Dorin Andrica, Dana Mangra and Cornel Pintea

Abstract

In this paper we first compute the circular version of the Morse-Smale characteristic of all closed surfaces. We also observe that the critical points of the real valued height functions alongside those of some S^1 valued functions on a surface $\Sigma \subset \mathbb{R}^3$, are the characteristic points with respect to some involutive distributions. We finally study the size of the characteristic set of the compact orientable surface of genus g, embedded in a certain way in the first Heisenberg group, with respect to the horizontal distribution of the Heisenberg group.

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1 Introduction and preliminary results

The minimum number of critical points of all Morse functions on a manifold M, equally called the *Morse-Smale characteristic* of M, is an important tool in differential topology as, for example, it is related to the minimum number of cells in the CW-decompositions of M up to homotopy. It is also a lower bound for the total curvature of M with respect to its embeddings in Euclidean spaces. In fact, the Chern-Lashof conjecture states that the Morse-Smale characteristic of a manifold M is precisely the infimum of the total curvatures of M with respect to its embeddings in Euclidean spaces. Recall that the Chern-Lashof conjecture holds true for several manifolds [7], [8].

The *Morse-Smale characteristic* of a compact smooth manifold is therefore worth to be studied. It is defined by

$$\gamma(M) = \min\{\text{card}(C(f)) : f \in \mathfrak{F}(M)\},$$

where $\mathfrak{F}(M)$ denotes the set of all real-valued Morse functions defined on M. For details, examples, properties and concrete computations we refer to monograph [1, pp.106-129].

The minimality of the number of cells in the CW-decompositions of M up to homotopy emphasizes the importance of $\gamma(M)$ and provides a serious reason why its computation is rather a hard problem in differential topology. On the other hand, the existence of F-perfect Morse functions on M, for some given field F, is characterized by the equality between $\gamma(M)$ and the sum of F-Betti numbers of M [1, Th. 4.2.3]. If M is additionally endowed with a symplectic structure, then the latter $\mathbb{Z}_2$-sum associated to a coisotropic submanifold of M is a lower bound for the number of leaf-wise fixed points of suitable Hamiltonian diffeomorphisms and coisotropic submanifolds of M [22].

For maps with higher dimensional target manifold having finitely many critical points we refer the reader to the recent work by Funar [11] and the references therein.

The circular version of the Morse-Smale characteristic was introduced in [2]

Definition 1.1. If M is a differential manifold, then the *circular Morse-Smale characteristic* of M is defined by

$$\gamma_{S^1}(M) := \min\{\mathrm{card}(C(f)) : f \in \mathcal{F}(M, S^1)\} \tag{1.1}$$

where $\mathcal{F}(M, S^1)$ stands for the collection of all circular Morse functions $f{:}M \to S^1$.

The number $\gamma_{S^1}(M)$ is a special case of φ-category of a pair of manifolds (M, N) corresponding to a family of smooth mappings $\mathfrak{F} \subseteq C^\infty(M, N)$ (see the recent expository paper [4]), where N is the circle S^1 and the family $\mathfrak{F}$ is given by the set of all circle-valued Morse functions $f : M \to S^1$ defined on M. The systematic study of circle-valued Morse functions was initiated by S.P.Novikov in 1980. The motivation came from a problem in hydrodynamics, where the application of the variational approach led to a multi-valued Lagrangian. The formulation of the circle-valued Morse theory as a new branch of differential topology with its own problems was outlined also by S.P.Novikov. For specific definitions and properties we refer to the recent monographs of M.Farber [10] and A.Pajitnov [17].

Some properties of the circular Morse-Smale characteristic are already proved in the papers [2] and [3]. For instance, for every closed manifold (i.e.compact and without boundary) we have the inequality

$$\gamma_{S^1}(M) \leq \gamma(M), \tag{1.2}$$

as every Morse real valued function composed with the universal cover $\exp : \mathbb{R} \longrightarrow S^1$ produces a circle valued Morse function. These property imply that $\gamma_{S^1}(M)$ is finite whenever M is compact.

