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A family of seventh-order schemes for solving nonlinear systems

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Abstract

This paper focuses on solving nonlinear systems numerically. We propose an efficient family of three-step iterative schemes with seventh-order of convergence. The proposed methods are obtained by using the weight functions procedure and they do not require the evaluation of second or higher Frechet derivatives per iteration to proceed. Numerical comparisons are made with other existing methods to confirm the theoretical results and to show the performance of the presented schemes.

Nonlinear systems, iterative methods, order of convergence,multipoint methods.
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1 Introduction

Many applied problems in different fields of science, engineering and technology require to find the solutions of a nonlinear equation $f(x) = 0$, with $f : D \subseteq \mathbb{R} \to \mathbb{R}$, or a nonlinear system of equations $F(x) = 0$ with $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$. The best known method for finding a solution $\bar{x} \in D$, for being very simple and effective, is the Newton's scheme, whose iterative expression is

$$x^{(k+1)} = x^{(k)} - [F'(x^{(k)})]^{-1}F(x^{(k)}), \;\; k = 0, 1, \ldots,$$

where $F'(x^{(k)})$ is the Jacobian matrix of the function F evaluated in the kth iteration.

Multipoint iterative methods for solving nonlinear equations appeared for the first time in Ostrowski's book [15] and then they were extensively studied in Traub's text [20] and some papers published in the 1960s and 1970s, for example the ones of Jarratt ([10]) and King ([11]). This class of methods has gained importance in recent years, which has resulted in the creation of many methods of this type. Some examples of multipoint methods have been recently developed by Cordero et al. in [5, 6], Bi et al. in [3], Petković et al. in [17] and B. Neta et al. in [13]. A very interesting review of multipoint iterative methods can be found in the book of Petković et al. [16]. The reason for the revived interest in this area is the property of multipoint methods to overcome theoretical limits of one-point methods concerning the convergence order and computational efficiency, which is of great practical importance. The multipoint methods are introduced in the beginning to get the highest possible order of convergence using a fixed number of functional evaluations. This is closely related to the optimal order of convergence in the sense of the Kung-Traub conjecture. In fact, studying the optimal convergence rate of multipoint schemes, Kung and Traub in [12] conjectured that multipoint methods without memory for solving nonlinear equations, based on m + 1 functional evaluations per iteration, have the order of convergence at most 2^m.

Although there exist some robust and efficient methods for solving nonlinear systems (see for example [2, 8, 18]), it would be very interesting to be able to transform the existing methods for nonlinear equations to the multivariate case, but that is not possible, in general.

By using weight functions procedure, Soleymani et al. in [19] designed a family of three-step iterative schemes, with seventh-order of convergence, for solving nonlinear equations. In this paper, we extend this family to the multivariate case keeping the order of convergence. We use in this transformation several n-dimensional tools, among them, the n-dimensional operator $[x, y; F]$ defined by Ortega et al. in [14]. In the following, we remember some known notions and results that we need in order to analyze the convergence of the developed methods.

Let $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be sufficiently Frechet differentiable in an open convex set D. By using the notation introduced in [7], the qth derivative of F at $u \in \mathbb{R}^n$, $q \geq 1$ is the q-linear function $F^{(q)}(u) : \mathbb{R}^n \times \mathbb{R}^n \times \ldots \times \mathbb{R}^n \to \mathbb{R}^n$ such that $F^{(q)}(u)(v_1, v_2, \ldots, v_q) \in \mathbb{R}^n$. It is easy to observe that

1. $F^{(q)}(u)(v_1, v_2, \ldots, v_{q-1}, \cdot) \in \mathcal{L}(\mathbb{R}^n)$,
2. $F^{(q)}(u)(v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(q)}) = F^{(q)}(u)(v_1, v_2, \ldots, v_q)$ for all permutation σ of $\{1, 2, \ldots, q\}$,

where $\mathcal{L}(\mathbb{R}^n)$ is the set of lineal operators of $\mathbb{R}^n$ in $\mathbb{R}^n$.

From the above properties we can use the following notation:

a) $F^{(q)}(u)(v_1, v_2, \ldots, v_q) = F^{(q)}(u)v_1 \cdot \ldots \cdot v_q$,

b) $F^{(q)}(u)v^{q-1}F^{(p)}v^p = F^{(q)}(u)F^{(p)}(u)v^{q+p-1}$.

