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Perfect edge-magic graphs

by
Susana-Clara López, Francisco Antonio Muntaner-Batle and Miquel Rius-Font

Abstract

The study of the possible valences for edge-magic labelings of graphs has motivated us to introduce the concept of perfect edge-magic graphs. Intuitively speaking, an edge-magic graph is perfect edge-magic if all possible theoretical valences occur. In particular, we prove that for each integer $m > 0$, that is the power of an odd prime, and for each natural number n, the crown product $C_m \odot \overline{K_n}$ is perfect edge-magic. Related results are also provided concerning other families of unicyclic graphs. Furthermore, several open questions that suggest interesting lines for future research are also proposed.

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1 Introduction

For the graph theory terminology and notation not defined in this paper we refer the reader to either one of the following sources [2, 3, 8, 18]. Kotzig and Rosa [12] introduced in 1970 the concepts of edge-magic graphs and edge-magic labelings as follows: a (p, q)-graph G is called *edge-magic* if there is a bijective function $f : V(G) \cup E(G) \to \{i\}_{i=1}^{p+q}$ such that the sum $f(u) + f(uv) + f(v) = k$ for any $uv \in E(G)$. Such a function is called an *edge-magic labeling* of G and k is called the *valence* or the *magic sum* of the labeling f. The purpose of this paper is to characterize the set of numbers that are valences for the edge-magic labelings of some families of unicyclic graphs.

Let $G = (V, E)$ be a (p, q)-graph, and define the set

$$T_G = \left\{ \frac{\sum_{u \in V} \deg(u) g(u) + \sum_{e \in E} g(e)}{q} : \ g : V \cup E \to \{i\}_{i=1}^{p+q} \ \text{ is a bijective function} \right\}.$$

If $\lceil \min T_G \rceil \leq \lfloor \max T_G \rfloor$ then the *magic interval* of G, denoted by J_G, is defined to be the set $J_G = [\lceil \min T_G \rceil, \lfloor \max T_G \rfloor] \cap \mathbb{N}$ and the *magic set* of G, denoted by τ_G, is the set $\tau_G = \{n \in J_G :$ n is the valence of some edge-magic labeling of $G\}$. It is clear that $\tau_G \subseteq J_G$. In this paper, we call G a *perfect edge-magic graph* if $\tau_G = J_G$.

A (p,q)-graph G is *super edge-magic* if there is an edge-magic labeling of G, namely $f: V(G) \cup E(G) \to \{i\}_{i=1}^{p+q}$, with the extra property that $f(V(G)) = \{i\}_{i=1}^{p}$. The function f is called a *super edge-magic labeling* of G. These concepts were introduced independently by Acharya and Hegde [1] and by Enomoto et al. in [4]. Figueroa-Centento et al. stated in [5] the following characterization for super edge-magic labelings.

Lemma 1.1. *Let G be a (p,q)-graph. Then G is super edge-magic if and only if there is a bijective function $g : V(G) \longrightarrow \{i\}_{i=1}^{p}$ such that the set $S = \{g(u) + g(v) : uv \in E(G)\}$ is a set of q consecutive integers. In this case, g can be extended to a super edge-magic labeling f with valence $p + q + \min S$.*

Let $f : V(G) \cup E(G) \to \{i\}_{i=1}^{p+q}$ be an edge-magic labeling of a (p,q)-graph G. The *complementary labeling* of f, denoted by $\overline{f}$, is the labeling defined by the rule: $\overline{f}(x) = p+q+1-f(x)$, for all $x \in V(G) \cup E(G)$. Notice that, if f is an edge-magic labeling of G with valence k, we have that $\overline{f}$ is also an edge-magic labeling of G with valence $\overline{k} = 3(p+q+1) - k$. Let $f : V(G) \cup E(G) \to \{i\}_{i=1}^{p+q}$ be a super edge-magic labeling of a (p,q)-graph G, with $p = q$. The *odd labeling* and the *even labeling* obtained from f, denoted respectively by $o(f)$ and $e(f)$, are the labelings $o(f), e(f) : V(G) \cup E(G) \to \{i\}_{i=1}^{p+q}$ defined as follows: (i) on the vertices: $o(f)(x) = 2f(x) - 1$ and $e(f)(x) = 2f(x)$, for all $x \in V(G)$, (ii) on the edges: $o(f)(xy) = 2\text{val}(f) - 2p - 2 - o(f)(x) - o(f)(y)$ and $e(f)(xy) = 2\text{val}(f) - 2p - 1 - e(f)(x) - e(f)(y)$, for all $xy \in E(G)$.

