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Perfect edge-magic graphs
by
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Abstract
The study of the possible valences for edge-magic labelings of graphs has motivated us to introduce the concept of perfect edge-magic graphs. Intuitively speaking, an edge-magic graph is perfect edge-magic if all possible theoretical valences occur. In particular, we prove that for each integer \( m > 0 \), that is the power of an odd prime, and for each natural number \( n \), the crown product \( C_m \odot K_n \) is perfect edge-magic. Related results are also provided concerning other families of unicyclic graphs. Furthermore, several open questions that suggest interesting lines for future research are also proposed.

Key Words: Edge-magic, perfect edge-magic, valence.

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1 Introduction

For the graph theory terminology and notation not defined in this paper we refer the reader to either one of the following sources [2, 3, 8, 18]. Kotzig and Rosa [12] introduced in 1970 the concepts of edge-magic graphs and edge-magic labelings as follows: a \((p, q)\)-graph \( G \) is called edge-magic if there is a bijective function \( f: V(G) \cup E(G) \to \{i\}_{i=1}^{p+q} \) such that the sum \( f(u) + f(uv) + f(v) = k \) for any \( uv \in E(G) \). Such a function is called an edge-magic labeling of \( G \) and \( k \) is called the valence or the magic sum of the labeling \( f \). The purpose of this paper is to characterize the set of numbers that are valences for the edge-magic labelings of some families of unicyclic graphs.

Let \( G = (V, E) \) be a \((p, q)\)-graph, and define the set

\[
T_G = \left\{ \frac{\sum_{u \in V} \deg(u)g(u) + \sum_{e \in E} g(e)}{q} : g : V \cup E \to \{i\}_{i=1}^{p+q} \text{ is a bijective function} \right\}.
\]

If \( \lfloor \min T_G \rfloor \leq \lceil \max T_G \rceil \) then the magic interval of \( G \), denoted by \( J_G \), is defined to be the set \( J_G = [\lfloor \min T_G \rfloor, \lceil \max T_G \rceil] \cap \mathbb{N} \) and the magic set of \( G \), denoted by \( \tau_G \), is the set \( \tau_G = \{n \in J_G : n \text{ is the valence of some edge-magic labeling of } G\} \). It is clear that \( \tau_G \subseteq J_G \). In this paper, we call \( G \) a perfect edge-magic graph if \( \tau_G = J_G \).
A \((p, q)\)-graph \(G\) is super edge-magic if there is an edge-magic labeling of \(G\), namely \(f : V(G) \cup E(G) \rightarrow \{i\}_{i=1}^{p+q}\), with the extra property that \(f(V(G)) = \{i\}_{i=1}^p\). The function \(f\) is called a super edge-magic labeling of \(G\). These concepts were introduced independently by Acharya and Hegde [1] and by Enomoto et al. in [4]. Figueroa-Centeno et al. stated in [5] the following characterization for super edge-magic labelings.

**Lemma 1.1.** Let \(G\) be a \((p, q)\)-graph. Then \(G\) is super edge-magic if and only if there is a bijective function \(g : V(G) \rightarrow \{i\}_{i=1}^p\) such that the set \(S = \{g(u) + g(v) : uv \in E(G)\}\) is a set of \(q\) consecutive integers. In this case, \(g\) can be extended to a super edge-magic labeling \(f\) with valence \(p + q + \text{min}S\).

Let \(f : V(G) \cup E(G) \rightarrow \{i\}_{i=1}^{p+q}\) be an edge-magic labeling of a \((p, q)\)-graph \(G\). The complementary labeling of \(f\), denoted by \(\overline{f}\), is the labeling defined by the rule: \(\overline{f}(x) = p + q + 1 - f(x)\), for all \(x \in V(G) \cup E(G)\). Notice that, if \(f\) is an edge-magic labeling of \(G\) with valence \(k\), we have that \(\overline{f}\) is also an edge-magic labeling of \(G\) with valence \(\overline{f} = 3(p + q + 1) - k\). Let \(f : V(G) \cup E(G) \rightarrow \{i\}_{i=1}^{p+q}\) be a super edge-magic labeling of a \((p, q)\)-graph \(G\), with \(p = q\). The odd labeling and the even labeling obtained from \(f\), denoted respectively by \(o(f)\) and \(e(f)\), are the labelings \(o(f), e(f) : V(G) \cup E(G) \rightarrow \{i\}_{i=1}^{p+q}\) defined as follows: (i) on the vertices: \(o(f)(x) = 2f(x) - 1\) and \(e(f)(x) = 2f(x)\), for all \(x \in V(G)\), (ii) on the edges:

\[
\begin{align*}
o(f)(xy) &= 2\text{val}(f) - 2p - 2 - o(f)(x) - o(f)(y) \\
e(f)(xy) &= 2\text{val}(f) - 2p - 1 - e(f)(x) - e(f)(y),
\end{align*}
\]

for all \(xy \in E(G)\).

**Lemma 1.2.** Let \(G\) be a \((p, q)\)-graph with \(p = q\) and let \(f : V(G) \cup E(G) \rightarrow \{i\}_{i=1}^{p+q}\) be a super edge-magic labeling of \(G\). Then, the odd labeling \(o(f)\) and the even labeling \(e(f)\) obtained from \(f\) are edge-magic labelings of \(G\) with valences \(\text{val}(o(f)) = 2\text{val}(f) - 2p - 2\) and \(\text{val}(e(f)) = 2\text{val}(f) - 2p - 1\).

**Proof:** Note that, since \(f\) is super edge-magic, the set \(S_n = \{o(f)(x) + o(f)(y) : xy \in E(G)\} = \{2(f(x) + f(y)) - 2 : xy \in E(G)\}\) is an arithmetic progression of difference 2, starting at \(2(\text{val}(f) - 2p)\). Thus, by assigning the even labels to the edges, we obtain an edge-magic labeling with valence \(\text{val}(o(f)) = 2\text{val}(f) - 2p\). The proof for \(e(f)\) is similar.  

When we say that a digraph has a labeling we mean that its underlying graph has such labeling, see [7].

The paper is organized as follows: in section 2 we prove that each element in the family \(C_m \odot \overline{K_n}\), where \(m\) is a power of an odd prime and \(\overline{K_n}\) denotes the complementary graph of the complete graph \(K_n\) (\(n \in \mathbb{N}\)), is a perfect edge-magic graph. In section 3, we prove that the magic set of irregular crowns is big by showing a general construction of edge-magic labelings, and that a subfamily of them is perfect edge-magic. In section 4, we establish a new relation among super edge-magic and even harmonious labelings. Finally, we end by a short section of conclusions and remarks.

## 2 A family of perfect edge-magic graphs

We begin our calculation of the magic interval \(J_{G_m \odot \overline{K_n}}\), for all \(m, n \in \mathbb{N}\). Let \(G_m \odot \overline{K_n} = (V, E)\), where \(V = \{v_i\}_{i=1}^m \cup \{v_i^j\}_{j=1}^n\) and \(E = \{v_i, v_{i+1}\}_{i=1}^{m-1} \cup \{v_i, v_{m}\} \cup \{v_i, v^j\}_{j=1}^{n}\).
For any bijective function \( g : V \cup E \rightarrow \{ i \}_{i=1}^{2m(n+1)} \), the corresponding element in \( T_G \) is 
\[
(1+n)\sum_{i=1}^{m} g(v_i) + \sum_{u \in V \cup E} g(u) / (mn + m).
\]
Thus, the minimum possible valence occurs when the labels \( \{1, 2, \ldots, m\} \) are assigned to the vertices of the cycle. Therefore,
\[
\min T_{C_m \circ K_n} = \frac{(1+n)\sum_{i=1}^{m} i + \sum_{i=1}^{2m(n+1)} i}{mn + m} = \frac{3+5m}{2} + 2mn.
\]
On the other hand, the maximum possible valence occurs when the labels \( \{2m(n+1) - m + 1, 2m(n+1) - m + 2, \ldots, 2m(n+1)\} \) are assigned to the vertices of the cycle. Hence, using similar calculations, we obtain that \( \max T_{C_m \circ K_n} = (3+7m)/2 + 4mn \).