On the other hand, for $\bar{x} + h \in \mathbb{R}^n$ lying in a neighborhood of a solution $\bar{x}$ of $F(x) = 0$ we can apply Taylor's expansion and assuming that the Jacobian matrix $F'(\bar{x})$ is nonsingular, we have

$$F(\bar{x} + h) = F'(\bar{x}) \left[h + \sum_{q=2}^{p-1} C_q h^q \right] + O(h^p),$$

where $C_q = (1/q!)[F'(\bar{x})]^{-1} F^{(q)}(\bar{x}), q \geq 2$. We observe that $C_q h^q \in \mathbb{R}^n$ since $F^{(q)}(\bar{x}) \in \mathcal{L}(\mathbb{R} \times \ldots \times \mathbb{R}, \mathbb{R})$ and $[F'(\bar{x})]^{-1} \in \mathcal{L}(\mathbb{R}^n)$.

In addition, we can express F' as

$$F'(\bar{x} + h) = F'(\bar{x}) \left[I + \sum_{q=2}^{p-1} q C_q h^{q-1} \right] + O(h^p) = F'(\bar{x}) D(h) + O(h^p),$$

where I is the identity matrix of size $n \times n$. Therefore, $q C_q h^{q-1} \in \mathcal{L}(\mathbb{R}^n)$. From the previous equation we obtain

$$[F'(\bar{x} + h)]^{-1} = D(h)^{-1} [F'(\bar{x})]^{-1} + O(h^p).$$

Then, if we assume that the inverse of $D(h)$ is

$$D(h)^{-1} = I + X_2 h + X_3 h^2 + X_4 h^3 + \ldots,$$

provided that X_i, $i = 2, 3, \ldots$ verify

$$D(h)D(h)^{-1} = D(h)^{-1}D(h) = I,$$

then X_i, $i = 2, 3, \ldots$ can be obtained by

$$\begin{aligned} X_2 &= -2C_2, \\ X_3 &= 4C_2^2 - 3C_3, \\ X_4 &= -8C_2^3 + 6C_2C_3 + 6C_3C_2 - 4C_4, \\ &\vdots \end{aligned}$$

We denote $e^{(k)} = x^{(k)} - \bar{x}$ the error in the kth iteration. The equation $e^{(k+1)} = Le^{(k)^p} + O(e^{(k)^{p+1}})$, where L is a p-linear function $L \in \mathcal{L}(\mathbb{R} \times \ldots \times \mathbb{R}, \mathbb{R})$, is called the *error equation* and p is the *order of convergence*. Observe that $e^{(k)^p}$ is $(e^{(k)}, e^{(k)}, \ldots, e^{(k)})$.

Now, we recall the definition given by Ortega and Rheinboldt [14] of a first divided difference of F on $\mathbb{R}^n$ as a mapping $[\cdot, \cdot; F] : D \times D \subset \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n)$ which satisfies $[x, y; F](x - y) = F(x) - F(y)$, $\forall x, y \in D$.

From the Genocchi-Hermite formula [9]

$$[x, x + h; F] = \int_0^1 F'(x + th)dt,$$

and by developing $F'(x + th)$ in Taylor's series at the point x, we obtain

$$\int_0^1 F'(x + th)dt = F'(x) + \frac{1}{2}F''(x)h + \frac{1}{6}F'''(x)h^2 + O(h^3). \tag{1.1}$$

If $e = x - \bar{x}$ and assuming that $F'(\bar{x})$ is nonsingular, we have

$$\begin{aligned} F(x) &= F'(\bar{x})[e + C_2e^2 + C_3e^3 + C_4e^4] + O(e^5), \\ F'(x) &= F'(\bar{x})[I + 2C_2e + 3C_3e^2 + 4C_4e^3] + O(e^4), \\ F''(x) &= F'(\bar{x})[2C_2 + 6C_3e + 12C_4e^2] + O(e^3), \\ F'''(x) &= F'(\bar{x})[6C_3 + 24C_4e] + O(e^2). \end{aligned}$$

Replacing these expressions in (1.1) and setting $y = x + h$ and $e_y = y - \bar{x}$, we get

$$[x, y; F] = F'(\bar{x})[I + C_2(e_y + e) + C_3e^2] + O(e^3).$$

In particular, if y is the Newton's approximation, we obtain

$$[x, y; F] = F'(\bar{x})[I + C_2e + (C_2^2 + C_3)e^2] + O(e^3).$$

The rest of this paper is organized as follows: in Section 2 we present our new family of iterative methods and analyze its seventh-order of convergence and in Section 3 we show several numerical tests that confirm the theoretical results and allow us to compare our schemes with other known ones.

2 Main results

In this section we display a new family of iterative schemes of order seven for solving nonlinear systems of equations, obtained by using the results of Soleymani et al. in [19], weight matricial functions and the divided difference operator $[x, y; F]$. In [19] Soleymani et al. presented the family of iterative schemes for solving nonlinear equations, whose iterative expression is

$$\begin{aligned} y_k &= x_k - \tfrac{f(x_k)}{f'(x_k)}, \\ z_k &= y_k - G(t_k)\tfrac{f(y_k)}{f[x_k,y_k]}, \\ x_{k+1} &= z_k - H(t_k)\tfrac{f(z_k)}{f[y_k,z_k]}, \end{aligned} \tag{2.1}$$

where $t_k = f(y_k)/f(x_k)$, G and H are real functions and $f[x, y]$ denotes the divided difference of order one. They proved that, under some conditions of functions G and H, the methods of family (2.1) have order of convergence seven.