Lemma 1.2. *Let G be a (p,q)-graph with $p = q$ and let $f : V(G) \cup E(G) \to \{i\}_{i=1}^{p+q}$ be a super edge-magic labeling of G. Then, the odd labeling $o(f)$ and the even labeling $e(f)$ obtained from f are edge-magic labelings of G with valences $val(o(f)) = 2val(f) - 2p - 2$ and $val(e(f)) = 2val(f) - 2p - 1$.*

Proof: Note that, since f is super edge-magic, the set $S_o = \{o(f)(x) + o(f)(y) : xy \in E(G)\} = \{2(f(x) + f(y)) - 2 : xy \in E(G)\}$ is an arithmetic progression of difference 2, starting at $2(\text{val}(f) - 2p) - 2$. Thus, by assigning the even labels to the edges, we obtain an edge-magic labeling with valence $\text{val}(o(f)) = 2\text{val}(f) - 2p - 2$. The proof for $e(f)$ is similar. $\square$

When we say that a digraph has a labeling we mean that its underlying graph has such labeling, see [7].

The paper is organized as follows: in section 2 we prove that each element in the family $C_m \odot \overline{K_n}$, where m is a power of an odd prime and $\overline{K_n}$ denotes the complementary graph of the complete graph K_n ($n \in \mathbb{N}$), is a perfect edge-magic graph. In section 3, we prove that the magic set of irregular crowns is big by showing a general construction of edge-magic labelings, and that a subfamily of them is perfect edge-magic. In section 4, we establish a new relation among super edge-magic and even harmonius labelings. Finally, we end by a short section of conclusions and remarks.

2 A family of perfect edge-magic graphs

We begin our calculation of the magic interval $J_{C_m \odot \overline{K_n}}$, for all $m, n \in \mathbb{N}$. Let $C_m \odot \overline{K_n} = (V, E)$, where $V = \{v_i\}_{i=1}^{m} \cup (\cup_{i=1}^{m} \{v_i^j\}_{j=1}^{n})$ and $E = \{v_i v_{i+1}\}_{i=1}^{m-1} \cup \{v_1 v_m\} \cup (\cup_{i=1}^{m} \{v_i v_i^j\}_{j=1}^{n})$.

For any bijective function $g : V \cup E \to \{i\}_{i=1}^{2m(n+1)}$, the corresponding element in T_G is $\left((1+n)\sum_{i=1}^{m} g(v_i) + \sum_{u \in V \cup E} g(u)\right)/(mn+m)$. Thus, the minimum possible valence occurs when the labels $\{1,2,\ldots,m\}$ are assigned to the vertices of the cycle. Therefore,

$$\min T_{C_m \odot \overline{K_n}} = \frac{(1+n)\sum_{i=1}^{m} i + \sum_{i=1}^{2m(n+1)} i}{mn+m} = \frac{3+5m}{2} + 2mn.$$

On the other hand, the maximum possible valence occurs when the labels $\{2m(n+1)-m+1, 2m(n+1)-m+2, \ldots, 2m(n+1)\}$ are assigned to the vertices of the cycle. Hence, using similar calculations, we obtain that $\max T_{C_m \odot \overline{K_n}} = (3+7m)/2 + 4mn$.

López et al. have shown in [15] that for each $r \in \mathbb{N}$, with $1 \leq r \leq mn+1$, there exists a super edge-magic labeling f_r with valence

$$\mathrm{val}(f_r) = r - 1 + \frac{3+5m}{2} + 2mn \tag{1}$$

of $C_m \odot \overline{K_n}$, when m is a power of a prime greater than 2. Taking the complementary labelings of these labelings, we get that all the natural numbers from $3mn + (3+7m)/2$ up to $4mn + (3+7m)/2$ also appear as valences of edge-magic labelings of $C_m \odot \overline{K_n}$. Therefore, in order to prove that $C_m \odot \overline{K_n}$ is perfect edge-magic, we only need to show that for each $k \in \mathbb{N}$, with $3mn + (3+5m)/2 < k < 3mn + (3+7m)/2$, there exists an edge-magic labeling with valence k. We do this using the odd and the even labelings of the labelings f_r introduced in [15].

