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Uniform exponential stability for discrete non-autonomous systems via discrete evolution semigroups

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Abstract

We prove that a discrete evolution family

$$\mathcal{U} = \{U(m,n)\}_{m\geq n\in\mathbb{Z}_+}$$

of bounded linear operators acting on a complex Banach space X is uniformly exponentially stable if and only if it is admissible in respect to the pair $(c_{00}(\mathbb{Z}_+, X), c_{00}(\mathbb{Z}_+, X))$, (i. e. the sequence $n \mapsto \sum_{k=0}^{n} U(n,k)f_k : \mathbb{Z}_+ \to X$ belongs to $c_{00}(\mathbb{Z}_+, X)$ for each $(f_k) \in c_{00}(\mathbb{Z}_+, X)$). The approach is based on the theory of discrete evolution semigroups associated to such families.

Key Words: Non-autonomous discrete problems; discrete evolution families of bounded linear operators, discrete evolution semigroups, spectrum of bounded linear operators.
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1 Introduction

The study of the asymptotic behavior of the non-autonomous discrete systems $x_{n+1} = A_n x_n$ or $y_{n+1} = A_n y_n + f_n$ is much more difficult than the corresponding study of the autonomous ones. For the systems in latter case there are a lot of spectral criteria which characterizes different types of stability (or other types of asymptotic behavior) of their solutions. However, only in some particular cases (for example in the case when the coefficients are periodic) such criteria work (partially) for time periodic systems. New difficulties appear in the study of the inhomogeneous systems, especially because the part of the solution generated by the forced term (f_n), i. e. $\sum_{k=\nu}^{n} U(n,k)f_k$, is not a convolution in the classical sense. These difficulties may by passed by using the so called evolution semigroups. Having in mind the well known results in the continuous case, see for example [6],[3], [4] and [7], we can say that this method is a very efficient one. Mention however, that is not very easy to apply such method. For example, even when the coefficients of the non-autonomous system are complex scalars, the elements of the associated evolution semigroup act in an infinite dimensional space. Here we develop the theory of discrete evolution semigroups on some spaces of bounded sequences. Results of this type in the continuous case may found in [5], [1], [2] and the references therein. However, by contrast with the continuous case, we didn't find in the existent literature papers written in the spirit of the present paper refereing to discrete evolution semigroups. Such results could be new and useful for people whose area of research is restricted to the difference equations.

2 Notations and Preliminary Results

Let X be a complex Banach space and let $\mathcal{L}(X)$ be the Banach algebra of all linear and bounded operators acting on X. The norms in X and in $\mathcal{L}(X)$ will be denoted by the same symbol $||\cdot||$. Let $\mathbb{Z}$ be the set of all integer numbers and let $\mathbb{Z}_+$ the set of all nonnegative integers. By $c_{00}(\mathbb{Z}_+, X)$ will denote the set of all X-valued sequences defined on $\mathbb{Z}_+$ which decays at 0 and at ∞. Also, $c_{00}(\mathbb{Z}, X)$ is defined as the set of all X-valued sequences which decay at infinities. Clearly these spaces became Banach spaces if endow them by the "sup" norm. Let Y be a Banach space. A family $\mathbf{T} = \{T(j)\}_{j\in\mathbb{Z}_+}$ of bounded linear operators acting on Y is called *discrete semigroups* if $T(0) = I$ (I being the identity operator on Y) and $T(k+j) = T(k)\circ T(j)$ for all $k, j \in \mathbb{Z}_+$. Clearly for each $j \in \mathbb{Z}_+$ have that $T(j) = T(1)^j$. We call $T(1)$ the *algebraic generator* of the semigroup $\mathbf{T}$. Having in mind the notion of infinitesimal generator for a strongly continuous semigroup, we define the "infinitesimal generator" for a discrete semigroup as being $G := T(1) - I$. The Taylor formula of order one for discrete semigroups may be written as:

$$T(j)f - f = \sum_{k=0}^{j-1} T(k)Gf \quad \forall j \in \mathbb{Z}_+, j \geq 1, \quad f \in Y. \tag{2.1}$$

In fact,

$$\sum_{k=0}^{j-1} T(k)Gf = \sum_{k=0}^{j-1}[T(k+1) - T(k)]f = T(j)f - f.$$

Let $J \in \{\mathbb{Z}, \mathbb{Z}_+\}$. A *discrete evolution family* on the Banach space X is a family of two parameters $\mathbf{U}_J := \{U(n,m) : n \geq m \in J\}$, having the properties: $U(m,m) = I$ and $U(m,n) = U(m,p)U(p,n)$ for any $m \geq p \geq n \in J$. Here I is the identity operator on X. Such family has *exponential growth* if there exist two real constants M and ω, such that

$$||U(n,m)|| \leq M\exp\left(\omega(n-m)\right) \quad \text{ for all } n \geq m \in J.$$