López et al. have shown in [15] that for each \( r \in \mathbb{N} \), with \( 1 \leq r \leq mn + 1 \), there exists a super edge-magic labeling \( f_r \) with valence
\[
\text{val}(f_r) = r - 1 + \frac{3+5m}{2} + 2mn
\]
of \( C_m \circ K_n \), when \( m \) is a power of a prime greater than 2. Taking the complementary labelings of these labelings, we get that all the natural numbers from \( 3mn + (3 + 7m)/2 \) up to \( 4mn + (3 + 7m)/2 \) also appear as valences of edge-magic labelings of \( C_m \circ K_n \). Therefore, in order to prove that \( C_m \circ K_n \) is perfect edge-magic, we only need to show that for each \( k \in \mathbb{N} \), with \( 3mn + (3 + 5m)/2 < k < 3mn + (3 + 7m)/2 \), there exists an edge-magic labeling with valence \( k \). We do this using the odd and the even labelings of the labelings \( f_r \) introduced in [15].

**Lemma 2.1.** Let \( m \) be a power of a prime greater than 2 and let \( n \) be any positive integer. Then, for each \( k \) with \( 2mn + 3m + 1 \leq k \leq 4mn + 3m + 2 \) there exists an edge-magic labeling of \( C_m \circ K_n \) with valence \( k \).

**Proof:** Notice that, by (1) the set \( \{\text{val}(f_r) : 1 \leq r \leq mn + 1\} \) is a set of consecutive integers. Thus, Lemma 1.2 implies that the set \( \{\text{val}(o(f_r)) : 1 \leq r \leq mn + 1\} \cup \{\text{val}(e(f_r)) : 1 \leq r \leq mn + 1\} \) contains all integers from \( \text{val}(o(f_1)) \) up to \( \text{val}(e(f_{mn+1})) \). That is, all integers from \( 2mn + 3m + 1 \) up to \( 4mn + 3m + 2 \).

Since \( 2mn + 3m + 1 \leq 3mn + (3 + 5m)/2 \) and \( 3mn + (3 + 7m)/2 \leq 4mn + 3m + 2 \), for \( n \geq 1 \), we obtain the next theorem.

**Theorem 2.1.** Let \( m = p^k \) where \( p \) is an odd prime and \( k \in \mathbb{N} \). Then the graph \( C_m \circ K_n \) is perfect edge-magic for all \( n \in \mathbb{N} \), \( n \geq 1 \).

### 3 Super edge-magic toroidal labelings

The purpose of this section is to introduce another family of perfect edge-magic graphs. This is a subfamily of the family of irregular crowns that we introduce in the next lines.

Let \( C(n; j_1, j_2, \ldots, j_n) = (V, E) \), where \( n \in \mathbb{N} \setminus \{1, 2\} \) and \( j_i \in \mathbb{N} \cup \{0\} \) for all \( i \in \{1, 2, \ldots, n\} \) be the irregular crown defined as follows: 
\[
V = \{v_{1k}\}_{k=1}^{n} \cup V_1 \cup V_2 \cdots \cup V_n, \quad V_k = \{v_{1k}^1, v_{1k}^2, \ldots, v_{1k}^{j_k}\},
\]
with \( j_k \neq 0 \) and \( V_k = \emptyset \) if \( j_k = 0 \), for each \( k \in \{1, 2, \ldots, n\} \) and
Moreover, the valence of \( g_v \) is \( 1 + \lambda_v(u) + 2p \), where \( u \) is the (only) vertex such that \((u, v) \in E(\overrightarrow{C}(n; j_1, j_2, \ldots, j_n)) \) and \( p = n + j_1 + j_2 + \ldots + j_n \).
Proof: Let \( S_i = \{ \lambda(v_i) + j : j \in N(v_i) \} \), for \( i = 1, \ldots, n \). The orientation chosen in \( C(n; j_1, j_2, \ldots, j_n) \) guarantees the existence of \( k \in \{ 1, 2, \ldots, n \} \) such that \((v_k, v) \in E(C(n; j_1, j_2, \ldots, j_n))\). By Remark 3.1, \( S_i \) is a set of consecutive integers, for each \( i \neq k \), and for \( i = k \), we have the partition \( S_k = S_k^1 \cup S_k^2 \) given by \( S_k^1 = \{ \lambda(v_k) + j : j \in N(v_k) \} \), where \( S_k^2 \) is also a set of consecutive integers, for \( \alpha = 1, 2 \). Then, the sequence \( S_k^1, S_{k+1}, \ldots, S_n, S_1, S_2, \ldots, S_{k-1}, S_k^2 \) verifies: (i) every element of the sequence is a set of consecutive integers and, (ii) the maximum in each element of the sequence is the minimum of the next element of the sequence. Therefore, by Lemma 1.1, the labeling \( \lambda_n \) can be extended to a super edge-magic labeling \( g_v \) of \( C(n; j_1, j_2, \ldots, j_n) \), for each \( v \in V \). Moreover, the valence of \( g_v \) is given by \( 1 + \lambda_n(v_k) + 2p \).