Proposition 1.2. *If* $\mathrm{Hom}\,(\pi_1(M), \mathbb{Z}) = 0$ *for some connected differential manifold* M*, then* $\gamma_{S^1}(M) = \gamma(M)$*. In particular* $\gamma_{S^1}(M) = \gamma(M)$ *whenever* M *is connected and simply-connected.*

Proof: Indeed, in this case every smooth circle valued function $f : M \longrightarrow S^1$ can be lifted to a smooth real valued function $\tilde{f} : M \longrightarrow \mathbb{R}$, i.e. $\exp \circ \tilde{f} = f$. Since the universal cover $\exp : \mathbb{R} \longrightarrow S^1$ is a local diffeomorphism, it follows that $C(f) = C(\tilde{f}) \Rightarrow \mathrm{card}\,(C(f)) = \mathrm{card}\left(C(\tilde{f})\right) \geq \gamma(M)$ for every smooth function $f : M \longrightarrow S^1$. This shows that $\gamma_{S^1}(M) \geq \gamma(M)$, which combined to the general inequality (1.2), leads to the desired equality. $\square$

Corollary 1.3. *If $m, n \geq 2$ are natural numbers, then $\gamma_{S^1}(S^n) = \gamma(S^n) = 2$ and $\gamma_{S^1}(\mathbb{RP}^n) = \gamma(\mathbb{RP}^n) = n+1$.*

Proof: While the equalities $\gamma_{S^1}(S^n) = \gamma(S^n)$, $\gamma_{S^1}(\mathbb{RP}^n) = \gamma(\mathbb{RP}^n)$ follow from Proposition 1.2, the equality $\gamma(S^n) = 2$ is obvious. On the other hand,

$$\gamma(\mathbb{RP}^n) \leq \mathrm{card}(C(f)) = n+1,$$

where $f : \mathbb{RP}^n \longrightarrow \mathbb{R}$ is the Morse function defined by

$$f_n([x_1, \ldots, x_{n+1}]) = \frac{x_1^2 + 2x_2^2 + \cdots + nx_n^2 + (n+1)x_{n+1}^2}{x_1^2 + x_2^2 + \cdots + x_n^2 + x_{n+1}^2}$$

whose critical set consists of $n+1$ critical points of indices $0, 1, \ldots, n$ (see e.g. [14, pp. 84,85]). Finally,

$$\gamma(\mathbb{RP}^n) \geq \mathrm{cat}(\mathbb{RP}^n) = n+1$$

where $\mathrm{cat}(\mathbb{RP}^n)$ stands for the Lusternik-Schnirelmann category of the projective space $\mathbb{RP}^n$ (see e.g. [18, pp. 190-192]). $\square$

Corollary 1.4. *If $m_1, \ldots, m_k \geq 2$ are natural numbers, then*

$$\begin{aligned}
&\gamma_{S^1}(S^{m_1} \times \cdots \times S^{m_k}) = \gamma(S^{m_1} \times \cdots \times S^{m_k}) = 2^k, \\
&\gamma_{S^1}(\mathbb{RP}^{m_1} \times \cdots \times \mathbb{RP}^{m_k}) = \gamma(\mathbb{RP}^{m_1} \times \cdots \times \mathbb{RP}^{m_k}) = (m_1+1) \cdots (m_k+1).
\end{aligned}$$

Proof: While the equalities $\gamma_{S^1}(S^{m_1} \times \cdots \times S^{m_k}) = \gamma(S^{m_1} \times \cdots \times S^{m_k})$ and $\gamma_{S^1}(\mathbb{RP}^{m_1} \times \cdots \times \mathbb{RP}^{m_k}) = \gamma(\mathbb{RP}^{m_1} \times \cdots \times \mathbb{RP}^{m_k})$ follow from Proposition 1.2, the equality $\gamma(S^{m_1} \times \cdots \times S^{m_k}) = 2^k$, which works for $m_1, \ldots, m_k \geq 1$, appears in [1, Ex. 4.2.9] and the equality $\gamma(\mathbb{RP}^{m_1} \times \cdots \times \mathbb{RP}^{m_k}) = (m_1+1) \cdots (m_k+1)$ follow from [1, Th. 4.2.7] combined with Corollary 1.3. $\square$

