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# Certain generalizations of Eneström-Kakeya theorem

by
Vinay Kumar Jain

**Abstract**

We extend the classical Eneström-Kakeya Theorem to some classes of lacunary polynomials with complex coefficients.

**Key Words**: Lacunary polynomials, increasing moduli of coefficents, increasing real parts of coefficents.
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## 1 Introduction

The following result ( [1], [3], [4, p. 136, Th. (30,3)]) is well known in the theory of the distribution of zeros of polynomials.

**Theorem A.** *(Eneström-Kakeya theorem). If* $f(z) = \sum_{k=0}^{n} a_k z^k$ *is a polynomial of degree* $n$ *such that*

$$a_n \geq a_{n-1} \geq a_{n-2} \geq \ldots \geq a_1 \geq a_0 > 0,$$

*then* $f(z)$ *will have all its zeros in* $|z| \leq 1$.

*In [2], Govil and Rahman considered polynomials, with complex coefficients, (i) with increasing moduli of coefficients and (ii) with increasing real parts of coefficients, (real parts of the coefficients, being assumed to be non-negative), and obtained the following generalizations of Eneström-Kakeya theorem.*

**Theorem B.** *Let* $f(z) = \sum_{k=0}^{n} a_k z^k$ *be a polynomial of degree* $n$ *with complex coefficients such that*

$$|\arg a_k - \beta| \leq \alpha \leq \pi/2, k = 0, 1, 2, \ldots, n,$$

*for some real* $\beta$ *and*

$$|a_n| \geq |a_{n-1}| \geq |a_{n-2}| \geq \ldots \geq |a_0|.$$

*Then $f(z)$ has all its zeros on or inside the circle*

$$|z| = \cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|}\sum_{k=0}^{n-1}|a_k|.$$

**Theorem C.** *Let $f(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial of degree $n$. If*

$$Re\ a_k = \alpha_k\ , Im\ a_k = \ \beta_k,\ k = 0, 1, 2, \ldots, n$$

*and*

$$\alpha_n \geq \alpha_{n-1} \geq \ldots \geq \alpha_1 \geq \alpha_0 \geq 0,\ \alpha_n > 0$$

*then $f(z)$ has all its zeros in or on the circle*

$$|z| = 1 + \left(\frac{2}{\alpha_n}\right)\left(\sum_{k=0}^{n}|\beta_k|\right).$$

In this paper we consider some classes of lacunary polynomials and obtain two results that are analogous to Theorem B and Theorem C. More precisely, we will prove the following two results.

**Theorem 1.** *Let $n, n_0, n_1, n_2, \ldots, n_k$ be non-negative integers such that*

$$0 = n_0 < n_1 < n_2 < \ldots < n_k = n \tag{1}$$

*and let*

$$p(z) = a_{n_0}z^{n_0} + a_{n_1}z^{n_1} + a_{n_2}z^{n_2} + \ldots + a_{n_k}z^{n_k}$$

*be a polynomial of degree $n$ such that*
*(i) $0 < |a_{n_0}| \leq |a_{n_1}| \leq |a_{n_2}| \leq \ldots \leq |a_{n_k}|$,* (2)
*(ii) for certain real numbers $\alpha$ and $\beta$*

$$|arg\ a_{n_j} - \beta| \leq \alpha \leq \pi/2, j = 0, 1, 2, \ldots, k \tag{3}$$

*and*
*(iii) no two adjacent coefficients $a_{n_j}$'s ($j = 0, 1, \ldots, k$) are equal i.e.*

$$a_{n_0} \neq a_{n_1}, a_{n_1} \neq a_{n_2}, \ldots, a_{n_{k-1}} \neq a_{n_k}. \tag{4}$$

*Further let*

$$M_{n_j} = \max_{|z|=1}\left|\frac{a_{n_j}z^{n_j} - a_{n_{j-1}}z^{n_{j-1}+1}}{a_{n_j} - a_{n_{j-1}}}\right|, j = 1, 2, \ldots, k, \tag{5}$$

*with*

$$M = \max_{1\leq j\leq k} M_{n_j}. \tag{6}$$

*Then all the zeros of $p(z)$ lie in*

$$|z| \leq M\left\{\cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|}\sum_{j=0}^{k-1}|a_{n_j}|\right\}.$$