In order to extend (2.1) to the multivariate case we need to take in account the following:

(a) The divided difference $f[x, y]$ must be replaced by the divided difference operator $[x, y; F]$.

(b) We observe that $f(x_k) = (x_k - y_k)f'(x_k)$. So,

$$t_k = \frac{f(y_k)}{f(x_k)} = \frac{f(y_k) - f(x_k) + f(x_k)}{(x_k - y_k)f'(x_k)} = -\frac{f[x_k, y_k]}{f'(x_k)} + 1.$$

Therefore, for a n-dimensional function F we are going to use

$$t = I - [F'(x)]^{-1}[x, y; F],$$

as a variable of the weight functions

From these comments we design the following scheme for solving a nonlinear system $F(x) = 0$:

$$\begin{aligned} y^{(k)} &= x^{(k)} - [F'(x^{(k)})]^{-1}F(x^{(k)}), \\ z^{(k)} &= y^{(k)} - G(t^{(k)})[x^{(k)}, y^{(k)}; F]^{-1}F(y^{(k)}), \\ x^{(k+1)} &= z^{(k)} - H(t^{(k)})[y^{(k)}, z^{(k)}; F]^{-1}F(z^{(k)}), \end{aligned} \tag{2.2}$$

where $t^{(k)} = I - [F'(x^{(k)})]^{-1}[x^{(k)}, y^{(k)}; F]$ and $G, H : \mathcal{M}_{n\times n}(\mathbb{R}) \to \mathcal{L}(\mathbb{R}^n)$, being $\mathcal{M}_{n\times n}$ the set of $n \times n$ real matrices.

In the next result we are going to prove that, under certain conditions of functions G and H, the convergence order of the methods described by (2.2) is seven.

Theorem 1. *Let $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a sufficiently differentiable function in an open neighborhood D of $\bar{x}$, that is a solution of the nonlinear system $F(x) = 0$. Let us suppose that $F'(x)$ is nonsingular in $\bar{x}$. Then, the sequence $\{x^{(k)}\}_{k\geq 0}$ obtained by using the iterative expression (2.2) converges to $\bar{x}$ with order of convergence seven, when the sufficiently differentiable functions G and H satisfy $G(0) = G'(0) = I$ and $\|G''(0)\| < \infty$, and also $H(0) = I$, $H'(0) = 0$, $H''(0) = 2I$ and $\|H'''(0)\| < \infty$.*

Proof: This proof is based in Taylor expansion of the different elements that appear in iterative expression (2.2). In these Taylor expansion we are going to omit index k for simplicity. Taylor expansion of $F(x)$ and $F'(x)$ about $\bar{x}$ gives

$$F(x) = F'(\bar{x})[e + C_2e^2 + C_3e^3 + C_4e^4 + C_5e^5 + C_6e^6 + C_7e^7] + O(e^8), \tag{2.3}$$

and

$$F'(x) = F'(\bar{x})[I + 2C_2e + 3C_3e^2 + 4C_4e^3 + 5C_5e^4 + 6C_6e^5 + 7C_7e^6] + O(e^7), \tag{2.4}$$

where $C_k = (1/k!)[F'(\bar{x})]^{-1}F^{(k)}(\bar{x}),\ \ k = 2, 3, \ldots$, and $e = x^{(k)} - \bar{x}$.

As we have stated in the Introduction, from (2.4) we obtain

$$[F'(x)]^{-1} = [I + X_2e + X_3e^2 + X_4e^3 + X_5e^4 + X_6e^5 + X_7e^6][F'(\bar{x})]^{-1} + O(e^7), \tag{2.5}$$

where

$$X_2 = -2C_2,\ \ X_3 = 4C_2^2 - 3C_3,\ \ X_4 = -8C_2^3 + 6C_2C_3 + 6C_3C_2 - 4C_4,$$
$$X_5 = 16C_2^4 - 12C_3C_2^2 - 12C_2C_3C_2 + 8C_4C_2 - 12C_2^2C_3 + 9C_3^2 + 8C_2C_4 - 5C_5,$$

$$\begin{aligned} X_6 \ = \ & -6C_6 + 10C_2C_5 + 10C_5C_2 - 16C_2^2C_4 - 16C_4C_2^2 - 16C_2C_4C_2 + 12C_3C_4 \\ & +12C_4C_3 + 24C_2^3C_3 + 24C_3C_2^3 + 24C_2C_3C_2^2 + 24C_2^2C_3C_2 - 18C_2C_3^2 \\ & -18C_3C_2C_3 - 18C_3^2C_2 - 32C_2^5, \end{aligned}$$