Clearly $M \geq 1$ and ω may be chosen a positive real number. By contrast with the semigroup case, the exponential growth condition is not automatically verified for discrete evolution families. For any operator A acting on X we denote by $\rho(A)$ its resolvent set, i.e. the set of all complex scalars z for which $zI - A$ is an invertible operator in $\mathcal{L}(X)$. By $\sigma(A) := \mathbb{C}\setminus\rho(A)$ we denote the spectrum of the operator A. The spectral radius of A, denoted by $r(A)$, is defined by $r(A) := \sup\{|z| : z \in \sigma(A)\}$. It is well known that

$$r(A) = \lim_{n\to\infty} ||A^n||^{\frac{1}{n}}.$$

As a consequence, a discrete semigroup $\{T(j)\}$ is uniformly exponentially stable if and only if the spectral radius of $T(1)$ is less than 1.

3 Evolution semigroups and uniform exponential stability for evolution families on $\mathbb{Z}_+$.

Let $\mathcal{U} = \{U(m,n) : m \geq n \geq 0\}$ be a discrete evolution family of bounded linear operators acting on a Banach space X having exponential growth. For each $j \in \mathbb{Z}_+$, the linear operator given by

$$(T(j)f)(n) = \begin{cases} U(n, n-j)f(n-j), & \text{for all } n \geq j; \\ 0, & \text{otherwise} \end{cases}$$

is well defined and acts on $Y := c_{00}(\mathbb{Z}_+, X)$. Moreover it is a bounded operator on Y and $||T(j)||_{\mathcal{L}(Y)} \leq M \exp(\omega j)$. The family $\mathbf{T} = \{T(j)\}_{j \in \mathbb{Z}_+}$ is called the *evolution semigroup* associated to $\mathcal{U}$ on $c_{00}(\mathbb{Z}_+, X)$. The following Lemma is the key tool in the proof of the main result of this section. It connects the "infinitesimal generator" of the evolution semigroup and a non-homogeneous discrete Cauchy Problem leading to the evolution family.

Lemma 3.1. *Let $\mathbf{T} = \{T(j)\}_{j \in \mathbb{Z}_+}$ be the evolution semigroup associated to the discrete evolution family $\mathcal{U}$ on the space $c_{00}(\mathbb{Z}_+, X)$ and let $x, f \in c_{00}(\mathbb{Z}_+, X)$. The following two statements are equivalent:*

- **(i)** $Gx = -f$.
- **(ii)** $x(j) = \sum_{k=0}^{j} U(j,k)f(k)$ *for* $j \in \mathbb{Z}_+$.

Proof: (**i**) $\Rightarrow$ (**ii**) For $j = 0$ the assertion is obvious. Let $j \in \mathbb{Z}_+, j \geq 1$. From (2.1) follows:

$$T(j)x - x = \sum_{k=0}^{j-1} T(k)Gx = -\sum_{k=0}^{j-1} T(k)f.$$

By applying both sides to j, obtain:

$$\begin{aligned} x(j) &= (T(j)x)(j) + \left(\sum_{k=0}^{j-1} T(k)f\right)(j) \\ &= U(j,0)x(0) + \sum_{k=0}^{j-1} U(j, j-k)f(j-k) \\ &= \sum_{r=0}^{j} U(j,r)f(r). \end{aligned}$$

(ii) $\Rightarrow$ **(i)** Let $n \geq 1$. Successively one has:

$$\begin{aligned}(Gx)(n) &= [(T(1)-I)x](n)\\ &= U(n,n-1)x(n-1)-x(n)\\ &= \sum_{j=0}^{n-1} U(n,j)f(j)-x(n)\\ &= \sum_{j=0}^{n} U(n,j)f(j)-U(n,n)f(n)-x(n)\\ &= -f(n).\end{aligned}$$

∎

Lemma 3.2. *Let $\mathcal{U}$ be a discrete evolution family of bounded linear operators acting on a Banach space X having exponential growth. If there exists a positive constant c such that*

$$(n-j+1)\|U(n,j)\| \leq c \text{ for all } n \geq j \geq 0 \tag{3.1}$$

then there exist two positive constants K and ν such that

$$\|U(n,j)\| \leq Ke^{-\nu(n-j)} \text{ for all } n \geq j \geq 0, \tag{3.2}$$

that is, the family is uniformly exponentially stable.

