Time-independent Schrödinger polyharmonic equation and applications
by
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Abstract
We prove that the time-independent Schrödinger polyharmonic equation $(-\Delta)^m u + q(x)u = \psi(x) > 0$, $x \in D$, where $D$ is an unbounded domain of $\mathbb{R}^n$ ($n \geq 2$) has a positive solution provided that the function $q$ belongs to a certain Kato class of functions $K_{\infty}^{m,n}(D)$. As applications, the existence and asymptotic behavior of positive solutions of some polyharmonic problems are established.

Key Words: Schrödinger polyharmonic equation, Green function, polyharmonic elliptic equation, positive solution.

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1 Introduction and statement of main results
Considerable attention has been given to the time-independent Schrödinger equation

$$-\Delta u + q(x)u = \psi(x), \quad x \in \Omega \subseteq \mathbb{R}^n,$$

where $\Omega$ is an open subset of $\mathbb{R}^n$ and the potential $q$ belongs to the Kato class $K_{\text{loc}}^{n,1}(\Omega)$. See, e.g., Aizenman and Simon [1], Chiarenza, Fabes and Garofalo [4], Fabes and Strook [9], Hinz and Kalf [14], Simader [17], Zhao [20-22] and the references therein. Following different approaches these authors have studied the existence and regularity of the solutions for the Dirichlet problem. In [12, Theorem 5.1], the authors considered the following Schrödinger polyharmonic equation:

$$(-\Delta)^m u + q(x)u = \psi(x), \quad x \in \Omega \subseteq \mathbb{R}^n,$$

where $\Omega$ is the unit ball in $\mathbb{R}^n$ with $n \geq 1$ and $m \geq 1$. They have proved that if the coefficient $q$ in (1.2) is continuous in $\overline{\Omega}$ and sufficiently small, $\psi$ positive implies that the solution $u$ of the Dirichlet problem for (1.2) is positive. The later result has been extended by the same authors in [13], by considering domains which are close to the unit ball and operators close to $(-\Delta)^m$. On the other hand in [5, Proposition 2.10], the authors studied the equation (1.2) in the case $m = 2$, $n > 4$ and where $\Omega = B(0,r)$ is the open ball of center 0 and radius $r$. 
They have showed that the problem (1.2) subject to either Dirichlet boundary conditions or Navier boundary conditions admits a nonnegative Green function on $B(0, r)$ provided that the function $q$ belongs to the Kato class $K^{n,2}_{loc}(B(0, r))$. For more related results we refer to [10]. For the convenience of the reader, we recall the definition of the functional class $K^{n,m}_{loc}$. 

**Definition 1.1.** [5] Given $n > 2m$ and $\Omega$ be an open subset of $\mathbb{R}^n$. The Kato class $K^{n,m}_{loc}(\Omega)$ is the set of functions $q \in L^1_{loc}(\Omega)$ such that for any compact set $K \subset \Omega$ the quantity

$$\Phi_q(r, K) = \sup_{x \in \mathbb{R}^n} \int_{B(x, r)} \frac{|q(y)| \chi_K(y)}{|x - y|^{n-2m}} dy$$

is finite (here, $\chi_K$ denotes the characteristic function of $K$) and

$$\lim_{r \to 0} \Phi_q(r, K) = 0.$$

We emphasize that the proofs presented by these authors are based in the following 3-G Theorem satisfied by be the Green function $G^B_{m,n}$ for the $m$-polyharmonic operator $u \to (-\Delta)^m u$ with Dirichlet boundary conditions on the unit ball $B$ in $\mathbb{R}^n$.

**Theorem 1.2.** [12, Proposition 4.1] Given $n > 2m$. There exists a constant $C_{m,n} > 0$ such that for all $x, y, z \in B$,

$$\frac{G^B_{m,n}(x, z) G^B_{m,n}(z, y)}{G^B_{m,n}(x, y)} \leq C_{m,n} [\|x - z\|^{2m-n} + \|z - y\|^{2m-n}].$$

We shall prove similar results remain valid for the equation (1.2) on the unbounded domain $D = \{ x \in \mathbb{R}^n : |x| > 1 \}$, where the function $q$ is assumed to belongs to the Kato class $K^{\infty}_{m,n}(D)$ (see Definition 1.3 below). As application we will answer the questions of existence and asymptotic behavior of positive solutions of some polyharmonic problems of the form:

$$\begin{cases}
(-\Delta)^m u + f(., u) = 0, & \text{in } D \quad \text{(in the sense of distributions)} \\
u > 0, \\
\lim_{x \to \zeta \in \partial D} \frac{u(x)}{|x|^{1-n} - 1} = \varphi(\zeta), \\
u(x) \simeq \rho_0(x) & \text{near } x = \infty,
\end{cases} \quad (1.3)$$

where $\varphi$ is a nonnegative continuous function on $\partial D$, $m$ is a positive integer and

$$\rho_0(x) = \begin{cases}
1, & \text{for } n > 2m \\
\ln |x|, & \text{for } n = 2m \\
|x|^{2m-n}, & \text{for } n < 2m.
\end{cases} \quad (1.4)$$

The notation $u(x) \simeq \rho_0(x)$, near $x = \infty$, means that for some constant $C > 0$,

$$\frac{1}{C} \rho_0(x) \leq u(x) \leq C \rho_0(x), \text{ when } x \text{ near } \infty.$$
Throughout this paper, we denote by $G_{m,n}^D$ a Green function of $(-\Delta)^m$ on $D$ with Dirichlet boundary conditions $(\frac{\partial}{\partial \nu})^j u = 0$, $0 \leq j \leq m - 1$.

