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Asymptotic constancy for impulsive differential equations with piecewise constant argument

by
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Abstract

After studying the existence of the unique solution $x(t)$ of initial value problem $(1.1)-(1.3)$, we prove that the limit of $x(t)$ is a real constant as $t \to \infty$. Also, we formulate this limit value in terms of the initial condition (1.3) and the solution of the integral equation (2.3).

Key Words: Asymptotic constancy, Impulsive differential equation, Differential equation with piecewise constant arguments.
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1 Introduction

The theory of differential equations with piecewise constant arguments ($DEPCA$) of the type

$$x'(t) = f\ (t, x(t), x(h(t)))$$

was initiated in ([14],[27]) where $h(t) = [t],\ [t-n],\ [t+n],$ *etc.* Systems described by $DEPCA$ exist in a large area such as biomedicine, chemistry, physics and mechanical engineering. Busenberg and Cooke [12] first established a mathematical model with a piecewise constant argument for analyzing vertically transmitted diseases. $DEPCA$ are also closely related to difference and differential equations. So, they describe hybrid dynamical systems and combine the properties of both differential and difference equations. The oscillation, periodicity and some asymptotic properties of various differential equations with piecewise constant arguments were methodically demonstrated in ([1]-[5],[17]-[19],[23]-[25],[28]). Also, Wiener's book [29] is a distinguished source in this area.

The theory of impulsive differential equations developed rapidly, in recent years. There are many works about impulsive differential equations. The monographs ([7], [26]) are good sources for impulsive differential equations. But, there are only a few papers on impulsive differential equations with piecewise constant arguments (*IDEPCA*) ([11], [21], [22], [30]). In [22], Li and Shen considered the problem

$$y'(t) = f\ (t, y[t-k]),\ t \neq n,\ t \in J,$$
$$\Delta y(n^+) = I_n(y(n)),\ n = 1, 2, \ldots, p,\ y(0) = y(T).$$

Using the method of upper and lower solutions, they proved that it has at least one solution. In [30], Wiener and Lakshmikantham established the existence and uniqueness of solutions of the initial value problem

$$x'(t) = f\ (x(t), x(g(t))),\ x(0) = x_0,$$

and they also studied the cases of oscillation and stability, where f is a continuous function and $g : [0, \infty) \to [0, \infty),\ g(t) \leq t$, is a step function. In [11] and [21], some qualitative aspects of advanced and delay *IDEPCA* are investigated.

Lately, the problem of the asymptotic constancy of solutions was studied for some functional differential equations ([6], [8],[9],[13],[15],[16]) and as well the same problem has been considered for some impulsive delay differential equations ([10],[20]). So, due to the practical reasons and the papers mentioned above one can be motivated to deal with the problem of asymptotic constancy of solutions of an impulsive differential equation with piecewise constant arguments. Let us note that in physical and engineering systems, the phenomena related to stepwise or piecewise constant variables or motions under piecewise constant forces can usually come out as first or second order differential equations with piecewise constant arguments.

In this paper, we consider the first order nonhomogeneous linear impulsive differential equation with piecewise constant argument

$$x'(t) = a(t)(x(t) - x([t])) + f(t), \quad t \neq n \in \mathbb{Z}^+ = \{1, 2, \ldots\},\ t \geq 0, \tag{1.1}$$

$$\Delta x(n) = d(n),\ n \in \mathbb{Z}^+, \tag{1.2}$$

and the initial condition

$$x(0) = x_0, \tag{1.3}$$

where $a(t)$ and $f(t)$ are continuous real valued functions on $[0, \infty)$, $d : \mathbb{Z}^+ \longrightarrow \mathbb{R}$, $x_0 \in \mathbb{R}$, $\Delta x(n) = x(n^+) - x(n^-)$, $x(n^+) = \lim_{t \to n^+} x(t)$, $x(n^-) = \lim_{t \to n^-} x(t)$ and $[.]$ denotes the greatest integer function.