Proof: Let $N \geq 1$ be an integer number such that $\frac{c}{n-j+1} \leq \frac{1}{2}$ for all $n-j \geq N$. From (3.1), we get:

$$\|U(n,j)\| \leq \frac{1}{2} \text{ for all } n \geq N+j.$$

Let m be the integer part of $\frac{n-j}{N} \in Z_+$. Then $m \geq 1$ and n may be represented as $n = j+mN+\rho N$, with $\rho \in [0,1)$. Thus

$$U(n,j) = U(j+Nm+\rho N, j+Nm)U(j+Nm,j).$$

By using exponential growth property, we get

$$\|U(n,j)\| \leq Me^{\omega N}\|U(j+Nm,j)\|. \tag{3.3}$$

On the other hand, using the evolution property, we may write

$$U(j+Nm,j) = U(j+Nm, j+N(m-1))U(j+N(m-1), j+N(m-2))\cdots U(j+N,j)$$

and hence

$$\|U(j+Nm,j)\| \leq \frac{1}{2^m}. \tag{3.4}$$

By combining (3.3) and (3.4) we get:

$$\|U(n,j)\| \leq \frac{1}{2^m} M e^{\omega N} \leq M e^{\omega N} \left(\frac{1}{2}\right)^{\frac{n-j}{N}-1} = K e^{-\nu(n-j)},$$

where $K = 2Me^{\omega N}$ and $\nu = \frac{\ln 2}{N}$. ∎

Theorem 3.3. *Let $\mathcal{U} = \{U(n,m)\}_{n \geq m \geq 0}$ be a discrete evolution family of bounded linear operators acting on a complex Banach space X having exponential growth. Let's consider the map $g_{\mathcal{U},f}$ given by*

$$g_{\mathcal{U},f}(n) := \sum_{k=0}^{n} U(n,k) f(k), \quad f \in c_{00}(\mathbb{Z}_+, X).$$

If for each f belonging to $c_{00}(\mathbb{Z}_+, X)$ have that $g_{\mathcal{U},f}$ belongs to $c_{00}(\mathbb{Z}_+, X)$ then the family $\mathcal{U}$ is uniformly exponentially stable.

Proof: We give the proof in three steps.

Step 1: Let us consider the linear operator

$$K : c_{00}(\mathbb{Z}_+, X) \to c_{00}(\mathbb{Z}_+, X)$$

defined by

$$(Kf)(n) := g_{\mathcal{U},f}(n) \quad n \in Z_+, \quad f \in c_{00}(\mathbb{Z}_+, X).$$

We prove that the operator K is bounded. In view of the Closed Graph Theorem it is enough to prove that the operator K is closed. For this purpose, let us choose $f_j, f, g \in c_{00}(\mathbb{Z}_+, X)$, $j \in \mathbb{Z}_+$ such that

$$f_j \to f \quad (\text{ as } j \to \infty) \text{ in } c_{00}(\mathbb{Z}_+, X)$$

and

$$Kf_j \to g \quad (\text{ as } j \to \infty) \text{ in } c_{00}(\mathbb{Z}_+, X).$$

Since $f_j \to f$ in $c_{00}(\mathbb{Z}_+, X)$, the X-valued sequence $\{f_j(k)\}_{j=0}^{\infty}$ converges, for each $k \in \mathbb{Z}_+$, to $f(k)$. By using the continuity of the operators $U(n,k)$ we also get:

$$\lim_{j \to \infty} (Kf_j)(n) = (Kf)(n), \quad \text{ for each } n \in \mathbb{Z}_+.$$

On the other hand $(Kf_j)(n) \to g(n) \quad (\text{ as } j \to \infty)$ for each fixed $n \in \mathbb{Z}_+$ and thus $Kf = g$. The continuity of the operator K assures the existence of a constant $c > 0$ such that

$$\|Kf\|_{c_{00}(\mathbb{Z}_+,X)} \leq c \text{ for all } f \in c_{00}(\mathbb{Z}_+, X) \text{ with } \|f\| \leq 1. \tag{3.5}$$