In [3, Theorem 2.6], the authors proved the following 3-G Theorem: there exists a constant $C_{m,n} > 0$ such that for each $x, y, z \in D$,

$$\frac{G_{m,n}^D(x, z)G_{m,n}^D(z, y)}{G_{m,n}^D(x, y)} \leq C_{m,n} \left[ \left( \frac{\rho(z)}{\rho(x)} \right)^m G_{m,n}^D(x, z) + \left( \frac{\rho(z)}{\rho(y)} \right)^m G_{m,n}^D(y, z) \right],$$

(1.5)

where

$$\rho(z) = \begin{cases} |z|^{-1} & \text{if } n \geq 2m \\ |z|^{1-\frac{n}{m}} (|z|-1) & \text{if } n < 2m. \end{cases}$$

(1.6)

This form of the 3-G Theorem has been exploited to introduce the Kato class $K_{\infty}^{m,n}(D)$ as follows:

**Definition 1.3.** A Borel measurable function $q$ in $D$ belongs to the class $K_{\infty}^{m,n}(D)$ if $q$ satisfies the following conditions

$$\lim_{r \to 0} \left( \sup_{x \in D} \int_{D \cap B(x, r)} \left( \frac{\rho(y)}{\rho(x)} \right)^m G_{m,n}^D(x, y) |q(y)| dy \right) = 0,$$

(1.7)

$$\lim_{M \to \infty} \left( \sup_{x \in D} \int_{|y| \geq M} \left( \frac{\rho(y)}{\rho(x)} \right)^m G_{m,n}^D(x, y) |q(y)| dy \right) = 0,$$

(1.8)

where $\rho$ is given by (1.6).

This class contains for example any function belonging to $L^s(D) \cap L^1(D)$ with $s > \frac{n}{2m} > 1$ (see Example 3.1).

We point out that the class $K_{\infty}^{m,n}(D)$ is well adapted to study various existence and multiplicity results for wide classes of polyharmonic boundary value problems including the case of equations with blow-up at infinity. In the later case, we develop a more careful analysis with respect to other recent papers in this field for $m = 1$ (see, e.g. [6, 11, 16]).

Next we shall often refer in this paper to $h_{m,n}$ the $m$-harmonic function defined in $D$ by

$$h_{m,n}(x) := |x|^{2m-n} G_{m,n}^B(j(x), 0) = k_{m,n} \int_{1}^{v} \frac{(v^2 - 1)^{m-1}}{v^{n-1}} dv,$$

(1.9)

where $j : D \cup \{\infty\} \to B$, $j(x) = |x|^{-2} x$ is the inversion and $k_{m,n} = \frac{\Gamma(\frac{n}{2})}{2^{2m-1} \pi^{\frac{n}{2}} (m-1)!^2}$.

Observe that,

$$h_{m,n}(x) \simeq \rho_0(x), \text{ near } x = \infty.$$  

(1.10)

We also let $H_D \varphi$ be the bounded continuous solution of the Dirichlet problem

$$\begin{cases}
\Delta u = 0, & \text{in } D \\
u = \varphi & \text{on } \partial D \\
\lim_{|x| \to \infty} u(x) = 0,
\end{cases}$$

(1.11)
where \( \varphi \) is a nonnegative nontrivial continuous on \( \partial D \) and \( h_{1,n} \) is the harmonic function defined by (1.9).

Note that from [8, p.427], the function \( H_D \varphi \) belongs to \( C(\overline{D} \cup \{\infty\}) \) and satisfies

\[
\lim_{|x| \to \infty} |x|^{n-2} H_D \varphi (x) = c > 0.
\] (1.12)

Our plan is organized as follows. In section 2, we will first study the existence and uniqueness of positive classical solution for the linear problem

\[
\begin{cases}
(-\Delta)^m u = \mathcal{F}, & \text{in } D \\
u > 0, & \\
(-\partial_j\nu)^j u = 0 & \text{on } \partial D \text{ for } j = 0, \ldots, m-2, \\
(-\partial_{\nu}^{m-1}) u = \phi & \text{on } \partial D,
\end{cases}
\]

subject to an asymptotic behavior at \( \infty \), where the functions \( \mathcal{F} \) and \( \phi \) are required to satisfy some convenient hypotheses.

In section 3, we collect some properties of functions belonging to \( K_{\infty,m,n} (D) \). In particular, we derive from the 3-G Theorem (1.5) that for each \( q \in K_{\infty,m,n} (D) \), we have

\[
\alpha_q := \sup_{x,y \in D} \int_D \frac{G_{m,n}^D(x,z)G_{m,n}^D(z,y)}{G_{m,n}^D(x,y)} |q(z)| \, dz < \infty.
\]

Next, we exploit again the inequality (1.5) to prove that on \( D \) the inverse of polyharmonic operators that are perturbed by a zero-order term, are positivity preserving. That is, if the coefficient \( q \in K_{\infty,m,n} (D) \) with \( \alpha_q \leq \frac{1}{2} \) and \( \psi \) is positive, then the equation

\[
(-\Delta)^m u + q(x) u = \psi(x), \quad x \in D
\] (1.13)

has a positive solution.

In section 4, we will establish two existence results for the problem (1.3), where the function \( f \) is closed to linear.

More precisely, first we consider the nonlinearity \( f(x,t) = tg(x,t) \), we let

\[
\omega(x) := h_{m,n}(x) + \left(|x|^2 - 1\right)^{m-1} H_D \varphi(x),
\] (1.14)

and we assume that

\((H_1)\) \( g \) is a nonnegative measurable function on \( D \times [0, \infty) \).

\((H_2)\) For each \( \lambda > 0 \), there exists a positive function \( q_\lambda = q \in K_{\infty,m,n} (D) \) with \( \alpha_q \leq \frac{1}{2} \) such that for each \( x \in D \), the map \( t \mapsto t (q(x) - g(x, \omega(x))) \) is continuous and nondecreasing on \( [0, \lambda] \).