Lemma 2.1. *Let m be a power of a prime greater than 2 and let n be any positive integer. Then, for each k with $2mn + 3m + 1 \leq k \leq 4mn + 3m + 2$ there exists an edge-magic labeling of $C_m \odot \overline{K_n}$ with valence k.*

Proof: Notice that, by (1) the set $\{\mathrm{val}(f_r) :\ 1 \leq r \leq mn+1\}$ is a set of consecutive integers. Thus, Lemma 1.2 implies that the set $\{\mathrm{val}(o(f_r)) :\ 1 \leq r \leq mn+1\} \cup \{\mathrm{val}(e(f_r)) :\ 1 \leq r \leq mn+1\}$ contains all integers from $\mathrm{val}(o(f_1))$ up to $\mathrm{val}(e(f_{mn+1}))$. That is, all integers from $2mn+3m+1$ up to $4mn+3m+2$. □

Since $2mn+3m+1 \leq 3mn+(3+5m)/2$ and $3mn+(3+7m)/2 \leq 4mn+3m+2$, for $n \geq 1$, we obtain the next theorem.

Theorem 2.1. *Let $m = p^k$ where p is an odd prime and $k \in \mathbb{N}$. Then the graph $C_m \odot \overline{K_n}$ is perfect edge-magic for all $n \in \mathbb{N}$, $n \geq 1$.*

3 Super edge-magic toroidal labelings

The purpose of this section is to introduce another family of perfect edge-magic graphs. This is a subfamily of the family of irregular crowns that we introduce in the next lines.

Let $C(n; j_1, j_2, \ldots, j_n) = (V, E)$, where $n \in \mathbb{N} \setminus \{1,2\}$ and $j_i \in \mathbb{N} \cup \{0\}$ for all $i \in \{1,2,\ldots,n\}$ be the irregular crown defined as follows: $V = \{v_i\}_{i=1}^{n} \cup V_1 \cup V_2 \cdots \cup V_n$, where $V_k = \{v_k^1, v_k^2, \ldots, v_k^{j_k}\}$, if $j_k \neq 0$ and $V_k = \emptyset$ if $j_k = 0$, for each $k \in \{1,2,\ldots,n\}$ and

$E = \{v_i v_{i+1}\}_{i=1}^{n-1} \cup \{v_1 v_n\} \cup (\cup_{k=1, j_k \neq 0}^{n} \{v_k v_k^l\}_{l=1}^{j_k})$. Choose an orientation either clockwise or counterclockwise of the unique cycle in $C(n; j_1, j_2, \ldots, j_n)$, obtaining the oriented cycle $\overrightarrow{C_n}$. In what follows, we denote by $\overrightarrow{C}(n; j_1, j_2, \ldots, j_n)$ the oriented digraph obtained from $C(n; j_1, j_2, \ldots, j_n)$ by considering the strong orientation $\overrightarrow{C_n}$ and in such a way that all vertices have indegree equal to 1. The orientation chosen allows us to arrange the vertices of $C(n; j_1, j_2, \ldots, j_n)$ into n ordered levels. For each k, with $1 \leq k < n$, we consider the ordered vertices $v_k^1, v_k^2, \ldots, v_k^{j_k}, v_{k+1}$, if $j_k \neq 0$ and v_{k+1} if $j_k = 0$. For $k = n$, we consider $v_n^1, v_n^2, \ldots, v_n^{j_n}, v_1$, if $j_n \neq 0$ and v_1 if $j_n = 0$.

At this point, assume that n is odd and choose a vertex $v \in V$. We define the labeling $\lambda_v : V \to \{1, 2, \ldots, n + \sum_{i=1}^{n} j_i\}$ recursively, as follows when n is odd. The vertex v receives the label 1. Next, we consider the next vertex in the level of v, that receives the label 2. If the level of v only contains v, then the label 2 is assigned to the first vertex of the level that contains all vertices at distance 2 from v in the digraph. In general, if a vertex receives the label i, for $1 \leq i < |V|$, the next vertex in the level receives the label $i + 1$. If the vertex that receives the label i is the biggest one in the level, then the label $i + 1$ is assigned to the first vertex of the level that contains all vertices at distance 2 from the vertex labeled with i in the digraph. We keep labeling the vertices in this way until all vertices have been labeled, and our labeling λ_v is completed. Two examples are showed in Figure 1.

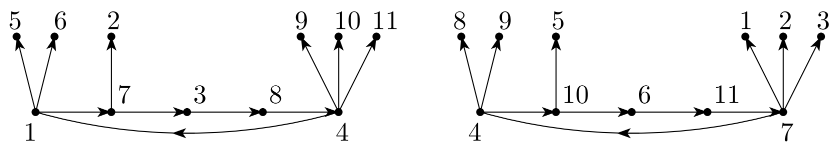

Figure 1: Two super edge-magic labelings of an oriented irregular crown.