Here, we want to give sufficient conditions for asymptotic constancy of the solution $x(t)$ of $(1.1) - (1.3)$, that is, $\lim_{t \to \infty} x(t) = \ell \in \mathbb{R}$. We also aim to calculate this limit value in terms of initial condition and the solution of an integral equation. As we know, this problem has not been studied, yet.

Definition 1. *A function $x(t)$ defined on $[0, \infty)$ is said to be a solution of $(1.1) - (1.3)$ if it satisfies the following conditions:*

(i) $x : [0, \infty) \to \mathbb{R}$ is continuous with the possible exception of the points $t \in \mathbb{Z}^+$,

(ii) $x(t)$ is right continuous and has left-hand limits at the points $t \in \mathbb{Z}^+$,

(iii) $x'(t)$ exists for every $t \in [0, \infty)$ with the possible exception of the points $t \in \mathbb{Z}^+$ where one-sided derivatives exist,

(iv) $x(t)$ *satisfies* (1.1) *for any* $t \in (0,\infty)$ *with the possible exception of the points* $t \in \mathbb{Z}^+$,
(v) $x(t)$ *satisfies* (1.2) *for every* $t = n \in \mathbb{Z}^+$,
(vi) $x(0) = x_0$.

Before given the main results we can prove the existence and uniqueness of solutions of (1.1) − (1.3) :

Theorem 1. *The initial value problem* (1.1) − (1.3) *has a unique solution on* $[0,\infty)$.

Proof: For $t \in [0,1)$, (1.1) can be written as

$$x'(t) = a(t)(x(t) - x_0) + f(t).$$

Integrating both sides from 0 to t,

$$x(t) = x_0 + \int_0^t \exp\left(\int_s^t a(u)\,du\right) f(s)\,ds. \tag{1.4}$$

For $t \in [1,2)$, the solution $x(t)$ of (1.1) − (1.3) is

$$x(t) = x_0 + \int_0^1 \exp\left(\int_s^1 a(u)\,du\right) f(s)\,ds + \int_1^t \exp\left(\int_s^t a(u)\,du\right) f(s)\,ds + d(1). \tag{1.5}$$

Following this way and using mathematical induction method for $t \in [n, n+1)$, we obtain that

$$x(t) = x(n) + \int_n^t \exp\left(\int_s^t a(u)\,du\right) f(s)\,ds$$

where

$$x(n) = x_0 + \sum_{r=0}^{n-1}\left(\int_r^{r+1} \exp\left(\int_s^{r+1} a(u)\,du\right) f(s)\,ds + d(r+1)\right), \quad n \in \mathbb{Z}^+. \tag{1.6}$$

Therefore, the problem (1.1) − (1.3) has the unique solution

$$x(t) = x_0 + \sum_{r=0}^{[t-1]}\left(\int_r^{r+1} \exp\left(\int_s^{r+1} a(u)\,du\right) f(s)\,ds + d(r+1)\right) + \int_{[t]}^t \exp\left(\int_s^t a(u)\,du\right) f(s)\,ds.$$

that defined on the interval $[0,\infty)$. □

In this while, we note that a straightforward verification shows that the solution of the initial value problem (1.1) − (1.3) satisfies the following integral equation

$$x(t) = x_0 + \int_0^t a(s)\,x(s)\,ds - \int_0^t a(s)\,x([s])\,ds + \int_0^t f(s)\,ds + \sum_{i=1}^{[t]} d(i) \tag{1.7}$$

which we use to prove our main results.

This paper is organized as follows. In Section 2, the main results are presented. Section 3 and Section 4 contain the proofs of our first and second results, respectively. At the end of the Section 4 we give an example to illustrate our results.

2 Main Results

Our main results are given as follows.

Theorem 2. *Let $a(t)$ and $f(t)$ be continuous functions on the interval $[0,\infty)$ and $d:\mathbb{Z}^+\to\mathbb{R}$. If*

(i) $\int\limits_0^\infty |a(s)|\,ds \le K_1 < \infty,$

(ii) $\int\limits_0^\infty |f(s)|\,ds \le K_2 < \infty,$

(iii) $\sum\limits_{i=1}^\infty |d(i)| \le L_1 < \infty,$

then, the solution $x(t)$ of $(1.1)-(1.3)$ tends to a constant as $t\to\infty$, where $K_1,\ K_2$ and L_1 are real positive constants.