Using the pointwise estimates for the Green function and a perturbation arguments, we prove the following.
Theorem 1.4. Under hypotheses \( (H_1)-(H_2) \), the problem
\[
\begin{cases}
(-\Delta)^m u + u g(\cdot, u) = 0, \text{ in } D \ (\text{in the sense of distributions}) \\
u > 0, \\
\lim_{x \to \zeta \in \partial D} \frac{u(x)}{|x|^2-1}^{m-1} = \varphi(\zeta), \\
u(x) \asymp \rho_0(x), \text{ near } x = \infty,
\end{cases}
\]
(1.15)
has at least one positive continuous solution \( u \) satisfying
\[
(1 - \alpha_q) \omega(x) \leq u(x) \leq \omega(x).
\]
Moreover, for \( n \geq 2m \) we obtain
\[
\lim_{|x| \to \infty} \frac{u(x)}{h_{m,n}(x)} = 1.
\]
This result extends Theorem 2 of [15] to the polyharmonic case.

To prove a second existence result for the problem (1.3), we fix a positive harmonic function \( h_0 \) in \( D \), which is continuous and bounded in \( D \), we let
\[
\omega_0(x) = h_{m,n}(x) + \left(|x|^2 - 1\right)^{m-1} h_0(x),
\]
and we assume that:

\( (A_1) \) \( f \) is a nonnegative Borel measurable function on \( D \times (0, \infty) \), which is continuous with respect to the second variable.

\( (A_2) \) There exists a positive function \( q \in K_{m,n}^\infty(D) \) with \( \alpha_q \leq \frac{1}{2} \) such that \( \forall x \in D \) and \( \forall t \geq s \geq \omega_0(x) \) we have
\[
\begin{cases}
f(x,t) - f(x,s) \leq q(x)(t-s) \text{ and} \\
0 \leq f(x,t) \leq tq(x).
\end{cases}
\]
Then we prove the following theorem, which extends Theorem 1.2 of [18] to the polyharmonic case.

Theorem 1.5. Assume \( (A_1)-(A_2) \), then there exists a constant \( c_1 > 1 \) such that if \( \bar{c} \geq c_1 \) and \( \varphi \geq c_1 h_0 \) on \( \partial D \), then problem (1.3) has at least one positive continuous solution \( u \) satisfying for each \( x \in D \)
\[
\omega_0(x) \leq u(x) \leq \omega(x),
\]
where \( \omega(x) = \bar{c} h_{m,n}(x) + \left(|x|^2 - 1\right)^{m-1} H_D \varphi(x) \).
Moreover, for \( n \geq 2m \) we have
\[
\lim_{|x| \to \infty} \frac{u(x)}{h_{m,n}(x)} = \bar{c}.
\]
A typical example of nonlinearity satisfying \( (A_1)-(A_2) \) :
\[
f(x,t) = p(x)t^\gamma, \text{ for } \gamma \in (0,1] \text{ and some appropriate } p \text{ admissible.}
\]
As usual, we denote by \( B^+(D) \), the set of of nonnegative Borel measurable functions in \( D \).
For $x, y \in \mathbb{R}^n$, we let
\[
[x, y]^2 = |x - y|^2 + \left( |x|^2 - 1 \right) \left( |y|^2 - 1 \right),
\]
and
\[
\theta(x, y) = [x, y]^2 - |x - y|^2 = \left( |x|^2 - 1 \right) \left( |y|^2 - 1 \right).
\]

For $\psi \in \mathcal{B}^+(D)$, we define
\[
V\psi(x) := V_{m,n}\psi(x) = \int_D G_{m,n}^D(x, y)\psi(y)dy, \text{ for } x \in D.
\]
and
\[
\|\psi\| := \sup_{x \in D} \int_D \left( \frac{\rho(y)}{\rho(x)} \right)^m G_{m,n}^D(x, y)\psi(y)dy.
\]

For a continuous function $\varphi$ on $\partial D$ we denote by $P\varphi$ the function defined in $D$ by
\[
P\varphi(x) = \int_{\partial D} P(x, \xi)\varphi(\xi)\sigma(d\xi),
\]
where $P(x, \xi) := \frac{|x|^2 - 1}{|x - \xi|^n}$ is the Poisson kernel on $D$ and $\sigma$ is the normalized measure on the unit sphere of $\mathbb{R}^n$. We remark that $P\varphi$ is a harmonic function in $D$ satisfying $\lim_{x \to \xi \in \partial D} P\varphi(x) = \varphi(\xi)$. We also denote by $\mathcal{H}$ the set of nonnegative harmonic functions $h$ defined in $D$ by
\[
h(x) = \int_{\partial D} P(x, \xi)\nu(d\xi),
\]
where $\nu$ is a nonnegative measure on $\partial D$ and $P(x, \xi)$ is the Poisson kernel on $D$.

Let $f$ and $g$ be two nonnegative functions on a set $S$. We call $f \preceq g$ on $S$ if and only if there exists a constant $C > 0$ such that
\[
f(x) \leq Cg(x) \text{ for all } x \in S.
\]
We say $f \simeq g$ on $S$ if and only if there exists a constant $C > 0$ such that
\[
\frac{1}{C}g(x) \leq f(x) \leq Cg(x) \text{ for all } x \in S.
\]
The letter $C$ will denote a generic positive constant which may vary from line to line.

2 The linear boundary value problem

First we consider the polyharmonic prototype Dirichlet problem:

\[
\begin{cases}
(-\Delta)^m u = f^*, \text{ in } B, \\
u > 0, \text{ in } B, \\
(-\frac{\partial}{\partial \nu})^j u = 0 \text{ on } \partial B \text{ for } j = 0, \ldots, m - 2, \\
(-\frac{\partial}{\partial \nu})^{m-1} u = \phi \text{ on } \partial B,
\end{cases}
\]
where $f^*$ is a positive function belonging to $C^{0,\gamma}(\mathcal{B})$ and $\phi$ is a positive function belonging to $C^{m+1,\gamma}(\partial\mathcal{B})$, for $0 < \gamma < 1$.

We recall the following existence results which is stated in [10].