Let $v \in V$ and $p = |V|$. Then, for any two vertices v' and $w \in V$, we have that $\lambda_{v'}(w) \in \{1, 2, \ldots, p\}$ and

$$\lambda_{v'}(w) \equiv \lambda_v(w) + 1 - \lambda_v(v') \pmod{p}. \tag{2}$$

Remark 3.1. *Let $x \in V$ and denote by $N_v(x) = \{\lambda_v(y) : xy \in E\}$. Notice that, by construction, if $|N_v(x) \cap \{1, p\}| < 2$ then $N_v(x)$ is a set of consecutive integers. In case, $|N_v(x) \cap \{1, p\}| = 2$ then $N_v(x)$ admits a partition, namely $N_v(x) = N_v^1(x) \cup N_v^2(x)$ such that, $1 \in N_v^1(x)$, $p \in N_v^2(x)$ and $N_v^i(x)$ is a set of consecutive integers, for $i = 1, 2$.*

The next lemma shows that any labeling λ_v is super edge-magic for any $v \in V$.

Lemma 3.1. *Let n be an odd integer and let $j_1, j_2, \ldots, j_n \in \mathbb{N} \cup \{0\}$. Then, the labeling λ_v of $\overrightarrow{C}(n; j_1, j_2, \ldots, j_n)$ can be extended to a super edge-magic labeling g_v, for each $v \in V$. Moreover, the valence of g_v is $1 + \lambda_v(u) + 2p$, where u is the (only) vertex such that $(u, v) \in E(\overrightarrow{C}(n; j_1, j_2, \ldots, j_n))$ and $p = n + j_1 + j_2 + \ldots + j_n$.*

Proof: Let $S_i = \{\lambda_v(v_i) + j : j \in N_v(v_i)\}$, for $i = 1, 2, \ldots, n$. The orientation chosen in $C(n; j_1, j_2, \ldots, j_n)$ guarantees the existence of $k \in \{1, 2, \ldots, n\}$ such that $(v_k, v) \in E(\overrightarrow{C}(n; j_1, j_2, \ldots, j_n))$. By Remark 3.1, S_i is a set of consecutive integers, for each $i \neq k$, and for $i = k$, we have the partition $S_k = S_k^1 \cup S_k^2$ given by $S_k^\alpha = \{\lambda_v(v_k) + j : j \in N_v^\alpha(v_k)\}$, where S_k^α is also a set of consecutive integers, for $\alpha = 1, 2$. Then, the sequence $S_k^1, S_{k+1}, \ldots, S_n, S_1, S_2, \ldots, S_{k-1}, S_k^2$ verifies: (i) every element of the sequence is a set of consecutive integers and, (ii) the maximum in each element of the sequence is the minimum of the next element of the sequence. Therefore, by Lemma 1.1, the labeling λ_v can be extended to a super edge-magic labeling g_v of $C(n; j_1, j_2, \ldots, j_n)$, for each $v \in V$. Moreover, the valence of g_v is given by $1 + \lambda_v(v_k) + 2p$. □

In what follows, we identify λ_v with the super edge-magic labeling g_v, for each $v \in V$.

Proposition 3.1. *Let n be an odd integer and let $j_1, j_2, \ldots, j_n \in \mathbb{N} \cup \{0\}$. Then the set $\{val(\lambda_v) : v \in V(\overrightarrow{C}(n; j_1, j_2, \ldots, j_n))\}$ is a set of consecutive integers.*

Proof: Let $D = \overrightarrow{C}(n; j_1, j_2, \ldots, j_n)$ and $p = n + j_1 + j_2 + \ldots + j_n$. By Lemma 3.1, the valence of the super edge-magic labeling λ_v is given by $1 + \lambda_v(u) + 2p$, where $(u, v) \in E(D)$. Thus, we should prove that the set $\{\lambda_{v'}(u') : v' \in V(D) \text{ and } (u', v') \in E(D)\}$ is a set of consecutive integers, which by (2), it is equivalent to prove that $\mathrm{Dif}_v := \{(\lambda_v(u') - \lambda_v(v'))^* : (u', v') \in E(D)\}$ is a set of consecutive integers, where a^* denotes the least nonnegative residue of $a \pmod p$.