**Theorem 2.** *Let* $n, n_0, n_1, n_2, \ldots, n_k$ *be non-negative integers such that*

$$0 = n_0 < n_1 < n_2 < \ldots < n_k = n$$

*and let*

$$p(z) = a_{n_0} z^{n_0} + a_{n_1} z^{n_1} + a_{n_2} z^{n_2} + \ldots + a_{n_k} z^{n_k}$$

*be a polynomial of degree* $n$ *such that*

$$(i)\ a_{n_j} = \alpha_{n_j} + i\beta_{n_j}, j = 0, 1, \ldots, k, \tag{7}$$

$$(ii)\ 0 \leq \alpha_{n_0} \leq \alpha_{n_1} \leq \alpha_{n_2} \leq \ldots \leq \alpha_{n_k}, \alpha_n > 0 \tag{8}$$

*and*
*(iii) no two adjacent coefficients* $a_{n_j}$*'s*$(j = 0, 1, \ldots, k)$ *are equal i.e.*

$$a_{n_0} \neq a_{n_1}, a_{n_1} \neq a_{n_2}, \ldots, a_{n_{k-1}} \neq a_{n_k}.$$

*Further let*

$$M_{n_j}, j = 1, 2, \ldots, k$$

*and* $M$, *be as in Theorem 1. Then all the zeros of* $p(z)$ *lie in*

$$|z| \leq M\left(1 + \frac{2}{\alpha_n}\sum_{j=0}^{k}|\beta_{n_j}|\right).$$

## 2 Required lemmas

For the proofs of the theorems we require the following lemmas.

*Lemma 1.* If $p(z)$ is a polynomial of degree $n$ then for $R \geq 1$

$$\max_{|z|=R} |p(z)| \leq R^n \left(\max_{|z|=1} |p(z)|\right),$$

with equality only for $p(z) = \lambda z^n$.

*Proof of Lemma 1.* It is a simple consequence of maximum modulus principle.

*Lemma 2.* If $a_j$ and $a_{j-1}$ are two complex numbers with

$$\begin{aligned} |Arg\ a_j - \beta| &\leq \alpha \leq \pi/2, \\ |Arg\ a_{j-1} - \beta| &\leq \alpha \leq \pi/2, \end{aligned}$$

for certain real $\beta$ and $\alpha$ then

$$|a_j - a_{j-1}|^2 \leq (|a_j| - |a_{j-1}|)^2 \cos^2\alpha + (|a_j| + |a_{j-1}|)^2 \sin^2\alpha.$$

This lemma is due to Govil and Rahman [2, Proof of Theorem 2].

*Lemma 3.* Under the same hypothesis as in Lemma 2

$$|a_j - a_{j-1}| \leq ||a_j| - |a_{j-1}|| \cos\alpha + (|a_j| + |a_{j-1}|) \sin\alpha.$$

*Proof of Lemma 3.* It follows easily from Lemma 2.

## 3 Proofs of the theorems

*Proof of Theorem 1.* We have

$$\begin{aligned}(1-z)p(z) &= -a_n z^{n+1} + (a_{n_k} z^{n_k} - a_{n_{k-1}} z^{n_{k-1}+1}) + \\ &\quad (a_{n_{k-1}} z^{n_{k-1}} - a_{n_{k-2}} z^{n_{k-2}+1}) + \ldots + \\ &\quad (a_{n_1} z^{n_1} - a_{n_0} z^{n_0+1}) + a_{n_0} z^{n_0}, && (9)\\ &= -a_n z^{n+1} + p_1(z), \text{ say.} && (10)\end{aligned}$$

Now for $|z| = R(> 1)$

$$\begin{aligned}|p_1(z)| &\leq \sum_{j=1}^{k} |a_{n_j} - a_{n_{j-1}}|\left|\frac{a_{n_j} z^{n_j} - a_{n_{j-1}} z^{n_{j-1}+1}}{a_{n_j} - a_{n_{j-1}}}\right| + |a_{n_0}|R^{n_0},\\ &\qquad \text{(by (9) and (10))},\\ &\leq \sum_{j=1}^{k} |a_{n_j} - a_{n_{j-1}}| M_{n_j} R^{n_j} + |a_{n_0}|R^{n_0},\\ &\qquad \text{(by (5) and Lemma 1)}. && (11)\end{aligned}$$