In what follows, we identify \( \lambda_n \) with the super edge-magic labeling \( g_v \), for each \( v \in V \).

Proposition 3.1. Let \( n \) be an odd integer and let \( j_1, j_2, \ldots, j_n \in \mathbb{N} \cup \{0\} \). Then the set \( \{ \text{val}(\lambda_n) : v \in V(C(n; j_1, j_2, \ldots, j_n)) \} \) is a set of consecutive integers.

Proof: Let \( D = \overrightarrow{C}(n; j_1, j_2, \ldots, j_n) \) and \( p = n + j_1 + j_2 + \ldots + j_n \). By Lemma 3.1, the valence of the super edge-magic labeling \( \lambda_n \) is given by \( 1 + \lambda_n(u) + 2p \), where \((u, v) \in E(D)\). Thus, we should prove that the set \( \{ \lambda_n(u') : v' \in V(D) \} \) is a set of consecutive integers, which by (2), it is equivalent to prove that \( D_{\lambda_n}(v) := \{ (\lambda_n(u') - \lambda_n(v'))^* : (u', v') \in E(D) \} \) is a set of consecutive integers, where \( \lambda_n(v) \) denotes the least nonnegative residue of \( \lambda_n(u) \) modulo \( p \).

Let us see first that the set \( D_{\lambda_n}(v) \) does not depend on the vertex \( v \). For any \( x \in V(D) \), using (2), we get:

\[
\lambda_n(u') - \lambda_n(v') \equiv (\lambda_n(u') + 1 - \lambda_n(x)) - (\lambda_n(v') + 1 - \lambda_n(x)) \pmod{p}
\]

\[
\equiv \lambda_n(u') - \lambda_n(v') \pmod{p}.
\]

Let \( D_{\lambda_n}(v) = \{ (\lambda_n(v_i) - \lambda_n(w))^* : \langle v_i, w \rangle \in E(D) \} \). As before, the set \( D_{\lambda_n}(v) \) does not depend on the vertex \( v \). Clearly, the following equality holds

\[
D_{\lambda_n}(v_1) = D_{\lambda_n}(v_1) \cup D_{\lambda_n}(v_2) \cup \cdots \cup D_{\lambda_n}(v_n).
\]

By Remark 3.1, \( D_{\lambda_n}(v_i) \) is a set of consecutive integers for each \( i \) with \( 1 \leq i \leq n \). Let \( a_i = \min D_{\lambda_n}(v_i) \) and \( b_i = \max D_{\lambda_n}(v_i) \). We will prove by induction on the number of leaves, \( j_1 + j_2 + \ldots + j_n \), that

\[
a_1 \leq b_3, \quad a_3 \leq b_5, \quad \ldots, \quad a_{n-2} \leq b_n, \quad a_n \leq b_2, \quad a_2 \leq b_4, \quad \ldots, \quad a_{n-1} \leq b_1.
\]