**Theorem 2.1.** [10] The problem (2.1) admits a unique classical solution $u$ given by

$$u(x) = \int_B G_{m,n}^B(x,y)f^*(y)dy + \int_{\partial B} L_{m,n}^B(x,\xi)\phi(\xi)d\omega(\xi), \quad x \in B$$

(2.2)

where the Poisson kernel $L_{m,n}^B$ is defined by

$$L_{m,n}^B(x,\xi) = \frac{\Gamma\left(\frac{n}{2}\right)}{2^m(m-1)!\pi^{\frac{n}{2}}} \frac{(1-|x|^2)^m}{|x-\xi|^n}, \quad \text{with } x \in B, \ \xi \in \partial B.$$  

(2.3)

**Proposition 2.2.** The unique positive solution $u$ of the problem (2.1), satisfies

$$(1-|x|)^m \leq u(x) \leq (1-|x|)^{m-1}, \quad \text{on } B.$$  

(2.4)

**Proof:** Let $f^*$ be a positive function belonging to $C^{0,\gamma}(\mathcal{B})$ and $\phi$ be a positive function belonging to $C^{m+1,\gamma}(\partial\mathcal{B})$, for $0 < \gamma < 1$. It is clear that

$$(1-|x|)^m \leq \int_{\partial B} L_{m,n}^B(x,\xi)\phi(\xi)d\omega(\xi) \leq (1-|x|)^{m-1}.$$  

(2.5)

Next, we aim at proving that

$$\int_B G_{m,n}^B(x,y)f^*(y)dy \simeq (1-|x|)^m, \quad \text{on } B,$$

(2.6)

where, $G_{m,n}^B$ is the Green function for the $m$-polyharmonic operator $u \rightarrow (-\Delta)^m u$ with Dirichlet boundary conditions on the unit ball $B$ in $\mathbb{R}^n$.

To this end, we claim that on $B^2$ (that is $(x,y) \in B^2$), we have

$$(1-|x|)^m (1-|y|)^m \leq G_{m,n}^B(x,y) \leq \begin{cases} (1-|x|)^m & \text{if } m \geq n, \\ \frac{(1-|x|)^m}{|x-y|^n} & \text{if } m < n. \end{cases}$$  

(2.7)

Indeed, from [12, Proposition 2.3], we have

$$G_{m,n}^B(x,y) \simeq \begin{cases} |x-y|^{2m-n} \min\left(1, \frac{(1-|x|)(1-|y|)^m}{|x-y|^{2m}}\right) & \text{for } n > 2m, \\ \log(1+ \frac{(1-|x|)(1-|y|)^m}{|x-y|^{2m}}) & \text{for } n = 2m, \\ \frac{(1-|x|)(1-|y|)^m-\frac{n}{2}}{|x-y|^n} \min\left(1, \frac{(1-|x|)(1-|y|)^m}{|x-y|^{2m}}\right) & \text{for } n < 2m. \end{cases}$$
Which implies that

$$G_{m,n}^B(x,y) \simeq \begin{cases} 
\frac{((1 - |x|)(1 - |y|))^m}{|x - y|^m} & \text{for } n > 2m, \\
\frac{((1 - |x|)(1 - |y|))^m}{|y|^n} & \text{for } n < 2m.
\end{cases}$$

So the lower inequality in (2.7) follows from (2.8) and the fact that for each $x, y \in B$, we have $|x - y| \leq |x, y| \leq 1$.

Now, if $m \geq n$, then using the fact that for each $x, y \in B$, we have $(1 - |y|) \leq |x, y|$, we deduce form (2.8) that

$$G_{m,n}^B(x,y) \simeq \frac{((1 - |x|)(1 - |y|))^m}{|x, y|^n} \leq ((1 - |x|))^m.$$ 

By similar argument we prove the upper inequality in (2.7) for the case $m < n$.

So using (2.7), we obtain

$$(1 - |x|)^m \leq \int_B G_{m,n}^B(x,y)f^*(y)dy \leq (1 - |x|)^m \int_B \frac{f^*(y)}{|x-y|^m}dy
\leq (1 - |x|)^m \int_{B(0,2)} \frac{1}{z^{\max(m-n,0)}}dz
\leq (1 - |x|)^m.$$ 

This proves (2.6).

Finally, the required inequality (2.4) follows from (2.2), (2.5) and (2.6).

The $m$-Kelvin transform of a function $u$, is defined by

$$v(y) = |y|^{2m-n} u\left(\frac{y}{|y|^2}\right), \text{ for } y \in D.$$ (2.9)

By direct computation, $v(y)$ satisfies

$$\Delta^m v(y) = |y|^{-2m-n} (\Delta^m u)\left(\frac{y}{|y|^2}\right).$$ (2.10)

See [19, p. 221]. This fact and Theorem 2.1 and Proposition 2.2 immediately imply the following result.

**Theorem 2.3.** Let $F$ be a nonnegative function such that $x \rightarrow |x|^{-2m-n} F\left(\frac{y}{|y|^2}\right) \in C^{\alpha,\gamma}(B)$ and $\phi$ is a positive function belonging to $C^{m+1,\gamma}(\partial B)$, for $0 < \gamma < 1$. Then the problem

$$\begin{cases} 
(-\Delta)^m v = F, \text{ in } D, \\
v > 0, \text{ in } D, \\
(-\frac{\partial}{\partial u})^j v = 0 \text{ on } \partial D \text{ for } j = 0, \ldots, m - 2, \\
(-\frac{\partial}{\partial u})^{m-1} v = \phi \text{ on } \partial D, \\
v(y) \simeq |y|^{2m-n} \text{ near } \infty
\end{cases}$$ (2.11)
admits a unique classical solution \( v \) satisfying

\[
|y|^{m-n} (|y|-1)^{m} \leq v(y) \leq |y|^{m-n+1} (|y|-1)^{m-1}, \quad \text{on } D.
\]  

(2.12)

3 The Kato class \( K_{m,n}^{\infty} (D) \) and Schrödinger polyharmonic equation

3.1 The Kato class \( K_{m,n}^{\infty} (D) \)

Example 3.1. Given \( s > \frac{n}{2m} > 1 \). Then \( L^{s} (D) \cap L^{1} (D) \subset K_{m,n}^{\infty} (D) \).