and

$$\begin{aligned} X_7 \ = \ & -7C_7 + 12C_2C_6 + 12C_6C_2 - 20C_2^2C_5 - 20C_2C_5C_2 - 20C_5C_2^2 + 15C_3C_5 + 15C_5C_3 \\ & +32C_2^3C_4 + 32C_4C_2^3 + 32C_2^2C_4C_2 + 32C_2C_4C_2^2 - 24C_2C_3C_4 - 24C_3C_2C_4 \\ & -24C_4C_2C_3 - 24C_2C_4C_3 - 24C_3C_4C_2 + 16C_4^2 - 24C_4C_3C_2 - 48C_2^4C_3 - 48C_3C_2^4 \\ & -48C_2^2C_3C_2^2 + 36C_3C_2^2C_3 + 36C_2C_3C_2C_3 + 36C_2^2C_3^2 + 36C_2C_3^2C_2 + 36C_3C_2C_3C_2 \\ & +36C_3^2C_2^2 - 27C_3^3 - 48C_2^3C_3C_2 - 48C_2C_3C_2^3 + 64C_2^6. \end{aligned}$$

Taylor's expansion of Newton's step y gives

$$\begin{aligned} y - \bar{x} \ = \ & C_2e^2 + (2C_3 - 2C_2^2)e^3 + (4C_2^3 - 4C_2C_3 - 3C_3C_2 + 3C_4)e^4 \\ & +(-6C_3^2 - 8C_2^4 + 8C_2^2C_3 + 6C_2C_3C_2 + 6C_3C_2^2 - 6C_2C_4 - 4C_4C_2 + 4C_5)e^5 \\ & +(5C_6 - 8C_2C_5 + 12C_2^2C_4 - 9C_3C_4 - 16C_2^3C_3 + 12C_2C_3^2 + 12C_3C_2C_3 \\ & -8C_4C_3 + 16C_2^5 - 12C_3C_2^3 - 12C_2C_3C_2^2 + 8C_4C_2^2 - 12C_2^2C_3C_2 + 9C_3^2C_2 \\ & +8C_2C_4C_2 - 5C_5C_2)e^6 + (6C_7 - 10C_2C_6 + 16C_2^2C_5 - 12C_3C_5 - 24C_2^3C_4 \\ & +18C_2C_3C_4 + 18C_3C_2C_4 - 12C_4^2 + 32C_2^4C_3 - 24C_3C_2^2C_3 - 24C_2C_3C_2C_3 \\ & +16C_4C_2C_3 - 24C_2^2C_3^2 + 18C_3^3 + 16C_2C_4C_3 - 10C_5C_3 - 6C_6C_2 + 10C_2C_5C_2 \\ & +10C_5C_2^2 - 16C_2^2C_4C_2 - 16C_4C_2^3 - 16C_2C_4C_2^2 + 12C_3C_4C_2 + 12C_4C_3C_2 \\ & +24C_2^3C_3C_2 + 24C_3C_2^4 + 24C_2C_3C_2^3 + 24C_2^2C_3C_2^2 - 18C_2C_3^2C_2 \\ & -18C_3C_2C_3C_2 - 18C_3^2C_2^2 - 32C_2^6)e^7 + O(e^8), \end{aligned} \tag{2.6}$$

and it allows us to obtain

$$\begin{aligned}
F(y) &= F'(\bar{x})\big[C_2e^2+(2C_3-2C_2^2)e^3+(5C_2^3-4C_2C_3-3C_3C_2+3C_4)e^4\\
&+(-6C_3^2-12C_2^4+10C_2^2C_3+8C_2C_3C_2+6C_3C_2^2-6C_2C_4-4C_4C_2+4C_5)e^5\\
&+(5C_6-8C_2C_5+15C_2^2C_4-9C_3C_4-24C_2^3C_3+16C_2C_3^2+12C_3C_2C_3-8C_4C_3\\
&+28C_2^5-11C_3C_2^3-19C_2C_3C_2^2+8C_4C_2^2-19C_2^2C_3C_2+9C_3^2C_2\\
&+11C_2C_4C_2-5C_5C_2)e^6+(6C_7-10C_2C_6+20C_2^2C_5-12C_3C_5-36C_2^3C_4\\
&+24C_2C_3C_4+18C_3C_2C_4-12C_4^2+56C_2^4C_3-18C_3C_2^2C_3-38C_2C_3C_2C_3\\
&+16C_4C_2C_3-38C_2^2C_3^2+18C_3^3+22C_2C_4C_3-10C_5C_3\\
&-6C_6C_2+14C_2C_5C_2+10C_5C_2^2-26C_2^2C_4C_2-16C_4C_2^3\\
&-26C_2C_4C_2^2+12C_3C_4C_2+12C_4C_3C_2+44C_2^3C_3C_2\\
&+18C_3C_2^4+44C_2C_3C_2^3+44C_2^2C_3C_2^2-30C_2C_3^2C_2\\
&-18C_3C_2C_3C_2-18C_3^2C_2^2-64C_2^6)e^7\big]+O(e^8).
\end{aligned} \tag{2.7}$$