Assume first that \( j_1 + j_2 + \ldots + j_n = 0 \), that is, \( D \) does not contain leaves. In this case, we have \( |D_{\lambda_n}| = 1 \) and there is nothing to prove. Assume now that the result is true for each digraph \( D' \) with \( j_1 + j_2 + \ldots + j_n = l \) and consider a digraph \( D' \) with \( j_1' + j_2' + \ldots + j_n' = l + 1 \). Let \((u, v)\) and \((u, v')\) be two arcs of \( D' \), where \( u \) and \( v \) are two vertices of the cycle. Let \( D = D' \setminus \{x\} \). We denote by \( \lambda_n' \) the labeling introduced before, when instead of \( D \) we consider the digraph \( D' \). Thus, we have \( \lambda_n'(w) = \lambda_n(w) \), for each \( w \in V(D) \) and \( \lambda_n'(x) = n + l + 1 \). Similarly, we consider the sets \( D_{\lambda_n'}(v_i) = \{ (\lambda_n'(v_i) - \lambda_n'(w))^* : v_i, w \in E(G) \} \) and \( a_i' = \min D_{\lambda_n'}(v_i) \) and \( b_i' = \max D_{\lambda_n'}(v_i) \).
for \( i = 1, 2, \ldots, n \). Without loss of restriction, we can assume that \( v = v_1 \). Notice that, by construction \( a'_n = a_n, a'_{2i-1} = a_{2i-1} + 1, a'_{2i} = a_{2i}, b'_{2i-1} = b_{2i-1} + 1, \) for \( i = 1, 2, \ldots, (n - 1)/2 \) and \( b'_{2i-1} = b_{2i-1} + 1, \) for \( i = 1, 2, \ldots, (n + 1)/2 \). Therefore, the induction hypothesis and an easy check show that \( a'_1 \leq b'_1, a'_2 \leq b'_2, \ldots, a'_{n-2} \leq b'_{n-2}, a'_n \leq b'_{n}, a'_2 \leq b'_2, \ldots, a'_{n-1} \leq b'_{n-1} \).

Now, we are ready to prove that \( \text{Dif}_0 \) is a set of consecutive integers. Assume to the contrary that there exists \( x \notin \text{Dif}_0 \) such that \( \min a_i \leq x \leq \max b_i \). Without loss of restriction assume that \( \min a_i = a_1 \). The condition \( x \notin \text{Dif}_0 \) and (3) imply that \( x > b_1 \) and thus, using (4) that \( x > a_{n-1} \). Again, the condition \( x \notin \text{Dif}_0 \) and (3) imply that \( x > b_{n-1} \). Repeating this reasoning recursively, we obtain that \( x > b_i \), for all \( i \), which contradicts the fact that \( x \leq \max b_i \). This proves the result. \( \square \)

3.1 Irregular crowns that are perfect edge-magic

Let \( C_m \) be the cycle of odd order \( m \), with \( V(C_m) = \{v_1\}^m_{i=1} \) and \( E(C_m) = \{v_i v_{i+1}\}^m_{i=1} \cup \{v_1 v_m\} \). We denote by \( C^m_m \) the graph obtained from \( C_m \) by attaching \( n \) leaves to each vertex \( v_{2i-1} \), for \( i = 1, 2, \ldots, (m+1)/2 \). That is, we have the identity \( C^m_m \cong C(m; j_1, j_2, \ldots, j_m) \), where \( j_{2i-1} = n \), for each \( i \) with \( 1 \leq i \leq (m+1)/2 \), and \( j_{2i} = 0 \), for each \( i \), \( 1 \leq i \leq (m-1)/2 \). Let us first calculate the magic interval of \( C^m_m \).

**Lemma 3.2.** The magic interval of \( C^m_m \) is \( J_{C^m_m} = [a, b] \), where \( a = 1 + (m+1)n + 2m + [(m + 3 + (2m(m-1))/((m+1)n + 2m))/4] \) and \( b = 1 + 2(m+1)n + 2m + [(7m + 1 - (2m(m-1))/((m+1)n + 2m))/4] \).