Let us see first that the set Dif_v does not depend on the vertex v. For any $x \in V(D)$, using (2), we get:

$$\begin{aligned}\lambda_x(u') - \lambda_x(v') &\equiv (\lambda_v(u') + 1 - \lambda_v(x)) - (\lambda_v(v') + 1 - \lambda_v(x)) \pmod p \\ &\equiv \lambda_v(u') - \lambda_v(v') \pmod p.\end{aligned}$$

Let $\mathrm{Dif}_v(v_i) = \{(\lambda_v(v_i) - \lambda_v(w))^* : (v_i, w) \in E(D)\}$. As before, the set $\mathrm{Dif}_v(v_i)$ does not depend on the vertex v. Clearly, the following equality holds

$$\mathrm{Dif}_v = \mathrm{Dif}_v(v_1) \cup \mathrm{Dif}_v(v_2) \cup \cdots \cup \mathrm{Dif}_v(v_n). \tag{3}$$

By Remark 3.1, $\mathrm{Dif}_v(v_i)$ is a set of consecutive integers for each i with $1 \leq i \leq n$. Let $a_i = \min \mathrm{Dif}_v(v_i)$ and $b_i = \max \mathrm{Dif}_v(v_i)$. We will prove by induction on the number of leaves, $j_1 + j_2 + \ldots + j_n$, that

$$a_1 \leq b_3,\ a_3 \leq b_5, \ldots, a_{n-2} \leq b_n,\ a_n \leq b_2,\ a_2 \leq b_4, \ldots, a_{n-1} \leq b_1. \tag{4}$$

Assume first that $j_1 + j_2 + \ldots + j_n = 0$, that is, D does not contain leaves. In this case, we have $|\mathrm{Dif}_v| = 1$ and there is nothing to prove. Assume now that the result is true for each digraph D, with $j_1 + j_2 + \ldots + j_n = l$ and consider a digraph D' with $j_1' + j_2' + \ldots + j_n' = l + 1$. Let (u, x) and (u, v) be two arcs of D', where u and v are two vertices of the cycle. Let $D = D' \setminus \{x\}$. We denote by λ_v' the labeling introduced before, when instead of D we consider the digraph D'. Thus, we have $\lambda_v'(w) = \lambda_v(w)$, for each $w \in V(D)$ and $\lambda_v'(x) = n + l + 1$. Similarly, we consider the sets $\mathrm{Dif}_v'(v_i) = \{(\lambda_v'(v_i) - \lambda_v'(w))^* : v_i w \in E(G)\}$ and $a_i' = \min \mathrm{Dif}_v'(v_i)$ and $b_i' = \max \mathrm{Dif}_v'(v_i)$, for $i = 1, 2, \ldots, n$. Without loss of restriction, we can assume that $v = v_1$. Notice that, by construction $a'_n = a_n$, $a'_{2i-1} = a_{2i-1} + 1$, $a'_{2i} = a_{2i}$, $b'_{2i} = b_{2i}$, for $i = 1, 2, \ldots, (n-1)/2$ and $b'_{2i-1} = b_{2i-1} + 1$, for $i = 1, 2, \ldots, (n+1)/2$. Therefore, the induction hypothesis and an easy check show that $a'_1 \leq b'_3$, $a'_3 \leq b'_5, \ldots, a'_{n-2} \leq b'_n$, $a'_n \leq b'_2$, $a'_2 \leq b'_4, \ldots, a'_{n-1} \leq b'_1$.

Now, we are ready to prove that Dif_v is a set of consecutive integers. Assume to the contrary that there exists $x \notin \mathrm{Dif}_v$ such that $\min_i a_i \leq x \leq \max_i b_i$. Without loss of restriction assume that $\min_i a_i = a_1$. The condition $x \notin \mathrm{Dif}_v$ and (3) imply that $x > b_1$ and thus, using (4) that $x > a_{n-1}$. Again, the condition $x \notin \mathrm{Dif}_v$ and (3) imply that $x > b_{n-1}$. Repeating this reasoning recursively, we obtain that $x > b_i$, for all i, which contradicts the fact that $x \leq \max_i b_i$. This proves the result. □

3.1 Irregular crowns that are perfect edge-magic

Let C_m be the cycle of odd order m, with $V(C_m) = \{v_i\}_{i=1}^m$ and $E(C_m) = \{v_i v_{i+1}\}_{i=1}^{m-1} \cup \{v_1 v_m\}$. We denote by C_m^n the graph obtained from C_m by attaching n leaves to each vertex v_{2i-1}, for $i = 1, 2, \ldots, (m+1)/2$. That is, we have the identity $C_m^n \cong C(m; j_1, j_2, \ldots, j_m)$, where $j_{2i-1} = n$, for each i with $1 \leq i \leq (m+1)/2$, and $j_{2i} = 0$, for each i, $1 \leq i \leq (m-1)/2$. Let us first calculate the magic interval of C_m^n.