Step 2: We prove that the family $\mathcal{U}$ is uniformly bounded. We put $U(n,m) = 0$ whenever $n < m$. Let $j \in \mathbb{Z}_+$, $j \geq 1$, and let f_j defined by

$$f_j(k) := 1_{\{j\}}(k) U(k,j) b, \quad b \in X, \quad \|b\| \leq 1.$$

As is usually by 1_S we denote the characteristic function of the non-empty set S. Clearly $f_j \in c_{00}(\mathbb{Z}_+, X)$ and $\|f\|_{c_{00}(\mathbb{Z}_+,X)} = \|b\| \leq 1$. Then (3.5) provides

$$c \geq \|Kf_j\|_{c_{00}(\mathbb{Z}_+,X)} \geq \|(Kf_j)(n)\| = \|U(n,j)b\|, \quad n \geq j \geq 1.$$

On the other hand

$$\|U(n,0)\| = \|U(n,1)U(1,0)\| \leq \|U(n,1)\|\|U(1,0)\| \leq c\|U(1,0)\|.$$

Finally, we may write:

$$\sup_{n\geq j\geq 0} \|U(n,j)\| \leq c_1 < \infty \text{ where } c_1 = max\{c, c\|U(1,0)\|\},$$

that is, the family $\mathcal{U}$ is uniformly bounded. Moreover, for each fixed $j \geq 1$ and all $b \in X$ with $||b|| \leq 1$ have that $U(n,j)b \to 0$ (as $n \to \infty$). This happens because $\|U(n,j)b\| = \|(Kf_j)(n)\| \to 0,$ as $n \to \infty$ and taking into account that $Kf_j \in c_{00}(\mathbb{Z}_+, X)$.

Step 3: We prove that

$$\sup_{n\geq j\geq 0} [(n-j+1)\|U(n,j)\|] = c_2 < \infty.$$

Let us consider $j \geq 1$ and $h_j(k) := \frac{1}{c_1} 1_{\{j,\cdots,n\}}(k)U(k,j)b$. Obviously h_j belongs to $c_{00}(\mathbb{Z}_+, X)$ and $\|h_j\|_{c_{00}(\mathbb{Z}_+,X)} \leq 1$ because $\|U(n,j)\| \leq c_1$ and $\|b\| \leq 1$. Using again (3.5) we get: $\|Kh_j(n)\| \leq \|Kh_j\|_{c_{00}(\mathbb{Z}_+,X)} \leq c$. On the other hand

$$\begin{aligned} (Kh_j)(n) &= \frac{1}{c_1}\sum_{k=0}^{n} U(n,k)1_{\{j,\cdots,n\}}(k)U(k,j)b && (3.6)\\ &= \frac{1}{c_1}\sum_{k=j}^{n} U(n,j)b = \frac{1}{c_1}(n-j+1)U(n,j)b && (3.7)\end{aligned}$$

and thus $(n-j+1)\|U(n,j)b\| \leq cc_1$ for all $n \geq j \geq 1$. Moreover

$$\begin{aligned} (n+1)\|U(n,0)\| &= (n+1)\|U(n,1)U(1,0)\| \leq (n+1)\|U(n,1)\|\|U(1,0)\| \\ &= n\frac{n+1}{n}\|U(n,1)\|\|U(1,0)\| \leq 2cc_1\|U(1,0)\|. \end{aligned}$$

Hence

$$\|U(n,j)\| \leq \frac{c_2}{n-j+1}$$

for all $n \geq j \geq 0$, where $c_2 = max\{cc_1, 2cc_1\|U(1,0)\|\}$. The assertion follows now by Lemma 3.2.

∎

Theorem 3.4. *The following four statements are equivalent.*

- **(i)** *The family $\mathcal{U}$ is uniformly exponentially stable.*
- **(ii)** *The evolution semigroup* $\mathbf{T}$ *associated to the family* $\mathcal{U}$ *on* $c_{00}(\mathbb{Z}_+, X)$ *is uniformly exponentially stable.*
- **(iii)** *The "infinitesimal generator"* $T(1) - I$ *of* $\mathbf{T}$ *is invertible.*
- **(iv)** *For each* $f \in c_{00}(\mathbb{Z}_+, X)$ *have that* $g_{\mathcal{U},f} \in c_{00}(\mathbb{Z}_+, X)$.

Proof: **(i)⇒(ii).** It is obvious.

(ii)⇒(iii). It is well known that the semigroup $\mathbf{T}$ is uniformly exponentially stable if and only if $r(T(1)) < 1$. Then the assumption assure that $1 \in \rho(T(1))$ and so $T(1) - I$ is invertible.