Theorem 3. *Suppose that all assumptions of Theorem 2 are satisfied. Let $x(t)$ be the solution of $(1.1)-(1.3)$ and $\lim\limits_{t\to\infty} x(t) = \ell(x_0)$.*
If

$$\int\limits_t^{[t+1]} |a(s)|\,ds \le \rho < 1, \tag{2.1}$$

then

$$\ell(x_0) = x_0 + \int\limits_0^\infty y(s)\,f(s)\,ds + \sum_{i=1}^\infty d(i) \tag{2.2}$$

where y is the unique solution of the integral equation

$$y(t) = 1 + \int\limits_t^{[t+1]} y(s)\,a(s)\,ds,\ \ t\ge 0. \tag{2.3}$$

3 Proof of Theorem 2

For the proof of Theorem 2 we need to prove the following lemma:

Lemma 1. *Assume that all hypotheses of Theorem 2 are satisfied. Then for the solution $x(n)$ of the corresponding difference equation*

$$x(n+1) = x(n) + \int_n^{n+1} \exp\left(\int_s^{n+1} a(u)\,du\right) f(s)\,ds + d(n+1),\ \ n \geq 0, \tag{3.1}$$

there is a positive constant L_2 such that

$$|x(n)| \leq L_2,\ \ n = 0, 1, 2, \ldots. \tag{3.2}$$

Proof: The solution $x(n)$ of Eq.(3.1) is

$$x(n) = x_0 + \sum_{r=0}^{n-1} \left(\int_r^{r+1} \exp\left(\int_s^{r+1} a(u)\,du\right) f(s)\,ds + d(r+1)\right),\ \ n \geq 0. \tag{3.3}$$

By the hypotheses of (i), (ii), (iii), it is easy to see that

$$\sum_{r=0}^{\infty} \left(\int_r^{r+1} \exp\left(\int_s^{r+1} a(u)\,du\right) f(s)\,ds + d(r+1)\right) < \infty.$$

This means that $\lim_{n\to\infty} x(n) \in \mathbb{R}$ and thus the solution $x(n)$ is bounded; that is, there is a $L_2 > 0$ such that (3.2) is satisfied. $\square$

Now, we can give the proof of Theorem 2.

Proof of Theorem 2. Let $x(t)$ be the solution of $(1.1)-(1.3)$. Then, from (1.7), we have

$$\begin{aligned} |x(t)| &\leq |x_0| + \int_0^t |a(s)|\,|x(s)|\,ds + \int_0^t |a(s)|\,|x([s])|\,ds + \int_0^t |f(s)|\,ds + \sum_{i=1}^{[t]} |d(i)| \\ &\leq |x_0| + \int_0^t |a(s)|\,|x(s)|\,ds + L_2 \int_0^\infty |a(s)|\,ds + \int_0^\infty |f(s)|\,ds + \sum_{i=1}^{\infty} |d(i)|. \end{aligned}$$

By using (i), (ii), (iii) and Lemma 1, we obtain

$$\begin{aligned} |x(t)| &\leq c + \int_0^t |a(s)|\,|x(s)|\,ds,\ \ t \geq 0, \\ &\leq c + L_2 \int_0^\infty |a(s)|\,ds \leq c + L_2 K_1 \end{aligned}$$

where $c = |x_0| + L_2 K_1 + K_2 + L_1$.
Hence,

$$|x(t)| \leq M, \; t \geq 0. \tag{3.4}$$

where $M = |x_0| + 2L_2 K_1 + K_2 + L_1$.