Indeed, let \( 0 < r < 1 \) and \( q \in L^{s} (D) \cap L^{1} (D) \) with \( s > \frac{n}{2m} > 1 \).

Since for each \( x, y \in D \), we have

\[
G_{m,n}^{D}(x, y) = |x|^{2m-n} |y|^{2m-n} G_{m,n}^{B}(j(x), j(y)),
\]  

(3.1)

then by using (2.8), there exists a constant \( C > 0 \), such that for each \( x, y \in D \)

\[
\left( \frac{\rho(y)}{\rho(x)} \right)^{m} G_{m,n}^{D}(x, y) \leq C \frac{1}{|x-y|^{n-2m}}.
\]  

(3.2)

This fact and the Hölder inequality imply that

\[
\int_{B(x,r) \cap D} \left( \frac{\rho(y)}{\rho(x)} \right)^{m} G_{m,n}^{D}(x, y) |q(y)| dy \leq C \int_{B(x,r) \cap D} \frac{|q(y)|}{|x-y|^{s-2m}} dy
\]

\[
\leq C \left( \int_{D} |q(y)|^{s} dy \right)^{\frac{1}{s}} \times \left( \int_{B(x,r)} |x-y|^{(2m-n)\frac{s-1}{s}} dy \right)^{\frac{s-1}{s}}
\]

\[
\leq C \left( \int_{0}^{r} t^{(2m-n)\frac{s-1}{s}} + n-1 dt \right)^{\frac{s-1}{s}} \to 0,
\]

as \( r \to 0 \), since \( (2m-n) \frac{s-1}{s} + n-1 < -1 \) when \( s > \frac{n}{2m} \).

This shows that \( q \) satisfies (1.7).

We claim that \( q \) satisfies (1.8). Indeed, let \( M > 0 \), then for each \( \varepsilon > 0 \), there exists \( r > 0 \), such that

\[
\int_{|y| \geq M} \left( \frac{\rho(y)}{\rho(x)} \right)^{m} G_{m,n}^{D}(x, y) |q(y)| dy
\]

\[
\leq \frac{\varepsilon}{2} + C \int_{|x-y| \geq r \cap |y| \geq M} \frac{|q(y)|}{|x-y|^{s-2m}} dy
\]

\[
\leq \frac{\varepsilon}{2} + C \int_{|y| \geq M} |q(y)| dy \to 0, \quad \text{as } M \to \infty.
\]

\[ \Box \]

Next we collect some properties of the Kato class \( K_{m,n}^{\infty} (D) \), which are useful to establish of our main results. For the proofs we refer to [3].
Proposition 3.2.  [3] Let \( q \) be a nonnegative function in \( K_{\infty}^{m,n}(D) \). Then we have

(i) \( \|q\| < \infty \).

(ii) \( x \to q(x) \in L_{\text{loc}}^1(D) \).

(iii) For each bounded function \( h \) in \( \mathfrak{F} \), the function

\[
x \to \int_D \frac{(|y|^2-1)^{m-1}}{(|x|^2-1)^{m-1}} h(y) G_{m,n}^{D}(x,y) |q(y)| \, dy,
\]

is continuous in \( \overline{D} \), vanishes at the boundary \( \partial D \).

(iv) The family of functions \( \left\{ \frac{1}{h_{m,n}()} \int_D G_{m,n}^{D}(\cdot,y) h_{m,n}(y) \zeta(y) \, dy : |\zeta| \leq q \right\} \) is relatively compact in \( C(\overline{D} \cup \{\infty\}) \).

Furthermore,

\[
\lim_{|x| \to \infty} \frac{1}{h_{m,n}(x)} \int_D G_{m,n}^{D}(x,y) h_{m,n}(y) q(y) \, dy = 0, \text{ for } n \geq 2m.
\]

Lemma 3.3. Let \( q \) be a nonnegative function in \( K_{\infty}^{m,n}(D) \). Then we have

(i)

\[
\alpha_q := \sup_{x,y \in D} \int_D \frac{G_{m,n}^{D}(x,z) G_{m,n}^{D}(z,y)}{G_{m,n}^{D}(x,y)} |q(z)| \, dz < \infty. \tag{3.3}
\]

(ii)

\[
V(qG_{m,n}^{D}(\cdot,y))(x) \leq \alpha_q G_{m,n}^{D}(x,y), \text{ for each } x,y \in D. \tag{3.4}
\]

Proof: (i) It follows from the 3-G Theorem (1.5) and proposition 3.2, that \( \alpha_q \leq 2C_{m,n} \|q\| < \infty \).

(ii) Inequality (3.4) follows immediately from the definitions of the potential function \( V \) and \( \alpha_q \).

Proposition 3.4. Let \( q \in K_{\infty}^{m,n}(D) \). Then for each \( x \in D \),

\[
\int_D G_{m,n}^{D}(x,y) h_{m,n}(y) |q(y)| \, dy \leq \alpha_q h_{m,n}(x). \tag{3.5}
\]

Proof: It follows from (3.1), that

\[
h_{m,n}(x) = \lim_{|y| \to \infty} |y|^{n-2m} G_{m,n}^{D}(x,y). \tag{3.6}
\]

In particular

\[
\lim_{|y| \to \infty} G_{m,n}^{D}(z,y) = h_{m,n}(z) h_{m,n}(x)^{-1}. \tag{3.7}
\]