Lemma 3.2. *The magic interval of C_m^n is $J_{C_m^n} = [a, b]$, where $a = 1 + (m+1)n + 2m + \lceil (m + 3 + (2m(m-1))/((m+1)n + 2m))/4 \rceil$ and $b = 1 + 2(m+1)n + 2m + \lfloor (7m + 1 - (2m(m-1))/((m+1)n + 2m))/4 \rfloor$.*

Proof: Let $C_m^n = (V, E)$, where $V = \{v_i\}_{i=1}^m \cup (\cup_{i=1}^{(m+1)/2} \{v_{2i-1}^j\}_{j=1}^n)$ and $E = \{v_i v_{i+1}\}_{i=1}^{m-1} \cup \{v_1 v_m\} \cup (\cup_{i=1}^{(m+1)/2} \{v_{2i-1}^j\}_{j=1}^n)$. For any bijective function $g : V \cup E \to \{i\}_{i=1}^{(m+1)n+2m}$, the corresponding element in T_G is

$$\frac{2}{(m+1)n + 2m} \left((1+n) \sum_{i=1}^{(m+1)/2} g(v_{2i-1}) + \sum_{i=1}^{(m-1)/2} g(v_{2i}) + \sum_{u \in V \cup E} g(u) \right).$$

Thus, the minimum possible valence occurs when the labels $\{1, 2, \ldots, (m+1)/2\}$ are assigned to the vertices of degree $2+n$ and the labels $\{(m+3)/2, (m+5)/2, \ldots, m\}$ are assigned to the remaining vertices of the cycle. Hence, the minimum possible valence is:

$$\begin{aligned} \min J_{C_m^n} &= \left\lceil \frac{2}{(m+1)n + 2m} \left((1+n) \sum_{i=1}^{(m+1)/2} i + \sum_{i=(m+3)/2}^{m} i + \sum_{i=1}^{(m+1)n+2m} i \right) \right\rceil \\ &= \left\lceil \frac{1}{4} \left(m + 3 + \frac{2m(m-1)}{(m+1)n + 2m} \right) \right\rceil + 1 + (m+1)n + 2m. \end{aligned}$$

On the other hand, the maximum possible valence occurs when the labels $\{(m+1)n + 2m, (m+1)n + 2m - 1, \ldots, (m+1)n + 2m - (m-1)/2\}$ are assigned to the vertices of degree $2+n$ and the labels $\{(m+1)n + 2m - (m+1)/2, (m+1)n + 2m - (m+3)/2, \ldots, (m+1)n + 2m - (m-1)\}$

are assigned to the remaining vertices of the cycle. Hence, the maximum possible valence is: $\max J_{C_m^n} = \lfloor (7m+1-(2m(m-1))/((m+1)n+2m))/4 \rfloor + 1 + 2(m+1)n + 2m$. □

Lemma 3.3. *Let m be an odd integer. Then, for any pair of integers n and k, with $(5m+3)/2+(m+1)n \leq k \leq (5m+3)/2+3(m+1)n/2$ there exists a super edge-magic labeling of C_m^n with valence k.*

Proof: An easy check shows that the labelings λ_{v_1} and $\lambda_{v_1^1}$ have valences $(5m+3)/2+(m+1)n$ and $(5m+3)/2+3(m+1)n/2$, respectively. By Lemma 3.1, the set $\{\mathrm{val}(\lambda_v) : v \in V(C_m^n)\}$ is a set of consecutive integers. Therefore, for each k with $(5m+3)/2+(m+1)n \leq k \leq (5m+3)/2+3(m+1)n/2$ there exists $v \in V(C_m^n)$ such that the valence of λ_v is equal to k. □

Corollary 3.1. *Let m be an odd integer. Then, for any pair of integers n and k with $3m+1+(m+1)n \leq k \leq 3m+2+2(m+1)n$ there exists an edge-magic labeling of C_m^n with valence k.*