On the other hand, by (1.7),

$$|x(t) - x(s)| \leq \int_s^t |a(u)| \, |x(u)| \, du + \int_s^t |a(u)| \, |x([u])| \, du + \int_s^t |f(u)| \, du + \sum_{i=[s]+1}^{[t]} |d(i)| \tag{3.5}$$

for $0 \leq s \leq t < \infty$.
Using (3.2) and (3.4), we have

$$|x(t) - x(s)| \leq (M + L_2) \int_s^\infty |a(u)| \, du + \int_s^\infty |f(u)| \, du + \sum_{i=[s]+1}^{\infty} |d(i)| \, .$$

Because of (i), (ii) and (iii), $\lim\limits_{s\to\infty} |x(t) - x(s)| = 0$. So, by the Cauchy convergence criterion, $\lim\limits_{t\to\infty} x(t) \in \mathbb{R}$ which completes the proof.

4 Proof of Theorem 3

The proof of Theorem 3 is based on the method presented in [9]. Therefore, it is necessary to prove the following theorem and lemmas.

Theorem 4. *Suppose $a(t)$ is continuous on $[0,\infty)$ and (2.1) is satisfied. Then, there is a unique bounded function $y \in PRC([0,\infty), \mathbb{R})$ such that Eq.(2.3) holds.*

Proof: Denote the set of piecewise right continuous functions by $PRC([0,\infty), \mathbb{R})$, that is, $\varphi \in PRC$ means that $\varphi : [0,\infty) \to \mathbb{R}$ is continuous for $t \in [0,\infty)$, $t \neq n \in \mathbb{Z}^+$, and is right handed continuous for $t = n \in \mathbb{Z}^+$.
Now, let us take the set

$$B = \{y : y \in PRC([0,\infty), \mathbb{R}) \text{ and bounded on the interval } [0,\infty)\}$$

which is a Banach space with respect to the norm

$$|y|_B = \sup_{t\geq 0} |y(t)| \, , \; y \in B.$$

Furthermore, for $y \in B$ with $|y|_B \leq \lambda$ and $t \geq 0$, let us define the operator T as

$$Ty(t) = 1 + \int_t^{[t+1]} y(s) \, a(s) \, ds.$$

where λ is a real number. Then, it can be easily shown that

$$Ty\left(n^{+}\right) = \lim_{t \to n^{+}} Ty\left(t\right) = Ty\left(n\right),\ n \in \mathbb{Z}^{+},$$

$$Ty\left(n^{-}\right) = \lim_{t \to n^{-}} Ty\left(t\right) = 1,$$

and

$$Ty\left(t_{*}^{+}\right) = Ty\left(t_{*}^{-}\right) = Ty\left(t_{*}\right),\ t_{*} \in \left(n, n+1\right).$$

So, $Ty \in PRC\left(\left[0, \infty\right), \mathbb{R}\right)$.
Moreover, from (2.1), it follows that

$$\left|Ty\right|_{B} \leq 1 + \rho\left|y\right|_{B} \leq 1 + \rho\lambda$$

i.e. Ty is bounded. Hence, T maps B into itself.
On the other hand, for y and $z \in B$, using (2.1), we have

$$\left|Ty - Tz\right|_{B} \leq \rho\left|y - z\right|_{B}.$$

Since $\rho < 1$, $T : B \to B$ is a contraction. Therefore, by the well known Banach fixed point theorem, there is a unique piecewise right continuous and bounded solution of Eq.(2.3). ∎

Lemma 2. *Under the hypotheses of Theorem 4, the solution y of the integral equation* (2.3) *satisfies the following adjoint equation:*

$$\begin{cases} y'\left(t\right) = -y\left(t\right)a\left(t\right),\ t \neq n,\ t \geq 0, \\ \Delta y\left(n\right) = \int\limits_{n}^{n+1} y\left(s\right)a\left(s\right)ds,\ n \in \mathbb{Z}^{+}. \end{cases} \tag{4.1}$$

Proof: Taking the derivative of (2.3) for $t \in \left(n, n+1\right)$, $n \in \mathbb{Z}^{+}$, we obtain

$$y'\left(t\right) = -y\left(t\right)a\left(t\right).$$

On the other hand,

$$\Delta y\left(n\right) = \int\limits_{n}^{n+1} y\left(s\right)a\left(s\right)ds.$$

So, the proof is complete. ∎

Now, for $t \geq 0$ let us denote

$$C\left(t\right) = y\left(t\right)x\left(t\right) - \int\limits_{t}^{\left[t+1\right]} y\left(s\right)a\left(s\right)x\left(\left[s\right]\right)ds \tag{4.2}$$

where y is the solution of Eq.(2.3) and x is the solution of $(1.1) - (1.3)$.