**Proof:** Let \( C^m_m = (V, E) \), where \( V = \{v_i\}^m_{i=1} \cup (\cup_{i=1}^{(m+1)/2} \{v_{2i-1}\})^n_{i=1} \) and \( E = \{v_i v_{i+1}\}^m_{i=1} \cup \{v_1 v_m\} \cup (\cup_{i=1}^{(m+1)/2} \{v_{2i-1}\})^n_{i=1} \). For any bijective function \( g : V \cup E \rightarrow \{1, 2, \ldots, (m+1)n + 2m, \} \), the corresponding element in \( T_G \) is

\[
\frac{2}{(m+1)n + 2m} \left( (1 + n) \sum_{i=1}^{(m+1)/2} g(v_{2i-1}) + \sum_{i=1}^{(m-1)/2} g(v_{2i}) + \sum_{u \in V \cup E} g(u) \right).
\]

Thus, the minimum possible valence occurs when the labels \( \{1, 2, \ldots, (m+1)/2\} \) are assigned to the vertices of degree \( 2 + n \) and the labels \( \{(m+3)/2, (m+5)/2, \ldots, m\} \) are assigned to the remaining vertices of the cycle. Hence, the minimum possible valence is:

\[
\min J_{C^m_m} = \left[ \frac{2}{(m+1)n + 2m} \left( (1 + n) \sum_{i=1}^{(m+1)/2} i + \sum_{i=(m+3)/2}^{m} i + \sum_{i=1}^{(m+1)n+2m} i \right) \right]
\]

\[
= \left[ \frac{1}{4} \left( m + 3 + \frac{2m(m-1)}{(m+1)n + 2m} \right) \right] + 1 + (m+1)n + 2m.
\]

On the other hand, the maximum possible valence occurs when the labels \( \{(m+1)n + 2m, (m+1)n + 2m - 1, \ldots, (m+1)n + 2m - (m-1)/2\} \) are assigned to the vertices of degree \( 2 + n \) and the labels \( \{(m+1)n + 2m - (m+1)/2, (m+1)n + 2m - (m+3)/2, \ldots, (m+1)n + 2m - (m-1)\} \)
are assigned to the remaining vertices of the cycle. Hence, the maximum possible valence is:
\[
\max J_{C^n_m} = \left(\frac{(7m+1 - (2m(m-1))/((m+1)n + 2m))}{4}\right) + 1 + 2(m+1)n + 2m.
\]

**Lemma 3.3.** Let \( m \) be an odd integer. Then, for any pair of integers \( n \) and \( k \), with \((5m + 3)/2 + (m + 1)n \leq k \leq (5m + 3)/2 + 3(m + 1)n/2\) there exists a super edge-magic labeling of \( C^n_m \) with valence \( k \).

**Proof:** An easy check shows that the labelings \( \lambda_v \) and \( \lambda_{v+1} \) have valences \((5m+3)/2 + (m+1)n\) and \((5m+3)/2 + 3(m+1)n/2\), respectively. By Lemma 3.1, the set \( \{\text{val}(\lambda_v) : v \in V(C^n_m)\} \) is a set of consecutive integers. Therefore, for each \( k \) with \((5m + 3)/2 + (m + 1)n \leq k \leq (5m + 3)/2 + 3(m + 1)n/2\) there exists \( v \in V(C^n_m) \) such that the valence of \( \lambda_v \) is equal to \( k \). \( \square \)

**Corollary 3.1.** Let \( m \) be an odd integer. Then, for any pair of integers \( n \) and \( k \) with \((m + 1)n \leq k \leq 3m + 2 + 2(m + 1)n\) there exists an edge-magic labeling of \( C^n_m \) with valence \( k \).

**Proof.** Notice that, by Lemma 3.3, for any pair of integers \( n \) and \( k \), with \((5m + 3)/2 + (m + 1)n \leq k \leq (5m + 3)/2 + 3(m + 1)n/2\) there exists a super edge-magic labeling of \( C^n_m \) with valence \( k \). Let \( g_r \) be a super edge-magic labeling of \( C^n_m \) with valence \((5m + 3)/2 + (m + 1)n + r - 1\), for \( r = 1, 2, \ldots, (m + 1)n/2 + 1 \). Thus, Lemma 1.2 implies that the set \( \{\text{val}(o(g_r)) : 1 \leq r \leq (m + 1)n/2 + 1\} \) contains all integers from \( \text{val}(o(g_1)) \) up to \( \text{val}(o(g((m+1)n/2+1))) \). That is, all integers from \((m + 1)n + 3m + 1\) up to \((m + 1)n + 3m + 2\). \( \square \)

**Corollary 3.2.** Let \( m \) be an odd integer. Then, for any pair of integers \( n \) and \( k \) with \((5m + 3)/2 + (m + 1)n \leq k \leq (7m + 3)/2 + 2(m + 1)n\) there exists an edge-magic labeling of \( C^n_m \) with valence \( k \).