Proof. Notice that, by Lemma 3.3, for any pair of integers n and k, with $(5m+3)/2+(m+1)n \leq k \leq (5m+3)/2+3(m+1)n/2$ there exists a super edge-magic labeling of C_m^n with valence k. Let g_r be a super edge-magic labeling of C_m^n with valence $(5m+3)/2+(m+1)n+r-1$, for $r = 1, 2, \ldots, (m+1)n/2+1$. Thus, Lemma 1.2 implies that the set $\{\mathrm{val}(o(g_r)) : 1 \leq r \leq (m+1)n/2+1\} \cup \{\mathrm{val}(e(g_r)) : 1 \leq r \leq (m+1)n/2+1\}$ contains all integers from $\mathrm{val}(o(g_1))$ up to $\mathrm{val}(e(g_{(m+1)n/2+1}))$. That is, all integers from $(m+1)n+3m+1$ up to $2(m+1)n+3m+2$. □

Corollary 3.2. *Let m be an odd integer. Then, for any pair of integers n and k with $n \geq 1$ and $(5m+3)/2+(m+1)n \leq k \leq (7m+3)/2+2(m+1)n$ there exists an edge-magic labeling of C_m^n with valence k.*

Proof: Let g_r be a super edge-magic labeling of C_m^n with valence $(5m+3)/2+(m+1)n+r-1$, for $r = 1, 2, \ldots, (m+1)n/2+1$. Such labelings exist by Lemma 3.3. Taking the complementary labelings of these labelings, we get that all the natural numbers from $(7m+3)/2+3(m+1)n/2$ up to $(7m+3)/2+2(m+1)n$ also appear as valences of edge-magic labelings of C_m^n. Since by Corollary 3.1, for any pair of integers n and k with $3m+1+(m+1)n \leq k \leq 3m+2+2(m+1)n$ there exists an edge-magic labeling of C_m^n with valence k, in order to complete the proof we only need to check that $3m+1+(m+1)n \leq (5m+3)/2+3(m+1)n/2$ and $(7m+3)/2+3(m+1)n/2 \leq 3m+2+2(m+1)n$. But, this is clear since the two inequalities are equivalent to the inequality $m-1 \leq (m+1)n$, which trivially holds for $n \geq 1$. □

Proposition 3.2. *The graph C_3^n is perfect edge-magic, for all $n \in \mathbb{N} \setminus \{1\}$.*

Proof: By Lemma 3.2, the magic interval of C_3^n is $J_{C_3^n} = [4n+9, 8n+12]$. Since by Corollary 3.2, the magic set $\tau_{C_3^n}$ contains the interval $[4n+9, 8n+12]$, we get the result. □

Theorem 3.2. *The graph C_5^n is perfect edge-magic for all $n \in \mathbb{N} \setminus \{1\}$.*

Proof: By Lemma 3.2, the magic interval of C_5^n is $J_{C_5^n} = [6n+14, 12n+19]$. Since by Corollary 3.2, the magic set $\tau_{C_5^n}$ contains the interval $[6n+14, 12n+19]$, we get the result. □

Using Lemma 3.2 and Corollary 3.2, a computer check shows other families of perfect edge-magic graphs.

Theorem 3.3. *The graph C_m^n is perfect edge-magic when (i) $m = 7$ and $1 \leq n \leq 3$, (ii) $m = 9$ and $n = 1$, and (iii) $m = 11$ and $n = 1$.*

4 Even harmonious labelings from super edge-magic labelings

A (p, q)-graph G with $p \leq q$ is called *harmonious* [10] if it is possible to label its vertices with distinct integers (mod q) in such a way that the edge sums are also distinct (mod q). When G is a tree, exactly one label may be used on two vertices. Variations of this concept have appeared recently in the literature. A (p, q)-graph G is said to be *odd harmonious* [13] if there exists an injection $f : V(G) \to \{i\}_{i=0}^{2q-1}$ such that the induced mapping $f^*(uv) = (f(u) + f(v))$ is a bijection from $E(G)$ onto the set $\{1, 3, 5, \ldots, 2q - 1\}$. Then f is called an *odd harmonious labeling* of G. Similarly, Sarasija and Binthiya introduced in [17] what they called an even harmonious graph. Let G be a (p, q)-graph. An injective function $f : V(G) \to \{i\}_{i=0}^{2q}$ such that the induced function $f^* : E(G) \to \{0, 2, 4, \ldots, 2(q-1)\}$ defined by $f^*(uv) = (f(u)+f(v)) \bmod (2q)$ is bijective. Then f is called an *even harmonious labeling* of G and G is called an *even harmonious graph*.