Another property relating the circular Morse-Smale characteristics of the total and base spaces of a finite-fold covering map is provided by the following:

Proposition 1.5. *If $\tilde{M}$ is a k-fold cover of M, then $\gamma_{S^1}(\tilde{M}) \leq k \cdot \gamma_{S^1}(M)$.*

Proof: Let $\pi : \tilde{M} \longrightarrow M$ be the k-fold covering map and $f : M \longrightarrow S^1$ be a Morse function. Then $f \circ \pi : \tilde{M} \longrightarrow S^1$ is obviously a Morse function and $\mathrm{card}(C(f \circ \pi)) = k \cdot \mathrm{card}(C(f))$, as $C(f \circ \pi)) = \pi^{-1}(C(f))$. This shows that the following relations hold

$$\mathrm{card}(C(f)) = \frac{1}{k}\mathrm{card}(C(f \circ \pi)) \geq \frac{1}{k}\gamma_{S^1}(\tilde{M}). \tag{1.3}$$

Consequently $\gamma_{S^1}(M) \geq \frac{1}{k}\gamma_{S^1}(\tilde{M})$, as the relations (1.3) are satisfied for every Morse function $f : M \longrightarrow S^1$. $\square$

It is an interesting and challenging problem to compute the circular Morse-Smale category for closed manifolds M for which $\mathrm{Hom}\,(\pi_1(M), \mathbb{Z}) \neq 0$. The main purpose of this paper is to do this computation for smooth, compact, connected surfaces of genus $g \geq 1$. Recall that a such surface is

$$\Sigma_g = T^2 \# T^2 \# \cdots \# T^2 \text{ or } g'\mathbb{RP}^2 = \mathbb{RP}^2 \# \mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$$

a connected sum of some g copies of the 2-dimensional torus $T^2 = S^1 \times S^1$ or a connected sum of some g' copies of the projective plane $\mathbb{RP}^2$. We can extend the definition for $g = 0$ by considering $\Sigma_0 = S^2$, the 2-dimensional sphere. From the classification theorem of surfaces, it follows that every smooth, compact, connected and orientable surface without boundary, is diffeomorphic to some Σ_g. Recall that the Morse-Smale characteristic of surfaces was completely determined by Kuiper [13] who proved the formula $\gamma(\Sigma) + \chi(\Sigma) = 4$ for every compact connected surface S. In this paper we will prove that for every closed surface Σ, except for the sphere S^2 and the projective plane $\mathbb{RP}^2$, one has $\gamma_{_{S^1}}(\Sigma) + \chi(\Sigma) = 0$.

2 The circular Morse-Smale characteristic of the compact surfaces

According to the results of the previous section, we have $\gamma_{_{S^1}}(\Sigma_0) = \gamma_{_{S^1}}(S^2) = \gamma(S^2) = 2$ and $\gamma_{_{S^1}}(\mathbb{RP}^2) = 3$, as follows from Corollary 1.3. Also $\gamma_{_{S^1}}(\Sigma_1) = \gamma_{_{S^1}}(T^2) = 0$, as the projection $T^2 = S^1 \times S^1 \to S^1$ is a submersion and it has no critical points. More generally, we shall prove the following:

Theorem 2.1. *The circular Morse-Smale characteristic of a closed surface $\Sigma \neq \mathbb{RP}^2$ is*

$$\gamma_{_{S^1}}(\Sigma) = |\chi(\Sigma)|. \tag{2.1}$$

Proof: We only need to consider the situation $b_1(\Sigma) \geq 2$, where $b_k(X)$ stands for the k-th Betti number of X with $\mathbb{Z}_2$-coefficients, as for the sphere the equality $\gamma_{_{S^1}}(S^2) = 2 = |\chi(S^2)|$ is obvious.

Let $f : \Sigma \longrightarrow S^1 = \mathbb{R}/2\pi\mathbb{Z}$ be a Morse function and $t_0 \in S^1$ be a regular value of f such that $[t_0 - \varepsilon, t_0 + \varepsilon] \subseteq S^1$ is an arc of regular values, for some $\varepsilon > 0$. We may assume, for simplicity, that $t_0 = 0 \in \mathbb{R}/2\pi\mathbb{Z}$.