But

$$M_{n_j} \geq 1, j = 1, 2, \ldots, k, \text{ (by (5))} \tag{12}$$

and therefore

$$M \geq 1, \text{ (by (6))},$$

which, by (11), (6) and (1), implies that for $|z| = R(> 1)$

$$\begin{aligned}|p_1(z)| &\leq MR^n \left\{ \sum_{j=1}^{k} |a_{n_j} - a_{n_{j-1}}| + |a_{n_0}| \right\}, && (13)\\ &\leq MR^n \Big\{ \sum_{j=1}^{k} (|a_{n_j}| - |a_{n_{j-1}}|) \cos\alpha + \sum_{j=1}^{k} (|a_{n_{j-1}}| + |a_{n_j}|) \sin\alpha + \\ &\quad |a_{n_0}|(\cos\alpha + \sin\alpha)\Big\}, \text{ (by (2), (3) and Lemma 3)},\\ &= MR^n \Big\{ |a_n|(\cos\alpha + \sin\alpha) + 2\sin\alpha (\sum_{j=0}^{k-1} |a_{n_j}|) \Big\}. && (14)\end{aligned}$$

Further, by (10), we have for $|z| = R(> 1)$

$$\begin{aligned} |(1-z)p(z)| &\geq |a_n|R^{n+1} - |p_1(z)|, \qquad (15)\\ &\geq |a_n|R^{n+1} - MR^n\Big\{|a_n|(\cos\alpha + \sin\alpha) + 2\sin\alpha\Big(\sum_{j=0}^{k-1}|a_{n_j}|\Big)\Big\}, \text{(by (14))},\\ &> 0, \end{aligned}$$

for

$$R > M\left\{\cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|}\left(\sum_{j=0}^{k-1}|a_{n_j}|\right)\right\},$$

and Theorem 1 follows.

***Remark* 1.** *In some cases, even if (iii) (i.e. relation (4)) is not satisfied, we can proceed, as in the proof, with a minor modification, to get (13), thereby allowing us to complete the proof of Theorem 1. For instance, let*

$$a_{n_0} = a_{n_1}, a_{n_1} \neq a_{n_2}, a_{n_2} \neq a_{n_3}, a_{n_3} = a_{n_4}, a_{n_4} = a_{n_5},$$

$$a_{n_5} \neq a_{n_6}, a_{n_6} \neq a_{n_7}, \ldots, a_{n_{k-2}} \neq a_{n_{k-1}}, a_{n_{k-1}} = a_{n_k}.$$

*For the possibility under consideration, we will not consider $M_{n_3}, M_{n_4}, M_{n_{k-1}}$ (involved in (5)) and will have*

$$\begin{aligned} M_{n_1} &= \max_{|z|=1}\left|\frac{a_{n_0} - a_{n_0}z^{n_0+1} + a_{n_1}z^{n_1}}{a_{n_0} - a_{n_0} + a_{n_1}}\right|, \textit{ (instead of the expression in (5))}, \qquad (16)\\ &\geq 1, \qquad (17)\\ M_{n_2}, &\quad \textit{as in (5)}, \qquad (18)\\ M_{n_5} &= \max_{|z|=1}\left|\frac{a_{n_5}z^{n_5} - a_{n_4}z^{n_4+1} + a_{n_4}z^{n_4} - a_{n_3}z^{n_3+1} + a_{n_3}z^{n_3} - a_{n_2}z^{n_2+1}}{a_{n_5} - a_{n_4} + a_{n_4} - a_{n_3} + a_{n_3} - a_{n_2}}\right|, \qquad (19)\\ &\quad \textit{(instead of the expression in (5))},\\ &\geq 1, \qquad (20) \end{aligned}$$

$$\left.\begin{array}{l} M_{n_6},\\ M_{n_7},\\ .\\ .\\ .\\ M_{n_{k-2}}, \end{array}\right\}, (\textit{as in } (5)), \qquad (21)$$

$$\begin{aligned}
M_{n_k} &= \max_{|z|=1} |\frac{a_{n_k} z^{n_k} - a_{n_{k-1}} z^{n_{k-1}+1} + a_{n_{k-1}} z^{n_{k-1}} - a_{n_{k-2}} z^{n_{k-2}+1}}{a_{n_k} - a_{n_{k-1}} + a_{n_{k-1}} - a_{n_{k-2}}}|, \qquad (22)\\
&\quad \textit{(instead of the expression in (5))},\\
&\geq 1, \qquad (23)\\
M &= \max(M_{n_1}, M_{n_2}, M_{n_5}, M_{n_6}, \ldots, M_{n_{k-2}}, M_{n_k}), \qquad (24)\\
&\quad \textit{(instead of the expression in (6))},\\
&\geq 1,\ (\textit{by } (17), (20), (23) \textit{ and } (12)),
\end{aligned}$$