Now, as we have commented in the Introduction, we obtain the Taylor's expression of operator $[x, y; F]$.

$$\begin{aligned}
[x,y;F] &= [F'(\bar{x})]\big[I+C_2e+(C_2^2-C_3)e^2+(C_4-2C_2^3+2C_2C_3+C_3C_2)e^3\\
&+(4C_2^4-4C_2^2C_3-3C_2C_3C_2+3C_2C_4+2C_3^2-C_3C_2^2+C_4C_2)e^4\\
&+(-4C_6-6C_2C_3^2-8C_2^5+8C_2^3C_3+6C_2^2C_3C_2+6C_2C_3C_2^2-6C_2^2C_4-4C_2C_4C_2\\
&+4C_2C_5-2C_3C_2C_3-C_3^2C_2+3C_3C_4+2C_4C_3-C_4C_2^2+5C_5C_2)e^5\\
&+(-14C_7+5C_2C_6+20C_6C_2-8C_2^3C_5-5C_2C_5C_2-15C_2C_2^2+12C_2^3C_4\\
&+8C_2C_4C_2^2+8C_2^2C_4C_2+C_4C_2^3-9C_2C_3C_4-8C_2C_4C_3-3C_3C_2C_4\\
&-C_3C_4C_2-2C_4C_2C_3-C_4C_3C_2-16C_2^4C_3-12C_2C_3C_2^3-12C_2^2C_3C_2^2\\
&-12C_2^3C_3C_2+4C_3C_2^4+12C_2^2C_3^2+12C_2C_3C_2C_3+9C_2C_3^2C_2\\
&-C_3C_2C_3C_2-C_3^2C_2^2+16C_2^6+8C_2C_4C_2^2+C_4C_2^3+8C_2^2C_4C_2\\
&-2C_3^3+4C_3C_5+10C_5C_3+3C_4^2)e^6\big]+O(e^7)
\end{aligned}$$

and the expression of the inverse of the divided difference $[x, y; F]$ is

$$\begin{aligned}
[x,y;F]^{-1} &= \big[I-C_2e-C_3e^2+(-C_4+3C_2^3-C_2C_3)e^3\\
&+(-C_5-9C_2^4+6C_2^2C_3+5C_2C_3C_2-2C_2C_4+2C_3C_2^2-C_3^2)e^4\\
&+(-C_6+9C_2C_3^2+18C_2^5-15C_2^3C_3-15C_2^2C_3C_2-11C_2C_3C_2^2+9C_2^2C_4\\
&+7C_2C_4C_2-3C_2C_5+4C_3C_2C_3+3C_3^2C_2-2C_3C_4+2C_4C_2^2\\
&-C_4C_3-4C_3C_2^3)e^5\big]\,[F'(\bar{x})]^{-1}+O(e^6).
\end{aligned} \tag{2.8}$$

Taking into account that $z^{(k)} = y^{(k)} - G(t^{(k)})[x^{(k)}, y^{(k)}; F]^{-1}F(y^{(k)})$, by using (2.8) and the

Taylor's expansion of function G about the zero matrix, we obtain

$$\begin{aligned} z-\bar{x} &= (I-G(0))C_2e^2+((-2I+3G(0)-G'(0))C_2^2+2(I-G(0))C_3)e^3 \\ &+\Big[(4I-7G(0)+6G'(0)-\frac{G''(0)}{2})C_2^3+(-4I+6G(0)-G'(0))C_2C_3 \\ &(-3I+4G(0)-2G'(0))C_3C_2+(3I-3G(0))C_4\Big]e^4+O(e^5). \end{aligned}$$

Then, by taking $G(0)=G'(0)=I$, we get

$$\begin{aligned} z-\bar{x} &= ((3I-\frac{G''(0)}{2})C_2^3-C_3C_2)e^4+((-18I+\frac{9G''(0)}{2}-\frac{G'''(0)}{6})C_2^4-2C_3^2 \\ &+(6I-G''(0))C_2C_3C_2+(8I-G''(0))C_3C_2^2-2C_4C_2)e^5 \\ &+((9I-\frac{3G''(0)}{2})C_2^2C_4-3C_3C_4+(-36I+9G''(0) \\ &-\frac{G'''(0)}{3})C_2^3C_3+(12I-2G''(0))C_2C_3^2 \\ &+(16I-2G''(0))C_3C_2C_3-4C_4C_3+(70I-25G''(0)+2G'''(0))C_2^5 \\ &+(-34I+8G''(0)-\frac{G'''(0)}{3})C_3C_2^3+(-31I+8G''(0) \\ &-\frac{G'''(0)}{3})C_2C_3C_2^2+(12I-\frac{3G''(0)}{2})C_4C_2^2 \\ &+(-29I+8G''(0)-\frac{G'''(0)}{3})C_2^2C_3C_2+(14I-2G''(0))C_3^2C_2 \\ &+(9I-\frac{3G''(0)}{2})C_2C_4C_2-3C_5C_2)e^6+O(e^7). \end{aligned} \tag{2.9}$$