(iii)⇒(iv). Let $T(1) - I$ be invertible. Then for each $f \in c_{00}(\mathbb{Z}_+, X)$ there exists a unique $u \in c_{00}(\mathbb{Z}_+, X)$ such that $[T(1) - I]u = -f$. From Lemma 3.1 this is equivalent to the fact that $u(j) = \sum_{k=0}^{j} U(j,k)f(k)$ for all $j \in \mathbb{Z}_+$, i. e. $g_{\mathcal{U},f} \in c_{00}(\mathbb{Z}_+, X)$.

(iv)⇒(i). Follows from Theorem 3.3. □

Theorem 3.5. *Let* $\mathcal{U} = \{U(n,m) : n \geq m \geq 0\}$ *be a discrete evolution family having exponential growth and let* $\mathbf{T} = \{T(j)\}_{j\geq 0}$ *be the evolution semigroup associated to* $\mathcal{U}$ *on* $c_{00}(\mathbb{Z}_+, X)$ *having* $G_0 = T(1) - I$ *as infinitesimal generator. The following two statements hold true:*

- **(i)** $\sigma(T(1)) = \{z \in \mathbb{C} : |z| \leq r(T(1))\}$.
- **(ii)** $\sigma(G_0) = \sigma(T(1)) - 1$.

Proof: We prove **(i)** in two steps.

Step 1. Consider the case when $\mathbf{T}$ is uniformly exponentially stable. First we prove that if $\lambda \in \rho(T(1))$ then $\mu \in \rho(T(1))$ for all $|\mu| \geq |\lambda|$. For this to be end let us consider a new evolution family defined by

$$U_\lambda(n,m) = \lambda^{-(n-m)}U(n,m).$$

Its associated evolution semigroup on the space $c_{00}(\mathbb{Z}_+, X)$ is

$$\mathbf{T}_\lambda(j) = \lambda^{-j}T(j).$$

Then $1 \in \rho(T_\lambda(1))$ if and only if $\lambda \in \rho(T(1))$. As a consequence, the assumption is equivalent with uniform exponential stability of the family $\{U_\lambda(n,m) : n \geq m \geq 0\}$ (say with the constants N and ν). Now, if $|\mu| \geq |\lambda|$, then

$$U_\mu(n,m) = \mu^{-(n-m)}U(n,m) \tag{3.8}$$

$$= \left(\frac{\lambda}{\mu}\right)^{n-m} \lambda^{-(n-m)}U(n,m) \tag{3.9}$$

$$= \left(\frac{\lambda}{\mu}\right)^{n-m} U_\lambda(n,m). \tag{3.10}$$

It follows

$$\|U_\mu(n,m)\| \leq Ne^{-\nu(n-m)} \text{ for all } n \geq m.$$

Since the family $\{U_\mu(n,m)\}$ is uniformly exponentially stable its associated evolution semigroup $\mathbf{T}_\mu$ is uniformly exponentially stable as well. Then the infinitesimal generator $\mu^{-1}T(1)-I$ is an invertible operator, i. e. $T(1)-\mu I$ is invertible, which provides $\mu \in \rho(T(1))$.

Step 2: Here we analyze the case when the evolution semigroup $\{T(j)\}_{j\geq 0}$ is not uniformly exponentially stable. Let $\omega \in \mathbb{R}$ such that $\|T(j)\| \leq e^{\omega j}$ for all $j \in \mathbb{Z}_+$. Define

$$S(j) := e^{-\nu j}T(j)$$

for a given $\nu > \omega$. Then $\|S(j)\| \leq e^{(\omega-\nu)j}$ for all $j \in \mathbb{Z}_+$ i. e. the semigroup $\{S(j)\}$ is uniformly exponentially stable. Let $\lambda \in \rho(T(1))$ and let μ be a complex number such that $|\mu| \geq |\lambda|$. Clearly $e^{-\nu}T(1) - \lambda e^{-\nu}I$ is invertible, i. e. $S(1) - \lambda e^{-\nu}I$ is invertible. Thus $\lambda e^{-\nu} \in \rho(S(1))$ and $|\mu e^{-\nu}| \geq |\lambda e^{-\nu}|$. The previous step assures that $\mu e^{-\nu} \in \rho(S(1))$. Hence $\mu \in \rho(T(1))$.