**Proof:** Let \( g_r \) be a super edge-magic labeling of \( C^n_m \) with valence \((5m + 3)/2 + (m + 1)n + r - 1\), for \( r = 1, 2, \ldots, (m + 1)n/2 + 1 \). Such labelings exist by Lemma 3.3. Taking the complementary labelings of these labelings, we get that all the natural numbers from \((7m + 3)/2 + 3(m + 1)n/2\) up to \((7m + 3)/2 + 2(m + 1)n\) also appear as valences of edge-magic labelings of \( C^n_m \). Since by Corollary 3.1, for any pair of integers \( n \) and \( k \) with \((m + 1)n \leq k \leq 3m + 2 + 2(m + 1)n\) there exists an edge-magic labeling of \( C^n_m \) with valence \( k \), in order to complete the proof we only need to check that \(3m + 1 + (m + 1)n \leq (5m + 3)/2 + 3(m + 1)n/2\) and \((7m + 3)/2 + 3(m + 1)n/2 \leq 3m + 2 + (m + 1)n\). But, this is clear since the two inequalities are equivalent to the inequality \(m - 1 \leq (m + 1)n\), which trivially holds for \( n \geq 1 \). \( \square \)

**Proposition 3.2.** The graph \( C^n_m \) is perfect edge-magic, for all \( n \in \mathbb{N} \setminus \{1\} \).

**Proof:** By Lemma 3.2, the magic interval of \( C^n_m \) is \( J_{C^n_m} = [4n + 9, 8n + 12] \). Since by Corollary 3.2, the magic set \( \tau_{C^n_m} \) contains the interval \([4n + 9, 8n + 12] \), we get the result. \( \square \)
Theorem 3.2. The graph $C^n_5$ is perfect edge-magic for all $n \in \mathbb{N} \setminus \{1\}$.

Proof: By Lemma 3.2, the magic interval of $C^n_5$ is $J_{C^n_5} = [6n + 14, 12n + 19]$. Since by Corollary 3.2, the magic set $\tau_{C^n_5}$ contains the interval $[6n + 14, 12n + 19]$, we get the result. \qed

Using Lemma 3.2 and Corollary 3.2, a computer check shows other families of perfect edge-magic graphs.

Theorem 3.3. The graph $C^n_m$ is perfect edge-magic when (i) $m = 7$ and $1 \leq n \leq 3$, (ii) $m = 9$ and $n = 1$, and (iii) $m = 11$ and $n = 1$.

4 Even harmonious labelings from super edge-magic labelings

A $(p, q)$-graph $G$ with $p \leq q$ is called harmonious [10] if it is possible to label its vertices with distinct integers (mod $q$) in such a way that the edge sums are also distinct (mod $q$). When $G$ is a tree, exactly one label may be used on two vertices. Variations of this concept have appeared recently in the literature. A $(p, q)$-graph $G$ is said to be odd harmonious [13] if there exists an injection $f : V(G) \rightarrow \{1, \ldots, 2q - 1\}$ such that the induced mapping $f^*(uv) = (f(u) + f(v)) \mod (2q)$ is a bijection from $E(G)$ onto the set $\{1, 3, 5, \ldots, 2q - 1\}$. Then $f$ is called an odd harmonious labeling of $G$. Similarly, Sarasija and Binthiya introduced in [17] what they called an even harmonious graph. Let $G$ be a $(p, q)$-graph. An injective function $f : V(G) \rightarrow \{1, \ldots, 2q - 1\}$ such that the induced function $f^* : E(G) \rightarrow \{0, 2, 4, \ldots, 2(q - 1)\}$ defined by $f^*(uv) = (f(u) + f(v)) \mod (2q)$ is bijective. Then $f$ is called an even harmonious labeling of $G$, and $G$ is called an even harmonious graph.