The codimension 1 submanifold $\Sigma_0 := f^{-1}(0)$ is cooriented (by the differential of f) and therefore defines a cohomology class in $H^1(\Sigma, \mathbb{Z})$. This cohomology class is actually f^*u, where u is the generator of $H^1(S^1, \mathbb{Z})$.

The complement $\Sigma_\varepsilon := \Sigma \setminus f^{-1}((-\varepsilon, \varepsilon))$ is a manifold whose boundary consists of two parts

$$\partial_\pm \Sigma_\varepsilon = \Sigma_0 \times \{\mp\varepsilon\}.$$

Moreover, Σ_ε is connected if and only if $f^*u \neq 0$, i.e. $f : \Sigma \longrightarrow S^1$ does not admit any lift to any smooth map $\Sigma \longrightarrow \mathbb{R}$. Here the identification between $H^1(\Sigma, \mathbb{Z})$ and $\mathrm{Hom}(\pi_1(\Sigma), \mathbb{Z})$ is being used.

The region $C_\varepsilon := f^{-1}([-\varepsilon, \varepsilon])$ is diffeomorphic to the cylinder $\Sigma_0 \times [-\varepsilon, \varepsilon]$ and the restriction $f_0 : C_\varepsilon \longrightarrow [-\varepsilon, \varepsilon]$, $f_0(x) = f(x)$ is a real-valued Morse function without critical points. The gradient flow of f_0 defines a diffeomorphism $\Phi : \partial_+\Sigma_\varepsilon \longrightarrow \partial_-\Sigma_\varepsilon$. The restriction of f to Σ_ε is a Morse function

$$f_\varepsilon : \Sigma_\varepsilon \longrightarrow [\varepsilon, 2\pi - \varepsilon] = S^1 \setminus (-\varepsilon, \varepsilon)$$

such that the gradient points inwardly on $\partial_-\Sigma_\varepsilon$ and outwardly on $\partial_+\Sigma_\varepsilon$. Note that on $\partial_+\Sigma_\varepsilon$ one gets $f = 2\pi - \varepsilon$.

The original function can be now recovered from the triplet $(\Sigma_\varepsilon, f_\varepsilon, \Phi)$. The number of critical points of f is equal to the numnber of critical points of f_ε, as $C(f) = C(f_\varepsilon)$.

The Morse inequalities imply that the number of critical points of f_ε is bounded from below by the sum

$$b(\Sigma_\varepsilon, \partial_+\Sigma_\varepsilon) := \sum_{k\geq 0} b_k(\Sigma_\varepsilon, \partial_+\Sigma_\varepsilon).$$

If $f^*u \neq 0$, then $b_1(M) \geq 2$, Σ_ε is connected and

$$b_0(\Sigma_\varepsilon, \partial_+\Sigma_\varepsilon) = b_2(\Sigma_\varepsilon, \partial_+\Sigma_\varepsilon) = 0.$$

Hence, in this case

$$b(\Sigma_\varepsilon, \partial_+\Sigma_\varepsilon) = b_1(\Sigma_\varepsilon, \partial_+\Sigma_\varepsilon) = |\chi(\Sigma_\varepsilon, \partial_+\Sigma_\varepsilon)|.$$

The boundary components $\partial_\pm\Sigma_\varepsilon$ are disjoint unions of circles and have trivial Euler-characteristics therefore. Using the additivity of the Euler characteristic with respect to the increasing squences $\emptyset \subset \partial_\pm\Sigma_\varepsilon \subset \Sigma_\varepsilon$ (see e.g. [18, p. 218]) we obtain

$$\chi(\Sigma_\varepsilon, \partial_+\Sigma_\varepsilon) = \chi(\Sigma_\varepsilon, \partial_-\Sigma_\varepsilon) = \chi(\Sigma_\varepsilon).$$

The Mayer-Vietoris principle applied to $M = \Sigma_\varepsilon \cup C_\varepsilon$ then yields

$$\chi(\Sigma_\varepsilon) = \chi(M).$$

Consequently we get successively:

$$\gamma_{_{S^1}}(\Sigma) \geq b(\Sigma_\varepsilon, \partial_+\Sigma_\varepsilon) = |\chi(\Sigma_\varepsilon, \partial_+\Sigma_\varepsilon)| = |\chi(\Sigma_\varepsilon)| = |\chi(\Sigma)|.$$

To prove the opposite inequality when $b_1(\Sigma) \geq 2$, i.e. $H^1(\Sigma, \mathbb{Z}) \neq 0$ we first observe that if Σ is a Riemann surface without boundary, then $\gamma(\Sigma) = b_1(\Sigma) + 2$.