Now we prove our required result. Let $\lambda \in \mathbb{C}$ such that $|\lambda| < r(T(1))$. Suppose for a contradiction that $\lambda \in \rho(T(1))$. Then, as stated before, all complex numbers μ with $|\mu| \geq |\lambda|$ belong to $\rho(T(1))$. Thus only the complex numbers w having modulus less than $|\lambda|$ may be in $\sigma(T(1))$. This shows that $|\lambda| \geq r(T(1))$ which is a contradiction. Hence if $z \in \mathbb{C}$ is such that $|z| < r(T(1))$ then $z \in \sigma(T(1))$. But $\sigma(T(1))$ is a closed set, so $\sigma(T(1)) = \{z \in \mathbb{C} : |z| \leq r(T(1))\}$.
(ii) The Spectral Mapping Theorem says that if $f : \mathbb{C} \to \mathbb{C}$ is an analytic function and $A \in \mathcal{L}(X)$ then $\sigma(f(A)) = f(\sigma(A))$, where by $f(\sigma(A))$ we denote the set$\{f(z) : z \in \sigma(A)\}$. For our purpose set $z \mapsto f(z) = z-1$, which is an analytic function and $f(A) = A-I$, so $\sigma(A-I) = \sigma(A)-1 = \{z-1 : z \in \sigma(A)\}$. ∎

In as follows we give a concrete example of operator whose spectrum is the closed unit disk.

Example 3.6. *Let us consider the Banach space $X := \mathbb{C}$ and consider the discrete evolution family $\mathcal{U}$ defined by $U(n,m)x := \frac{m+1}{n+1}x$ for all $x \in \mathbb{C}$ and for all $n \geq m \geq 0$. Clearly $\|U(n,m)\| \leq 1$ for all $n \geq m \geq 0$. In particular, the family $\mathcal{U}$ is uniformly bounded. Let $\{T(j)\}_{j\in\mathbb{Z}_+}$ be the evolution semigroup associated to the family $\mathcal{U}$ on $c_{00}(\mathbb{Z}_+,\mathbb{C})$. Let's consider $f_j(k) := 1_{\{j\}}(k)U(k,j)b, b \in \mathbb{C}, j \geq 1, k \in \mathbb{Z}_+, |b| = 1$, and let $\nu \in \mathbb{Z}_+$, be fixed. Then*

$$(T(\nu)f_j)(j+\nu) = U(j+\nu,j)b = \frac{j+1}{j+\nu+1}b.$$

This shows that $\|T(\nu)f_j\| \to 1($ as $j \to \infty)$ and so $\|T(\nu)\| = 1$. Thus $r(T(1)) = 1$ and by Theorem 3.5 follows that $\sigma(T(1))$ is the closed unit disk.

The next example is a concrete application of the Theorem 3.4.

Example 3.7. *Let X be a complex Banach space and let $\{A_j\}_{j\in\mathbb{Z}_+}$ be a family of bounded linear operators acting on X which is uniformly bounded, i.e. $\sup_{j\in\mathbb{Z}_+} ||A_j|| < \infty$. Consider the following two discrete Cauchy problems.*

$$\begin{cases} x_{j+1} = A_j x_j, \quad j \in \mathbb{Z}_+, \quad j \geq k \\ x_k = b \text{ (for fixed) } k \in \mathbb{Z}_+ \end{cases} \qquad (A_j, k, b)$$

and

$$\begin{cases} y_{j+1} = A_j y_j + f_{j+1}, & j \in \mathbb{Z}_+ \\ y_0 = 0. \end{cases} \qquad (A_j, f_j, 0)$$

The solutions of (A_j, k, b) *and* $(A_j, f_j, 0)$ *are (respectively) given by:* $x_j = U(j,k)b$ *and* $y_j = \sum_{k=0}^{j} U(j,k)f(k)$*. Here* $U(j,k) := A_{j-1} \cdots A_k$ *when* $j > k$*. The following two statements are equivalent:*

- *For each* $b \in X$ *the solution of* (A_j, k, b) *decays exponentially or, equivalently, there exist two positive constants* K *and* ν *such that*

$$\sup_{n>k} [Ke^{\nu(n-k)} ||A_{n-1} \cdots A_k||] < \infty.$$

- *For each* $f_j \in c_{00}(\mathbb{Z}_+, X)$ *the solution of* $(A_j, f_j, 0)$ *belongs to* $c_{00}(\mathbb{Z}_+, X)$*.*