Super edge-magic labelings are known to be a powerful link among different types of labelings. In [7] the digraph product $\otimes_h$ was introduced, and this product together with super edge-magic labelings, has been used in order to establish further relations among labelings, see for instance [7, 11, 14, 16]. In this section we establish a new relationship among super edge-magic labelings and even harmonious labelings.

Lemma 4.1. Let $G$ be a $(p, q)$-graph with $q \geq p - 1$. If $G$ is super edge-magic then $G$ is even harmonious.

Proof: Let $f$ be any super edge-magic labeling of a $(p, q)$-graph $G$, with $q \geq p - 1$. Consider the labeling $e^*(f) : V(G) \rightarrow \{1, \ldots, 2q\}$ defined by the rule $e^*(f)(u) = 2f(u) - 2$, for all $u \in V(G)$. Then, using a similar proof as in Lemma 1.2, it is clear that $e^*(f)$ is an even harmonious labeling of $G$. \qed

From this result we get that all super edge-magic graphs are even harmonious. In particular, we can obtain some of the results introduced in [17].

Corollary 4.1. [17] The following graphs are even harmonious: (i) the path $P_n$, with $n \geq 2$, (ii) the star $K_{1,n}$, with $n \geq 1$, and (iii) the cycle of odd order $C_n$, with $n \geq 3$. 
5 Conclusions and remarks

In this paper we have proved that the family $C_m \circ K_n$, where $m$ is a power of a prime greater than 2, is perfect edge-magic for all $n \in \mathbb{N} \setminus \{1\}$. In fact, it is the first non-trivial infinite family known to be perfect edge-magic. We also have proved that $C_3^n$ and $C_5^n$ are perfect edge-magic and that the magic set of the family $C_m^n$ contains a big interval. The problem of finding families of graphs that are perfect edge-magic seems to be a hard one, and we want to encourage other researchers to continue this line of research. Next we want to introduce some open problems in this direction.

Open question 5.1. Characterize the set $\Sigma_n$ defined by

$$\Sigma_m = \{m \in \mathbb{N} : C_{2m+1} \circ K_n \text{ is perfect edge-magic for all } n \in \mathbb{N}\}.$$

About open question 5.1, we have made some progress, continuing a work started in [15]. However, nothing is known about the next question.

Open question 5.2. Characterize the set $\Upsilon_n$ defined by $\Upsilon_m = \{m \in \mathbb{N} \setminus \{1\} : C_{2m} \circ K_n \text{ is perfect edge-magic for all } n \in \mathbb{N}\}$.

It is well known that stars are not perfect edge-magic. In fact the set $\tau_{K_1,n}$ contains only 3 elements for every $n \in \mathbb{N} \setminus \{1\}$ (see [6, 18]). This fact motivates the following two questions.

Open question 5.3. Find examples of infinite families of graphs which are edge-magic but not perfect edge-magic.

Open question 5.4. Characterize the set of caterpillars which are not perfect edge-magic. In particular, characterize the set of paths which are perfect edge-magic and characterize the set of caterpillars with the same number of leaves attached at each vertex of the spine which are perfect edge-magic.

The concept of perfect edge-magic graphs was motivated by the concept of perfect super edge-magic graphs introduced in [15]. Furthermore, the concept of perfect super edge-magic graphs was motivated by the following conjecture introduced in [9] by Godbold and Slater, that “as far as we know” remains unsolved up to the present.

Conjecture 5.1. [9]. For $n = 2t + 1 \geq 7$ and $5t + 4 \leq j \leq 7t + 5$ there is an edge-magic labeling of $C_n$, with valence $k = j$. For $n = 2t \geq 4$ and $5t + 2 \leq j \leq 7t + 1$ there is an edge-magic labeling of $C_n$, with valence $k = j$.

In this paper we want to renew the interest for this question, and encourage researches to work towards a final solution of the question. For any reader interested in it, the book of Wallis [18] constitutes an excellent source of information about this question. For related problems on graph labelings we direct the reader to [2] and [8].

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