Now, from the Taylor expansion of $F(z^{(k)})$

$$
\begin{aligned}
F(z) &= F'(\bar{x})[((3I-\frac{G''(0)}{2})C_2^3-C_3C_2)e^4\\
&+((-18I+\frac{9G''(0)}{2}-\frac{G'''(0)}{6})C_2^4-2C_3^2\\
&+(6I-G''(0))C_2C_3C_2+(8I-G''(0))C_3C_2^2-2C_4C_2)e^5\\
&+((9I-\frac{3G''(0)}{2})C_2^2C_4-3C_3C_4+(-36I+9G''(0)\\
&-\frac{G'''(0)}{3})C_2^3C_3+(12I-2G''(0))C_2C_3^2\\
&+(16I-2G''(0))C_3C_2C_3-4C_4C_3+(70I-25G''(0)+2G'''(0))C_2^5\\
&+(-34I+8G''(0)-\frac{G'''(0)}{3})C_3C_2^3+(-31I+8G''(0)\\
&-\frac{G'''(0)}{3})C_2C_3C_2^2+(12I-\frac{3G''(0)}{2})C_4C_2^2\\
&+(-29I+8G''(0)-\frac{G'''(0)}{3})C_2^2C_3C_2+(14I-2G''(0))C_3^2C_2\\
&+(9I-\frac{3G''(0)}{2})C_2C_4C_2-3C_5C_2)e^6]+O(e^7).
\end{aligned}
$$

we obtain the expression of the operator $[y^{(k)}, z^{(k)}; F]$

$$
\begin{aligned}
[y,z;F] &= F'(\bar{x})\left[I+C_2^2e^2+(-2C_2^3+2C_2C_3)e^3\right.\\
&+((7I-\frac{G''(0)}{2})C_2^4-4C_2^2C_3+C_3C_2^2-4C_2C_3C_2+3C_2C_4)e^4\\
&+((-26I+\frac{9G''(0)}{2}-\frac{G'''(0)}{6})C_2^5-8C_2C_3^2+(14I-G''(0))C_2^3C_3\\
&-4C_3C_2^3+(12I-G''(0))C_2^2C_3C_2+2C_3^2C_2+(14I-G''(0))C_2C_3C_2^2-6C_2^2C_4\\
&\left.-6C_2C_4C_2-4C_2C_5-2C_3C_2C_3)\right]e^5+O(e^6)
\end{aligned}
$$

and the expression of its inverse is

$$
\begin{aligned}
[y,z;F]^{-1} &= \left[I-C_2^2e^2+(2C_2^3-2C_2C_3)e^3\right.\\
&+((-6I+\frac{G''(0)}{2})C_2^4+4C_2^2C_3-C_3C_2^2+4C_2C_3C_2-3C_2C_4)e^4\\
&+((22I-\frac{9G''(0)}{2}+\frac{G'''(0)}{6})C_2^5+8C_2C_3^2+(-12I+G''(0))C_2^3C_3\\
&+4C_3C_2^3+(-12I+G''(0))C_2^2C_3C_2-2C_3^2C_2\\
&+(-12I+G''(0))C_2C_3C_2^2+6C_2^2C_4+6C_2C_4C_2\\
&\left.+4C_2C_5+2C_3C_2C_3)e^5\right][F'(\bar{x})]^{-1}+O(e^6).
\end{aligned}
\tag{2.10}
$$

Therefore,

$$
\begin{aligned}
x^{(k+1)} - \bar{x} \quad = \quad & \Big((3(I - H(0)) - \frac{(I - H(0))G''(0)}{2})C_2^3 + (H(0) - I)C_3C_2\Big)e^4 \\
& + M_5e^5 + M_6e^6 + O(e^7),
\end{aligned}
\tag{2.11}
$$