Thus by Fatou’s lemma and (3.7), we deduce that, for \( x \in D \)

\[
\int_D G_{m,n}^{D}(x,y) h_{m,n}(y) |q(y)| \, dy \leq \lim_{|z| \to \infty} \int_D \frac{G_{m,n}^{D}(x,y) G_{m,n}^{D}(y,z)}{G_{m,n}^{D}(x,z)} |q(y)| \, dy \leq \alpha_q.
\]
Proposition 3.5. For all \( q \in K_{m,n}^\infty (D) \) and any \( h \in \mathcal{J} \), we have for each \( x \in D \),
\[
\int_D G_{m,n}^D(x,z) \left( |z|^2 - 1 \right)^{m-1} h(z) |q(z)| \, dz \leq \alpha_q \left( |x|^2 - 1 \right)^{m-1} h(x).
\]

Proof: Let \( h \in \mathcal{J} \). Then there exists a nonnegative measure \( \nu \) on \( \partial D \) such that
\[
h(x) = \int_{\partial D} P(x,\xi) \nu(d\xi).
\]
So we need only to verify (3.8) for \( h(y) = P(y,\xi) \) uniformly in \( \xi \in \partial D \).

From (3.1) and [12, lemma 2.1], we deduce that the Green function \( G_{m,n}^D \) satisfies
\[
G_{m,n}^D(x,y) = k_{m,n} |x-y|^{2m-n} \int_0^{\frac{|x-y|}{|x-y|^2}} \frac{(v^2-1)^{m-1}}{v^{m-1}} \, dv.
\]

Using the transformation \( v^2 = 1 + \frac{\theta(x,y)}{|x-y|^2} (1-t) \) in (3.9), we obtain
\[
G_{m,n}^D(x,y) = \frac{k_{m,n}}{2} \frac{(\theta(x,y))^m}{|x,y|^n} \int_0^1 \frac{(1-t)^{m-1}}{(1-t \frac{\theta(x,y)}{|x,y|^2})^\frac{n}{2}} \, dt.
\]
This implies that for each \( x,z \in D \) and \( \xi \in \partial D \),
\[
\lim_{y \to \xi} \frac{G_{m,n}^D(z,y)}{G_{m,n}^D(x,y)} = \frac{(|z|^2 - 1)^{m-1} P(z,\xi)}{(|x|^2 - 1)^{m-1} P(x,\xi)}.
\]
Thus by Fatou’s lemma and (3.10), we deduce that, for \( x \in D \), and \( \xi \in \partial D \),
\[
\int_D G_{m,n}^D(x,z) \left( |z|^2 - 1 \right)^{m-1} P(z,\xi) |q(z)| \, dz \leq \liminf_{y \to \xi} \int_D \frac{G_{m,n}^D(x,z)G_{m,n}^D(z,y)}{G_{m,n}^D(x,y)} |q(z)| \, dz \leq \alpha_q.
\]

3.2 The Schrödinger polyharmonic equation

For a nonnegative function \( q \) in \( K_{m,n}^\infty (D) \) such that \( \alpha_q \leq \frac{1}{2} \), we put
\[
G_{m,n}(x,y) = \begin{cases} \sum_{k \geq 0} (-1)^k (V(q))^k \left( G_{m,n}^D(\cdot,y) \right)(x) & \text{if } x \neq y \\ +\infty & \text{if } x = y. \end{cases}
\]
Then we have
Lemma 3.6. Let $q$ be a nonnegative function in $K_{m,n}^\infty(D)$ such that $\alpha_q \leq \frac{1}{2}$. Then for each $x, y$ in $D$, we have
\[
(1 - \alpha_q) G_{m,n}^D(x,y) \leq G_{m,n}(x,y) \leq G_{m,n}^D(x,y).
\]

Proof: Since $\alpha_q \leq \frac{1}{2}$, we deduce from (3.4), that for $x \neq y$
\[
|G_{m,n}(x,y)| \leq \sum_{k \geq 0} (\alpha_q)^k G_{m,n}^D(x,y) = \frac{1}{1 - \alpha_q} G_{m,n}^D(x,y).
\]
On the other hand, from the expression for $G_{m,n}$, we deduce that for $x \neq y$
\[
G_{m,n}(x,y) = G_{m,n}^D(x,y) - V(qG_{m,n}(.,y))(x).
\]
Using these facts and (3.4), we obtain that
\[
G_{m,n}(x,y) \geq G_{m,n}^D(x,y) - \frac{\alpha_q}{1 - \alpha_q} G_{m,n}^D(x,y) = \frac{1 - 2\alpha_q}{1 - \alpha_q} G_{m,n}^D(x,y) \geq 0.
\]
Hence the result follows from (3.12) and (3.4).

In the sequel, for a given nonnegative function $q \in K_{m,n}^\infty(D)$ such that $\alpha_q \leq \frac{1}{2}$, we define the operator $V_q$ on $B^+(D)$ by
\[
V_q \psi(x) = \int_D G_{m,n}(x,y) \psi(y)dy, \quad x \in D.
\]
Then, we have the following Lemma.

Lemma 3.7. Let $q$ be a nonnegative function in $K_{m,n}^\infty(D)$ such that $\alpha_q \leq \frac{1}{2}$ and $\psi \in B^+(D)$. Then $V_q \psi$ satisfies the following resolvent equation:
\[
V \psi = V_q \psi + V_q(qV \psi) = V_q \psi + V(qV_q \psi).
\]

Proof: From the expression for $G_{m,n}$, we deduce for $\psi \in B^+(D)$ such that $V \psi < \infty$,
\[
V_q \psi = \sum_{k \geq 0} (-1)^k (V(q.))^k V \psi.
\]
So we obtain that
\[
V_q(qV \psi) = \sum_{k \geq 0} (-1)^k (V(q.))^k [V(qV \psi)]
= - \sum_{k \geq 1} (-1)^k (V(q.))^k V \psi
= V \psi - V_q \psi.
\]
The second equality is proved by integrating (3.12).
Proposition 3.8. Let $q$ be a nonnegative function in $K_{m,n}^\infty(D)$ such that $\alpha_q \leq \frac{1}{2}$ and let $\psi \in L^1_{\text{loc}}(D)$ be such that $V\psi \in L^1_{\text{loc}}(D)$. Then $V_q\psi$ is a solution of the time-independent Schrödinger polyharmonic equation (1.13).