Super edge-magic labelings are known to be a powerful link among different types of labelings. In [5] many relations among labelings were established in a direct way. Later on, in [7] the digraph product $\otimes_h$ was introduced, and this product together with super edge-magic labelings, has been used in order to establish further relations among labelings, see for instance [7, 11, 14, 16]. In this section we establish a new relationship among super edge-magic labelings and even harmonious labelings.

Lemma 4.1. *Let G be a (p, q)-graph with $q \geq p - 1$. If G is super edge-magic then G is even harmonious.*

Proof: Let f be any super edge-magic labeling of a (p, q)-graph G, with $q \geq p - 1$. Consider the labeling $e^*(f) : V(G) \to \{i\}_{i=0}^{2q}$ defined by the rule $e^*(f)(u) = 2f(u) - 2$, for all $u \in V(G)$. Then, using a similar proof as in Lemma 1.2, it is clear that $e^*(f)$ is an even harmonious labeling of G. □

From this result we get that all super edge-magic graphs are even harmonious. In particular, we can obtain some of the results introduced in [17].

Corollary 4.1. *[17] The following graphs are even harmonious: (i) the path P_n, with $n \geq 2$, (ii) the star $K_{1,n}$, with $n \geq 1$, and (iii) the cycle of odd order C_n, with $n \geq 3$.*

5 Conclusions and remarks

In this paper we have proved that the family $C_m \odot \overline{K_n}$, where m is a power of a prime greater than 2, is perfect edge-magic for all $n \in \mathbb{N} \setminus \{1\}$. In fact, it is the first non-trivial infinite family known to be perfect edge-magic. We also have proved that C_3^n and C_5^n are perfect edge-magic and that the magic set of the family C_m^n contains a big interval. The problem of finding families of graphs that are perfect edge-magic seems to be a hard one, and we want to encourage other researches to continue this line of research. Next we want to introduce some open problems in this direction.

Open question 5.1. *Characterize the set Σ_n defined by*
$\Sigma_m = \{m \in \mathbb{N} : C_{2m+1} \odot \overline{K_n}$ *is perfect edge-magic for all* $n \in \mathbb{N}\}$.

About open question 5.1, we have made some progress, continuing a work started in [15]. However, nothing is known about the next question.

Open question 5.2. *Characterize the set Υ_n defined by $\Upsilon_m = \{m \in \mathbb{N} \setminus \{1\} : C_{2m} \odot \overline{K_n}$ is perfect edge-magic for all $n \in \mathbb{N}\}$.*

It is well known that stars are not perfect edge-magic. In fact the set $\tau_{K_{1,n}}$ contains only 3 elements for every $n \in \mathbb{N} \setminus \{1\}$ (see [6, 18]). This fact motivates the following two questions.

Open question 5.3. *Find examples of infinite families of graphs which are edge-magic but not perfect edge-magic.*

Open question 5.4. *Characterize the set of caterpillars which are not perfect edge-magic. In particular, characterize the set of paths which are perfect edge-magic and characterize the set of caterpillars with the same number of leaves attached at each vertex of the spine which are perfect edge-magic.*

The concept of perfect edge-magic graphs was motivated by the concept of perfect super edge-magic graphs introduced in [15]. Furthermore, the concept of perfect super edge-magic graphs was motivated by the following conjecture introduced in [9] by Godbold and Slater, that "as far as we know" remains unsolved up to the present.

Conjecture 5.1. *[9]. For $n = 2t+1 \geq 7$ and $5t+4 \leq j \leq 7t+5$ there is an edge-magic labeling of C_n, with valence $k = j$. For $n = 2t \geq 4$ and $5t+2 \leq j \leq 7t+1$ there is an edge-magic labeling of C_n, with valence $k = j$.*

In this paper we want to renew the interest for this question, and encourage researches to work towards a final solution of the question. For any reader interested in it, the book of Wallis [18] constitutes an excellent source of information about this question. For related problems on graph labelings we direct the reader to [2] and [8].
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S.C. López
Departament de Matemàtica Aplicada IV
Universitat Politècnica de Catalunya,
C/Esteve Terrades 5
08860 Castelldefels, Spain
E-mail: `susana@ma4.upc.edu`

F.A. Muntaner-Batle
Graph Theory and Applications Research Group
School of Electrical Engineering and Computer Science
Faculty of Engineering and Built Environment
The University of Newcastle
NSW 2308 Australia
E-mail: `famb1es@yahoo.es`

M. Rius-Font
Departament de Matemàtica Aplicada IV
Universitat Politècnica de Catalunya,
C/Esteve Terrades 5
08860 Castelldefels, Spain
E-mail: `mrius@ma4.upc.edu`