Lemma 3. *If the hypotheses of Theorem 4 hold, then*

$$C\left(t\right)=C\left(0\right)+\int_{0}^{t}y\left(s\right)f\left(s\right)ds+\sum_{i=1}^{[t]}d\left(i\right). \tag{4.3}$$

Proof: To obtain (4.3), we should prove that $C\left(t\right)$ defined by (4.2) satisfies

$$\begin{cases} C'\left(t\right)=y\left(t\right)f\left(t\right),\ t\neq n,\ t\geq 0, \\ \Delta C\left(n\right)=d\left(n\right),\ n\in\mathbb{Z}^{+}. \end{cases} \tag{4.4}$$

For $t\in\left(n,n+1\right),\ n\in\mathbb{Z}^{+}$, (4.2) can be written as

$$C\left(t\right)=y\left(t\right)x\left(t\right)-\left(\int_{t}^{n+1}y\left(s\right)a\left(s\right)ds\right)x\left(n\right). \tag{4.5}$$

Differentiating (4.5), we get

$$C'\left(t\right)=y\left(t\right)f\left(t\right).$$

Moreover, from (4.2),

$$\Delta C\left(n\right)=y\left(n\right)x\left(n\right)-\left(\int_{n}^{n+1}y\left(s\right)a\left(s\right)ds\right)x\left(n\right)-y\left(n^{-}\right)x\left(n^{-}\right). \tag{4.6}$$

Substituting

$$y\left(n^{-}\right)=y\left(n\right)-\int_{n}^{n+1}y\left(s\right)a\left(s\right)ds \ \text{ and } x\left(n^{-}\right)=x\left(n\right)-d\left(n\right)$$

into (4.6), we obtain that $\Delta C\left(n\right)=d\left(n\right),\ n\in\mathbb{N}$, which completes the proof of (4.4). Integrating both sides of (4.4) from 0 to t, we obtain (4.3). ∎

We are now ready to prove the second main result.

Proof of Theorem 3. Let $x\left(t\right)$ be the solution of $(1.1)-(1.3)$. It is sufficient to show that

$$\lim_{t\to\infty}x\left(t\right)=C\left(0\right)+\int_{0}^{\infty}y\left(s\right)f\left(s\right)ds+\sum_{i=1}^{\infty}d\left(i\right) \tag{4.7}$$

where C is defined by (4.2). By (4.3), we have for $t \geq 0$

$$
\begin{aligned}
&x(t) - C(0) - \int_0^\infty y(s) f(s)\, ds - \sum_{i=1}^{\infty} d(i) \\
&\quad = x(t) - \left(C(0) + \int_0^t y(s) f(s)\, ds + \sum_{i=1}^{[t]} d(i) \right) - \int_t^\infty y(s) f(s)\, ds - \sum_{i=[t]+1}^{\infty} d(i) \\
&\quad = x(t) - C(t) - \int_t^\infty y(s) f(s)\, ds - \sum_{i=[t]+1}^{\infty} d(i).
\end{aligned}
$$

Using (4.2), it follows for $t \geq 0$

$$
\begin{aligned}
&x(t) - C(0) - \int_0^\infty y(s) f(s)\, ds - \sum_{i=1}^{\infty} d(i) \\
&\qquad = x(t) - y(t)\, x(t) + \int_t^{[t+1]} y(s)\, a(s)\, x([s])\, ds \\
&\qquad - \int_t^\infty y(s) f(s)\, ds - \sum_{i=[t]+1}^{\infty} d(i).
\end{aligned}
\tag{4.8}
$$