Proof: Using the resolvent equation (3.13), we have

$$V_q\psi = V\psi - V(qV_q\psi).$$

Applying the operator $(-\Delta)^m$ on both sides of the above equality, we obtain that

$$(-\Delta)^m(V_q\psi) = \psi - qV_q\psi$$

(in the sense of distributions).

This completes the proof. \qed

4 Proofs of Theorems 1.4 and 1.5

Proof of Theorem 1.4.

Let $\varphi$ be a nonnegative continuous function on $\partial D$ and $H_D\varphi$ the bounded continuous solution of the Dirichlet problem (1.11). We recall that

$$\omega(x) = h_{m,n}(x) + \left(|x|^2 - 1 \right)^{m-1} H_D\varphi.$$  

Since $g$ satisfies $(H_2)$, there exists a nonnegative function $q \in K_{m,n}^\infty(D)$ such that $\alpha_q \leq \frac{1}{2}$ and for each $x \in D$, the map $t \rightarrow t(g(x) - g(x,t\omega(x)))$ is continuous and nondecreasing on $[0,1]$. We consider the closed convex set $\Lambda$ given by

$$\Lambda := \{ v \in B^+(D) : (1 - \alpha_q) \leq v \leq 1 \}.$$  

We define the operator $T$ on $\Lambda$ by

$$Tv(x) := \frac{1}{\omega(x)} [\omega(x) - V_q(q\omega)(x)] + \frac{1}{\omega(x)} V_q[(q - g(.,\omega))\omega v](x), \quad \text{for } x \in D. \quad (4.1)$$

By $(H_2)$, we deduce that

$$0 \leq g(x,t\omega(x)) \leq q(x), \quad \text{for each } x \in D \text{ and } t \in [0,1].$$

Hence,

$$0 \leq g(.,\omega v) \leq q, \quad \text{for all } v \in \Lambda. \quad (4.2)$$

So the operator $T$ is well defined on $\Lambda$. On the other hand, using (3.5), (3.8) and (3.3) we have

$$\frac{1}{\omega} V_q(q\omega) \leq \alpha_q < \infty. \quad (4.3)$$

We claim that $\Lambda$ is invariant under $T$. Indeed, using (4.1) and (4.3) we have for $v \in \Lambda$,

$$Tv \leq \frac{1}{\omega} [\omega - V_q(q\omega)] + \frac{1}{\omega} V_q(q\omega v) \leq 1.$$
Furthermore, from (4.1), (4.2) and (4.3), we obtain
\[ Tv \geq \frac{1}{\omega} [\omega - V_{q}(\omega)] \geq (1 - \alpha_{q}). \]

Next, we will prove that the operator \( T \) is nondecreasing on \( \Lambda \). Indeed, let \( u, v \in \Lambda \) be such that \( u \leq v \). Since the map \( t \to t (q(x) - g(x, t\omega(x))) \) is nondecreasing on \([0, 1]\), for \( x \in D \), we obtain
\[ Tv - Tu = \frac{1}{\omega} V_{q} [v (q - g (\omega)) - u (q - g (\omega))] \geq 0. \]

Now, we consider the sequence \( (v_{k}) \) defined by \( v_{0} = (1 - \alpha_{q}) \in \Lambda \) and \( v_{k+1} = Tv_{k} \) for \( k \in \mathbb{N} \). Since \( \Lambda \) is invariant under \( T \), then \( v_{1} = Tv_{0} \geq v_{0} \), and so from the monotonicity of \( T \), we deduce that
\[ (1 - \alpha_{q}) = v_{0} \leq v_{1} \leq \ldots \leq v_{k} \leq v_{k+1} \leq 1. \]

Furthermore, by \( (H_{2}) \) it is clear for each \( x \in D \) that the map \( t \to t g (\omega(x)) \) is continuous on \([0, \infty)\). Which together with the dominated convergence theorem imply that the sequence \( (v_{k}) \) converges to a function \( v \in \Lambda \) which is a fixed point of \( T \). We let \( u(x) = \omega(x) v(x) \), for each \( x \in D \).

Then \( u \) satisfies \((1 - \alpha_{q}) \omega \leq u \leq \omega \) and
\[ u = (I - V_{q}(q)) \omega + V_{q} ([q - g (\omega, u)] u). \]

That is
\[ (I - V_{q}(q)) u = (I - V_{q}(q)) \omega - V_{q} (ug(\omega, u)). \]

Applying the operator \((I + V(q))\) on both sides of the above equality and using (3.13) we deduce that \( u \) satisfies
\[ u = \omega - V (ug(\omega, u)). \tag{4.4} \]

Finally, we need to verify that \( u \) is a positive continuous solution for the problem (1.3). Indeed, from (4.2) we obtain
\[ ug(\omega, u) \leq \omega q. \tag{4.5} \]

We deduce by Proposition 3.2(ii), that \( ug(\omega, u) \in L_{loc}^{1}(D) \) and by (3.5) and (3.8) that \( V(ug(\omega, u)) \leq V(\omega q) \leq \alpha_{q} \omega \in L_{loc}^{1}(D) \).