where

$$
\begin{aligned}
M_5 \quad = \quad & (18(H(0) - I) + \frac{9G''(0)}{2}(I - H(0)) \\
& + \frac{G'''(0)}{6}(H(0) - I) - 3H'(0))C_2^4 \\
& + 2(H(0) - I)C_3^2 + (6(I - H(0)) + (H(0) - I)G''(0))C_2^2C_3 \\
& + (6(I - H(0)) + (H(0) - I)G''(0) + H'(0))C_2C_3C_2 + (8(I - H(0)) \\
& + (H(0) - I)G''(0))C_3C_2^2 + (2(H(0) - I))C_4C_2 \\
& + \frac{3}{2}H'(0)C_2G''(0)C_2^3
\end{aligned}
$$

and

$$
\begin{aligned}
M_6 \quad = \quad & (9(I - H(0)) + \frac{3}{2}(H(0) - I)G''(0))C_2^2C_4 + 3(H(0) - I)C_3C_4 \\
& + (36(H(0) - I) + 9(I - H(0))G''(0) + \frac{1}{3}(H(0) - I)G'''(0) \\
& + H'(0)(G''(0) - 6I))C_2^3C_3 + (12(I - H(0)) + 2(H(0) - I)G''(0) \\
& + 2H'(0))C_2C_3^2 + (16(I - H(0)) + 2(H(0) - I)G''(0))C_3C_2C_3 \\
& + 4(H(0) - I)C_4C_3 + (70I - 67H(0) - \frac{3}{2}H''(0) \\
& + (-25 + \frac{49}{2}H(0) - 6H'(0) + \frac{1}{4}H''(0))G''(0) + 2(I - H(0) \\
& + \frac{1}{12}H'(0))G'''(0) + 27H'(0))C_2^5 + (34(H(0) - I) + 8(I - H(0))G''(0) \\
& + \frac{1}{3}(H(0) - I)G'''(0) + H'(0)(G''(0) - 6I))C_3C_2^3 + (31(H(0) - I) \\
& + 8(I - H(0))G''(0) + \frac{1}{3}(H(0) - I)G'''(0) + (G''(0) - 8I)H'(0))C_2C_3C_2^2 \\
& + (12(I - H(0)) + \frac{3}{2}(H(0) - I)G''(0))C_4C_2^2 + (-29I + 28H(0) \\
& + \frac{H''(0)}{2} + 8(I - H(0))G''(0) + \frac{1}{3}(H(0) - I)G'''(0) \\
& + (G''(0) - 9I)H'(0))C_2^2C_3C_2 + (14(I - H(0)) + 2(H(0) - I)G''(0) \\
& + 2H'(0))C_3^2C_2 + (9(I - H(0)) + \frac{3}{2}(H(0) - I)G''(0) + 2H'(0))C_2C_4C_2 \\
& + 3(H(0) - I)C_5C_2.
\end{aligned}
$$

Finally, taking $H(0) = I$, $H'(0) = 0$, and $H''(0) = 2I$ in (2.11) we obtain the thesis of the theorem. □

An example of functions G and H that satisfy the conditions of Theorem 1 is

$$G(t) = I+t = 2I-F'(x)^{-1}[x,y;F], \qquad H(t) = I+t^2 = 2I-2F'(x)^{-1}[x,y;F]+(F'(x)^{-1}[x,y;F])^2.$$

We denote the resulting method of using these specific weight functions by M7 and it will be used in the numerical section.

3 Numerical results

In this section, we compare the described procedure M7 with the classical Newton and Jarratt's schemes and the method presented by the authors in [1]. We recall that the Jarratt's method [10] is a fourth-order iterative scheme for solving nonlinear equations which can be easily extended to nonlinear systems

$$\begin{aligned} y^{(k)} &= x^{(k)} - (2/3)[F'(x^{(k)})]^{-1}F(x^{(k)}),\\ x^{(k+1)} &= y^{(k)} - (1/2)[3F'(y^{(k)}) - F'(x^{(k)})]^{-1}[3F'(y^{(k)}) + F'(x^{(k)})][F'(x^{(k)})]^{-1}F(x^{(k)}). \end{aligned}$$

On the other hand, the authors in the context of the Global Positioning System, designed in [1] a fifth-order iterative scheme for solving nonlinear systems, whose expression is

$$\begin{aligned} z^{(k)} &= x^{(k)} - F'(x^{(k)})^{-1}(F(x^{(k)}) + F(y^{(k)})),\\ x^{(k+1)} &= z^{(k)} - F'(y^{(k)})^{-1}F(z^{(k)}), \end{aligned}$$

where $y^{(k)}$ is the kth iteration of Newton's method. This scheme is denoted by M5. These methods are employed to solve the following nonlinear systems. We show the initial approximation used to obtain the numerical results presented in Table 1 and the exact or approximate root.

$$(1) \quad \left.\begin{array}{l} x_2x_3 + x_4(x_2+x_3) = 0\\ x_1x_3 + x_4(x_1+x_3) = 0\\ x_1x_2 + x_4(x_1+x_2) = 0\\ x_1x_2 + x_1x_3 + x_2x_3 - 1 = 0 \end{array}\right\},$$