Indeed, the classical description of the Riemann surface Σ with $b_1(\Sigma) = n > 0$ as a polygon with $2n$ edges identified according to the description

$$\begin{array}{l} a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1},\ n = 2g,\ \Sigma \text{ orientable} \\ a_1 a_1 \cdots a_n a_n,\ n = 2g,\ \Sigma \text{ is nonorientable.} \end{array}$$

This description produces a handle decomposition of the Riemann surface Σ with a single 0-handle, n 1-handles and a single 2-handle. As shown in [15], any handle decomposition of a manifold corresponds to a Morse function on the manifold with one critical point for each handle. In our case there is a Morse function on the Riemann surface Σ with $b_1(\Sigma) + 2$ critical points.

If $H^1(\Sigma,\mathbb{Z}) \neq 0$, i.e. $b_1(\Sigma) \geq 2$, then one can find a nonseparating circle C in Σ representing a primitive homology class in $H_1(\Sigma,\mathbb{Z})$. Cut Σ along C to obtain a connected Riemann surface Σ' with two boundary components C and $-C$. A simple computation shows that $\chi(\Sigma') = \chi(\Sigma)$. Cap the components C and $-C$ of the boundary of Σ' with two disks $D_\pm$ to obtain a closed surface S satisfying

$$\chi(S) = \chi(\Sigma') + 2 = \chi(\Sigma) + 2.$$

In particular

$$b_1(S) = b_1(\Sigma) - 2.$$

We now choose a Morse function $h : S \longrightarrow \mathbb{R}$ which has a minimal number of critical points $b_1(S) + 2$. This function has a unique maximum point p_+ and a unique minimum point p_-. By composing h with a suitable diffeomorphism of S we may assume that $p_\pm$ is the center of $D_\pm$ and that h has no other critical points inside these disks. The restriction of h to Σ' has $b_1(S) = b_1(\Sigma) - 2 = |\chi(\Sigma)|$ critical points. We now extend h to a Morse function $\Sigma \longrightarrow S^1$ with the same number of critical points by using the same type of arguments we used in the first part of the proof. □

3 On the minimum number of characteristic points

If $\Sigma \subset \mathbb{R}^3$ is a surface and $f : \mathbb{R}^3 \longrightarrow N$ is a submersion, where N is either the real line or the circle S^1, then the critical points of the restriction $f|_\Sigma$ are the *characteristic points*, equally called *tangency points* [6] in the higher codimension case, of the surface Σ with respect to the involutive distribution $\{\ker(df)_x\}_{x\in\mathbb{R}^3}$ of the tangent planes to the fibers of f. In other words the *characteristic point* looks like an extended concept for the *critical point* of real or S^1 valued functions. Indeed, while the distributions behind the notion of *critical point* of real or circle valued functions are some particular involutive distributions, the notion of *characteristic point* is defined in relation with an arbitrary distribution such as highly noninvolutive distributions coming, for instance, from contact forms [9]. In fact we will be dealing, in this section, with the minimum number of characteristic points with respect to the highly noninvolutive horizontal distribution of the Heisenberg group $\mathbb{H}^n = (\mathbb{R}^{2n+1}, *)$, namely

$$\begin{aligned}\mathcal{H}_n &= \operatorname{Span}\{X_1,\dots,X_n,Y_1,\dots,Y_n\}\\ &= \{\mathcal{H}_{n,p} := \operatorname{Span}\{X_{1,p},\dots,X_{n,p},Y_{1,p},\dots,Y_{n,p}\}\}_{p\in\mathbb{H}^n},\end{aligned}$$

where $X_i = \partial_{x_i} + 2y_i\partial_t$ and $Y_i = \partial_{y_i} - 2x_i\partial_t$ for $i = 1,\dots,n$. Some special attention will be payed to the minimum characteristic number of the compact orientable surface of genus g, embedded into the first Heisenberg group $\mathbb{H}^1$ with respect to its horizontal distribution $\mathcal{H}_1 = \operatorname{Span}\{X,Y\}$, where $X = \partial_x + 2y\partial_t$, $Y = \partial_y - 2x\partial_t$.