Hence we conclude by [7], that \( u \) satisfies (in the sense of distributions) the elliptic differential equation
\[ (-\Delta)^{m} u + u g(\omega, u) = 0 \text{ in } D. \]

Finally, since by Proposition 3.2, (3.5) and (3.8) the function \( x \mapsto \frac{V(\omega q)(x)}{\omega(x)} \), is continuous and bounded, then by writing
\[ \frac{1}{\omega} V(\omega q) = \frac{1}{\omega} [V(ug(\omega, u)) + V(\omega q - ug(\omega, u))], \]

we deduce that \( u \in C(D) \). Using (4.4), (4.5), (1.12) and again Proposition 3.2, we obtain that
\[ \lim_{x \to \zeta \in \partial D} \frac{u(x)}{|x|^{2} - 1} = \varphi(\zeta) \text{ and for } n \geq 2m, \lim_{|x| \to \infty} \frac{u(x)}{h_{m,n}(x)} = 1. \] This ends the proof. \( \square \)
Example 4.1. Let $\gamma, \sigma \in \mathbb{R}_+$ and $\lambda < 2m \leq 2m + \max (0, 2m - n) < \mu$.
Let $\varphi$ be a nonnegative continuous bounded function on $\partial D$. Put $\omega (x) = h_{m,n}(x) + \left( |x|^2 - 1 \right)^{m-1} H_D \varphi$.
Assume that $p$ is a nonnegative Borel measurable function on $D$ satisfying
\[ p(x) \leq \frac{\nu}{|x|^\mu - \lambda \left( |x| - 1 \right)^\lambda \omega (x) (1 + \omega (x))}, \]
where $\nu$ is a sufficiently small positive constant. Then the problem
\[
\begin{cases}
(-\Delta)^m u + p(x)u^\gamma \log (1 + u) = 0, & \text{in } D \text{ (in the sense of distributions)} \\
\lim_{x \to \xi \in \partial D} \frac{u(x)}{|x|^2 - 1}^{m-1} = \varphi (\xi), \\
u(x) \sim \rho_0(x), \text{ near } x = \infty,
\end{cases}
\]
has a continuous positive solution $u$ satisfying
\[ u(x) \sim h_{m,n}(x) + \left( |x|^2 - 1 \right)^{m-1} H_D \varphi. \]
Moreover, for $n \geq 2m$ we have
\[ \lim_{|x| \to \infty} \frac{u(x)}{h_{m,n}(x)} = 1. \]

Proof of Theorem 1.5.

We recall that $h_0$ is a fixed positive harmonic function in $D$, which is continuous and bounded in $D$. Let $\varphi$ be a nonnegative nontrivial continuous bounded function on $\partial D$ and let $H_D \varphi$ be the bounded continuous solution of the Dirichlet problem (1.11).
Let $q \in K_{m,n}^\infty (D)$ be given by $(A_2)$ and put $c_1 = \frac{1}{1 - \alpha_q} > 1$. Let $\bar{c} \geq c_1$ and assume that
\[ (A_4) \quad \varphi (x) \geq c_1 h_0(x), \quad \forall x \in \partial D. \]
Put $\omega_0(x) = h_{m,n}(x) + \left( |x|^2 - 1 \right)^{m-1} h_0(x)$ and $\omega(x) = \bar{c} h_{m,n}(x) + \left( |x|^2 - 1 \right)^{m-1} H_D \varphi(x)$.
We consider the closed convex set $S$ given by
\[ S := \{ u \in B^+ (D) : \omega_0(x) \leq u(x) \leq \omega (x), \quad \text{for all } x \in D \}. \]
Since $H_D \varphi = \varphi$ on $\partial D$ and $h_0$ is continuous and bounded in $D$, we obtain by $(A_3)$ that $H_D \varphi \geq c_1 h_0$ on $D$. So $S$ is a well defined nonempty set in $B^+ (D)$.
By $(A_2)$, we deduce that
\[ 0 \leq f(\cdot, u) \leq q u, \quad \text{for any } u \in S. \quad (4.6) \]
So we define the operator $L$ on $S$ by
\[ Lu := \omega - V_q (q \omega) + V_q [q u - f(\cdot, u)]. \quad (4.7) \]
It is easy to verify that $S$ is invariant under $L$ and that the operator $L$ is nondecreasing on $S$.
Now, we consider the sequence $(u_k)$ defined by $u_0 = \omega_0 \in S$ and $u_{k+1} = Lu_k$ for $k \in \mathbb{N}$. Then we have
\[ \omega_0 \leq u_1 \leq \ldots \leq u_k \leq u_{k+1} \leq \omega. \]
Using $(A_2)$ and similar argument as in the proof of Theorem 1.4, we prove that the sequence $(u_k)$ converges to a function $u \in S$, which satisfies

$$u = \omega - Vf(\cdot, u).$$

Finally, we verify that $u$ is the required solution. □

**Example 4.2.** Let $\gamma \in (0,1) \ n > 2m$ and $\lambda < 2m < \mu$. Let $\varphi$ be a nonnegative continuous bounded function on $D$ and $h_0$ be a positive harmonic function in $D$, which is bounded and continuous in $\overline{D}$. Then from [2, p.258], there exists a constant $C > 0$, such that for each $x \in D$,

$$\frac{C (|x| - 1)}{(|x| + 1)^{n-1}} \leq h_0(x).$$

Suppose that $p$ is a nonnegative Borel measurable function on $D$ satisfying

$$p(x) \leq \frac{\nu}{|x|^\mu - \lambda (\gamma - 1)(u-m)} \frac{(|x|^1 - 1)^{\lambda + (\gamma - 1)m}}{\lambda + (\gamma - 1)m}$$

where $\nu$ is a sufficiently small positive constant. Then there exists a constant $c_1 > 1$ such that if $c \geq c_1$ and $\varphi \geq c_1 h_0$ on $\partial D$, the problem

$$\begin{cases}
\begin{align*}
(-\Delta)^m u + p(x) u^\gamma(x) &= 0, &\text{in } D \text{ (in the sense of distributions)} \\
\lim_{x \to \zeta \in \partial D} \frac{u(x)}{|x|^\gamma - 1} &= \varphi(\zeta), \\
\lim_{|x| \to \infty} \frac{u(x)}{h_{m,n}(x)} &= \bar{c}.
\end{align*}
\end{cases}$$

has a continuous positive solution $u$ satisfying for each $x \in D$

$$\omega_0(x) \leq u(x) \leq \omega(x).$$

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