$x^{(0)} = (0.5, 0.5, 0.5, 0.5)^T$, $\bar{x} = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}})^T$

$$(2) \quad \left.\begin{array}{l} \cos x_2 - \sin x_1 = 0\\ x_3^{x_1} - 1/x_2 = 0\\ e^{x_1} - x_3^2 = 0 \end{array}\right\},$$

$x^{(0)} = (1, 0.5, 1.5)^T$, $\bar{x} \approx (0.909569, 0.661227, 1.575834)^T$

$$(3) \quad \left.\begin{array}{l} x_1 + e^{x_2} - \cos x_2 = 0\\ 3x_1 - x_2 - \sin x_2 = 0 \end{array}\right\},$$

$x^{(0)} = (0.5, 0.5)^T$, $\bar{x} = (0, 0)^T$

$$(4)\quad \left.\begin{array}{l} \sin x_1 + x_2^2 + \log x_3 - 7 = 0 \\ 3x_1 + 2^{x_2} - x_3^{-3} + 1 = 0 \\ x_1 + x_2 - x_3 - 5 = 0 \end{array}\right\},$$

$x^{(0)} = (-3.5, 3, -5)^T$, $\bar{x} \approx (-3.936800, 3.433547, -5.503253)^T$

$$(5)\quad \left.\begin{array}{ll} x_i x_{i+1} - 1 = 0, & i = 1, 2, \ldots, n-1 \\ x_n x_1 - 1 = 0 & \end{array}\right\},$$

$x^{(0)} = (2, 2, \ldots, 2)^T$, $\bar{x} = (1, 1, \ldots, 1)^T$

Table 1: Numerical results for the different systems and schemes

	Newton	Jarratt	M5	M7
System 1				
$\|F(x^{(1)})\|$	0.2534	0.0026	0.0012	7.7960e-4
$\|F(x^{(2)})\|$	0.0026	1.9140e-16	1.6685e-22	4.3561e-30
$\|F(x^{(3)})\|$	1.3560e-7	9.4865e-71	1.7043e-119	1.6868e-203
ACOC	2.3085	4.7015	6.0028	7.9269
System 2				
$\|F(x^{(1)})\|$	0.0628	0.0059	0.0084	0.0025
$\|F(x^{(2)})\|$	0.0098	2.4645e-9	3.3843e-9	7.1527e-15
$\|F(x^{(3)})\|$	1.9156e-4	2.1864e-35	4.3549e-41	7.4347e-59
ACOC	4.9812	4.1598	5.0367	6.2483
System 3				
$\|F(x^{(1)})\|$	0.1911	0.0074	0.0056	5.6664e-4
$\|F(x^{(2)})\|$	0.0095	1.2908e-10	3.2920e-13	4.9795e-25
$\|F(x^{(3)})\|$	3.1420e-5	1.2311e-41	2.5970e-64	5.0130e-123
ACOC	1.8292	3.6465	4.5444	6.5057
System 4				
$\|F(x^{(1)})\|$	0.4744	0.0043	6.0860e-4	6.6380e-5
$\|F(x^{(2)})\|$	0.0029	1.1620e-13	1.5391e-21	4.9817e-32
$\|F(x^{(3)})\|$	6.2011e-7	1.4796e-55	1.0709e-108	3.3104e-98
ACOC	2.9952	3.8280	5.0968	5.9903
System 5				
$\|F(x^{(1)})\|$	1.6875	0.1519	0.1034	0.0208
$\|F(x^{(2)})\|$	0.1519	2.7876e-7	2.0520e-9	6.4500e-18
$\|F(x^{(3)})\|$	0.0018	3.4950e-30	7.0170e-48	4.3080e-90
ACOC	1,8352	3.6019	4.3379	6.3026

Numerical computations have been carried out using variable precision arithmetic, with 2000 digits. In all cases we show the value of F in the three first steps and the approximate computational order of convergence (ACOC), according to (see [4])

$$p \approx ACOC = \frac{\ln\left(\|x^{(k+1)} - x^{(k)}\|/\|x^{(k)} - x^{(k-1)}\|\right)}{\ln\left(\|x^{(k)} - x^{(k-1)}\|/\|x^{(k-1)} - x^{(k-2)}\|\right)}.$$

The value of ACOC that appears in Table 1 is obtained only with the three first iterations. The behavior of the methods in the different examples is in concordance with the theoretical results. There are no significative differences taking into account that the methods have order of convergence 2 (Newton), 4 (Jarratt), 5 (M5) and 7 (M7), respectively.

In Example 5, we must distinguish n even or odd. For n even the methods do not converge since the Jacobian matrix is singular at the solution. For n odd the behavior of each method is independent of the size of the system. In Table 1 we show the numerical results for $n = 9$.

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