On the other hand, multiplying (2.3) by $x(t)$, we obtain for $t \geq 0$

$$
x(t) = y(t)\, x(t) - \int_t^{[t+1]} y(s)\, a(s)\, x(t)\, ds.
$$

Substituting the last expression into (4.8), we find for $t \geq 0$

$$
\begin{aligned}
&x(t) - C(0) - \int_0^\infty y(s) f(s)\, ds - \sum_{i=1}^{\infty} d(i) \\
&\qquad = \int_t^{[t+1]} y(s)\, a(s)\, (x([s]) - x(t))\, ds - \int_t^\infty y(s) f(s)\, ds - \sum_{i=[t]+1}^{\infty} d(i).
\end{aligned}
\tag{4.9}
$$

From (4.9), together with (3.2), (3.4) and the boundedness of $y(t)$ on $[0, \infty)$, we get for $t \geq 0$

$$
\begin{aligned}
&\left| x(t) - C(0) - \int_0^\infty y(s) f(s)\, ds - \sum_{i=1}^{\infty} d(i) \right| \\
&\qquad \leq |y|_B (L_2 + M) \int_t^{[t+1]} |a(s)|\, ds + |y|_B \int_t^\infty |f(s)|\, ds + \sum_{i=[t]+1}^{\infty} |d(i)|
\end{aligned}
$$

where $|y|_B = \sup_{t \geq 0} |y(t)|$. Thus, it follows that (4.7) is correct. Taking into account (4.2), it is easily verified that the limit relation (4.7) reduce to (2.2). So, the proof is completed.

Now, let us give the following example to illustrate our results.

Example 1. *Consider the linear impulsive differential equation with piecewise constant argument*

$$x'(t) = \frac{1}{(1+2t)^2}(x(t) - x([t])) + \frac{1}{(1+2t)^2}, \quad t \neq n \in \mathbb{Z}^+, t \geq 0, \tag{4.10}$$

$$\Delta x(n) = \frac{1}{2^n}, \quad n \in \mathbb{Z}^+ \tag{4.11}$$

with the initial condition

$$x(0) = 1, \tag{4.12}$$

that is a special case of $(1.1) - (1.3)$ *with* $a(t) = f(t) = \frac{1}{(1+2t)^2}$, $d(n) = \frac{1}{2^n}$ *and* $x_0 = 1$. *Here, all hypotheses of Theorem 2 and 3 are satisfied. So, as* $t \to \infty$, *the solution* $x(t)$ *of* $(4.10) - (4.12)$ *tends to a real constant, say* $\ell(1)$, *which can be calculated by* (2.2) *as*

$$\ell(1) = 2 + \int_0^\infty y(s) \frac{1}{(1+2s)^2} ds \tag{4.13}$$

where $y(t)$ *satisfies the integral equation*

$$y(t) = 1 + \int_t^{[t+1]} y(s) \frac{1}{(1+2s)^2} ds.$$

On the other hand, from the exact solution $x(t)$ *of* $(4.10) - (4.12)$ *we conclude that* $\lim_{t \to \infty} x(t) \in \mathbb{R}$ *where*

$$x(t) = e^{\frac{t-[t]}{(1+2t)(1+2[t])}} + \sum_{r=0}^{[t-1]} \left(e^{\frac{1}{(1+2r+2)(1+2r)}} - 1 \right) + \sum_{r=1}^{[t]} \frac{1}{2^r}.$$

Moreover, in this example, if we take $a(t) = 0$, *from* (2.3) *and* (2.2) *we can obtain easily* $y(t) = 1$ *and* $\ell(1) = \frac{5}{2}$, *respectively.*
We also note that, it is possible to find the same result from the exact solution

$$x(t) = 1 + \frac{t-[t]}{(1+2t)(1+2[t])} + \sum_{r=0}^{[t-1]} \frac{1}{(1+2r)(3+2r)} + \sum_{r=1}^{[t]} \frac{1}{2^r}.$$

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