4 Evolution semigroups and uniform exponential stability for evolution families on $\mathbb{Z}$.

Let $\mathcal{U} = \{U(m,n) : m \geq n \in \mathbb{Z}\}$ be a discrete evolution family of bounded linear operators acting on a Banach space X. For each $j \in \mathbb{Z}_+$, the linear operator given by

$$(T(j)f)(n) = U(n, n-j)f(n-j) \text{ for all } n \in \mathbb{Z}$$

is well defined and acts on the Banach space $Z := c_{00}(\mathbb{Z}, X)$. When the evolution family $\mathcal{U}$ has exponential growth (say with the constants M and ω) the operator $T(j)$ is a bounded on Z and

$$||T(j)||_{\mathcal{L}(Z)} \leq M \exp(\omega j).$$

The family $\mathcal{T} = \{T(j)\}_{j \in \mathbb{Z}_+}$ is called the *evolution semigroup* associated to $\mathcal{U}$ on $c_{00}(\mathbb{Z}, X)$. The following Lemma is similar to the above Lemma 3.1. However, in the proof of the next lemma we use the uniform boundedness of the family $\mathcal{U}$.

Lemma 4.1. *Let* $\mathcal{U} = \{U(m,n) : m \geq n \in \mathbb{Z}\}$ *be a uniformly bounded discrete evolution family and let* $\mathcal{T} = \{T(j)\}_{j \in \mathbb{Z}_+}$ *be the evolution semigroup associated to* $\mathcal{U}$ *on the space* $c_{00}(\mathbb{Z}, X)$ *"generated" by* G*. Let* $f, x \in c_{00}(\mathbb{Z}, X)$ *be fixed. The following two statements are equivalent:*

- **(i)** $Gx = -f$*.*
- **(ii)** *For each* $n \in \mathbb{Z}$ *there exists*

$$u(n) := \lim_{k \to -\infty} \sum_{\rho=k}^{n} U(n, \rho) f(\rho),$$

and $x(n) = u(n)$ *for all* $n \in \mathbb{Z}$*.*

Proof: **(i)⇒(ii)**. As we know

$$T(j)x - x = \sum_{k=0}^{j-1} T(k)Gx = -\sum_{k=0}^{j-1} T(k)f$$

and thus for any $n \in \mathbb{Z}$, one has

$$\begin{aligned} x(n) &= T(j)x(n) + \sum_{k=0}^{j-1} T(k)f(n) \\ &= U(n, n-j)x(n-j) + \sum_{k=0}^{j-1} U(n, n-k)f(n-k) \\ &= U(n, n-j)x(n-j) + \sum_{\rho=n-j+1}^{n} U(n,\rho)f(\rho). \end{aligned}$$

Using the uniform boundedness of the family $\{U(n, m)\}$, obtain:

$$\|U(n, n-j)x(n-j)\| \leq K\|x(n-j)\| \to 0 \text{ (as } j \to \infty).$$

It follows that $\lim_{j\to\infty} \sum_{\rho=n-j+1}^{n} U(n, \rho)f(\rho)$. exists, and

$$x(n) = \sum_{\rho=-\infty}^{n} U(n, \rho)f(\rho)$$

(ii) ⇒ (i). In view of the above definitions have that:

$$(Gx)(n) = (T(1) - I)x(n) = U(n, n-1)x(n-1) - x(n).$$

Based on the continuity of the operator $U(n, n-1)$, obtain:

$$(Gx)(n) = U(n, n-1) \sum_{j=-\infty}^{n-1} U(n-1, j)f(j) - x(n)$$

$$= \sum_{j=-\infty}^{n-1} U(n, j)f(j) - x(n) = -f(n).$$

□

Theorem 4.2. *Let $\mathcal{U} = \{U(m, n) : m \geq n \in \mathbb{Z}\}$, be a uniformly bounded discrete evolution family of bounded linear operators acting on the Banach space X and let $\mathcal{T}$ be the evolution semigroup associated to $\mathcal{U}$ on the space $c_{00}(\mathbb{Z}, X)$. The following four statements are equivalent:*

- **(i)** *The family $\mathcal{U}$ is uniformly exponentially stable.*

- **(ii)** *The evolution semigroup $\mathcal{T}$ is uniformly exponentially stable.*
- **(iii)** *The "infinitesimal generator" $G = T(1) - I$ of $\mathcal{T}$ is invertible.*
- **(iv)** *For each $f \in c_{00}(\mathbb{Z}, X)$ and each $n \in \mathbb{Z}$, there exists*

$$x(n) := \lim_{k\to-\infty} \sum_{\nu=k}^{n} U(n,\nu)f(\nu).$$

Moreover, $x(\cdot)$ belongs to $c_{00}(\mathbb{Z}, X)$.