Let us consider a C^1 smooth hypersurface $S \subseteq \mathbb{R}^{2n+1}$. The *characteristic set* of S is defined by

$$C(S,\mathcal{H}_n) := \{p \in S \,:\, T_pS = \mathcal{H}_{n,p}\}.$$

While the reference [5] provides upper bounds for the size of $C(S,\mathcal{H}_n)$ is terms of Hausdorff dimension, we emphasize here the possibility to get finite characteristic sets $C(\Sigma_g,\mathcal{H}_1)$ for suitable embeddings of the closed connected surface Σ_g of genus g and provide some bounds on its cardinality. Note that the characteristic set $C(\Sigma, \mathcal{H}_1)$ of some surface $\Sigma \subset \mathbb{R}^3$ with respect to the horizontal distribution $\mathcal{H}_1$ is the set of singularities of the vector field Z_Σ on Σ obtained by projecting orthogonally $X \wedge Y$ on the tangent spaces of Σ, i.e $C(\Sigma, \mathcal{H}_1) = \mathrm{Sing}(Z_\Sigma)$.

Definition 3.1. *If S is a C^1 smooth hypersurface of $\mathbb{R}^{2n+1}$, then the* minimum characteristic number *of S relative to the distribution $\mathcal{H}_n$ on $\mathbb{R}^{2n+1}$ is defined by*

$$mcn(S, \mathcal{H}_n) := \min\{\mathrm{card}\,(C(f(S), \mathcal{H}_n)) : f \in \mathrm{Embed}(S, \mathbb{R}^{2n+1})\},$$

where $\mathrm{Embed}(S, \mathbb{R}^{2n+1})$ *stands for the set of all C^1 embeddings of S into $\mathbb{R}^{2n+1}$.*

Remark 3.2. *If M is a compact orientable $2n$-manifold of non-zero Euler-Poincaré characteristic, then, according to [6, Ex. 8.9], $mcn(M, \mathcal{H}_n) \geq 2$. In fact $mcn(S^{2n}, \mathcal{H}_n) = 2$, as the Euler-Poincaré characteristic of the sphere S^{2n} is two and it admits an embedding into $\mathbb{H}^n$ with exactly two characteristic points. The image of this embedding is the well-known Korányi sphere. On the other hand the standard torus $T^{2n} \subset \mathbb{H}^n$ has no characteristic points at all (see [21]), i.e. $mcn(T^{2n}, \mathcal{H}_n) = 0$.*

Theorem 3.3. *If $g \geq 2$, then $2g - 2 \leq mcn(\Sigma_g, \mathcal{H}_1) \leq 4g - 4$.*

The lower bound $2g-2$ for $mcn(\Sigma_g, \mathcal{H}_1)$ follows from [12, Theorem 4.6.14] combined with a stability result on Morse-Smale vector fields by Peixoto (see [19, 20]). Another argument towards this lower bound relies on the Poincaré-Hopf theorem

$$2 - 2g = \chi\left(\Sigma_g\right) = \sum_{x \in \mathrm{Sing}(Z_{\Sigma_g})} \mathrm{index}_z(Z_{\Sigma_g}).$$

applied to the vector field Z_Σ, whose singularities are generally of index ± 1 [9]. The details on the upperbound $4g - 4$ are rather elementary.

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Department of Mathematics,
"Babeş-Bolyai" University
400084 M. Kogălniceanu 1,
Cluj-Napoca, Romania
E-mails: dandrica@math.ubbcluj.ro
mdm_dana@yahoo.com
cpintea@math.ubbcluj.ro