Proof: The proof of **(i)**⇒**(ii)**⇒**(iii)** are similar as already given in Theorem 3.4.

(iii) ⇒ **(iv)** : As $T(1) - I$ is invertible, for each $f \in c_{00}(\mathbb{Z}, X)$ there exists an unique $x \in c_{00}(\mathbb{Z}, X)$ such that $[T(1) - I]x = -f$. Then from Lemma 4.1, obtain:

$$x(n) := \lim_{k\to-\infty} \sum_{\nu=k}^{n} U(n,\nu)f(\nu),$$

exists and $x(n) \in c_{00}(\mathbb{Z}, X)$.

(iv)⇒**(i)**: We adapt the technique used in Theorem 3.3 and prove this result in two steps.

Step 1: Let us consider the linear operator

$$K : c_{00}(\mathbb{Z}, X) \to c_{00}(\mathbb{Z}, X)$$

defined by $(Kf)(n) = x(n)$ where

$$x(n) := \lim_{k\to-\infty} \sum_{\nu=k}^{n} U(n,\nu)f(\nu), \quad n \in \mathbb{Z}.$$

We prove that K is bounded operator. As in the previous section we use the Closed Graph Theorem. It is enough to prove that the operator K is closed. Let $f_j, f, g \in c_{00}(\mathbb{Z}, X)$, $j \in \mathbb{Z}_+$ such that

$$f_j \to f \text{ in } c_{00}(\mathbb{Z}, X) \text{ (as } j \to \infty) \text{ and}$$

$$Kf_j \to g \text{ in } c_{00}(\mathbb{Z}, X) \text{ as } (j \to \infty).$$

For any fixed integers N, n with $N < n$, let consider the following sequences in $c_{00}(\mathbb{Z}, X)$.

$$f^0(k) = \begin{cases} f(k), & \text{when } k \geq N \\ 0, & \text{when } k < N \end{cases}$$

and

$$f_j^0(k) = \begin{cases} f_j(k), & \text{when } k \geq N \\ 0, & \text{when } k < N. \end{cases}$$

Clearly $f_j^0(k) \to f^0(k)$ and $Kf_j^0(k) \to g(k)$ for every $k \geq N$. On the other hand

$$\begin{aligned} Kf_j^0(n) - Kf^0(n) &= \sum_{K=N}^{n} U(n,k)(f_j^0(k) - f^0(k)) \\ &= \sum_{K=N}^{n} U(n,k)(f_j(k) - f(k)) \to 0 \text{ as } (j \to \infty), \end{aligned}$$

But $Kf_j^0(n) \to g(n)$. Hence $Kf^0(n) = g(n)$. Taking into the account our assumption, obtain:

$$(Kf)(n) - g(n) = \sum_{k=-\infty}^{N-1} U(n,k)f(k) \to 0 \text{ as } (N \to -\infty).$$

Then $(Kf)(n) - g(n) = 0$ for all $n \in \mathbb{Z}$ which assures the boundedness of the operator K. As a consequence, there exists a positive constant L such that $\|Kf\|_{c_{00}(\mathbb{Z},X)} \leq L$ for all $f \in c_{00}(\mathbb{Z}, X)$ with $\|f\| \leq 1$.

Step 2: For any $j \in \mathbb{Z}$, we define: $f_j(k) = 1_{\{j\}}(k)U(k,j)b$ where $b \in X$ with $\|b\| \leq 1$. Clearly $f_j \in c_{00}(\mathbb{Z}, X)$ and

$$(Kf_j)(n) = \begin{cases} 0 & \text{for } n < j \\ U(n,j)b & \text{for } n \geq j. \end{cases}$$

Since

$$(Kf_j)(n) \to 0 \quad (\text{as } n \to \infty)$$

have that

$$\lim_{n\to\infty} \|U(n,j)b\| = 0 \text{ for all } j \in \mathbb{Z}.$$

Set $h_j(k) := \frac{1}{M} 1_{\{j,\cdots,n\}}(k)U(k,j)b$ where $M = \sup_{n\geq m} ||U(n,m)||$. Obviously, $h_j \in c_{00}(\mathbb{Z}, X)$ and $\|h_j\| \leq 1$. Then $\|(Kh_j)(n)\| \leq L$. On the other hand $(Kh_j)(n) = \frac{1}{M}(n-j+1)U(n,j)b$, hence

$$(n-j+1)\|U(n,j)b\| \leq c \text{ for all } n \geq j,$$

where $c = LM$. The assertion follows now by Lemma 3.2. ∎

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