$$\begin{aligned}
(1-z)p(z) &= -a_n z^{n+1} + (a_{n_k} z^{n_k} - a_{n_{k-1}} z^{n_{k-1}+1} + a_{n_{k-1}} z^{n_{k-1}} - a_{n_{k-2}} z^{n_{k-2}+1})\\
&\quad + (a_{n_{k-2}} z^{n_{k-2}} - a_{n_{k-3}} z^{n_{k-3}+1}) + \ldots + (a_{n_6} z^{n_6} - a_{n_5} z^{n_5+1})\\
&\quad + (a_{n_5} z^{n_5} - a_{n_4} z^{n_4+1} + a_{n_4} z^{n_4} - a_{n_3} z^{n_3+1} + a_{n_3} z^{n_3} - a_{n_2} z^{n_2+1})\\
&\quad + (a_{n_2} z^{n_2} - a_{n_1} z^{n_1+1}) + (a_{n_1} z^{n_1} - a_{n_0} z^{n_0+1} + a_{n_0}),\\
&= -a_n z^{n+1} + p_1(z),
\end{aligned}$$

*thereby helping us to write, (as in proof of Theorem 1), for* $|z| = R(>1)$

$$\begin{aligned}
|p_1(z)| &\leq |a_{n_k} - a_{n_{k-1}} + a_{n_{k-1}} - a_{n_{k-2}}| M_{n_k} R^{n_k} +\\
&\quad \sum_{j=6}^{k-2} |a_{n_j} - a_{n_{j-1}}| M_{n_j} R^{n_j} +\\
&\quad |a_{n_5} - a_{n_4} + a_{n_4} - a_{n_3} + a_{n_3} - a_{n_2}| M_{n_5} R^{n_5} + |a_{n_2} - a_{n_1}| M_{n_2} R^{n_2}\\
&\quad + |a_{n_1} - a_{n_0} + a_{n_0}| M_{n_1} R^{n_1},\\
&\qquad (\textit{by } (16), (18), (19), (21), (22), (5) \textit{ and Lemma 1}),\\
&\leq MR^n \Big\{ |a_{n_k} - a_{n_{k-1}} + a_{n_{k-1}} - a_{n_{k-2}}| + \sum_{j=6}^{k-2} |a_{n_j} - a_{n_{j-1}}| +\\
&\quad |a_{n_5} - a_{n_4} + a_{n_4} - a_{n_3} + a_{n_3} - a_{n_2}| + |a_{n_2} - a_{n_1}| +\\
&\quad |a_{n_1} - a_{n_0} + a_{n_0}| \Big\}, (\textit{by } (24)),\\
&\leq MR^n \left\{ \sum_{j=1}^{k} |a_{n_j} - a_{n_{j-1}}| + |a_{n_0}| \right\},
\end{aligned}$$

*i.e. (13).*

*Proof of Theorem 2.* As in the proof of Theorem 1, we have for $|z| = R(> 1)$

$$\begin{aligned}
|p_1(z)| &\le MR^n \left\{ \sum_{j=1}^{k} |a_{n_j} - a_{n_{j-1}}| + |a_{n_0}| \right\}, \text{ (by (13))}, \\
&\le MR^n \Big\{ \sum_{j=1}^{k} (\alpha_{n_j} - \alpha_{n_{j-1}}) + \sum_{j=1}^{k} (|\beta_{n_{j-1}}| + |\beta_{n_j}|) + (\alpha_{n_0} + |\beta_{n_0}|) \Big\}, \\
&\quad \text{(by (7) and (8))}, \\
&\le MR^n (\alpha_n + 2\sum_{j=0}^{k} |\beta_{n_j}|). \qquad (25)
\end{aligned}$$

Further, as in the proof of Theorem 1, we have for $|z| = R(> 1)$

$$\begin{aligned}
|(1-z)p(z)| &\ge |a_n| R^{n+1} - |p_1(z)|, \text{ (by (15))}, \\
&\ge \alpha_n R^{n+1} - MR^n \left\{ \alpha_n + 2\sum_{j=0}^{k} |\beta_{n_j}| \right\}, \\
&\quad \text{(by (7), (8) and (25))}, \\
&> 0,
\end{aligned}$$

for

$$R > M \left( 1 + \frac{2}{\alpha_n} \sum_{j=0}^{k} |\beta_{n_j}| \right),$$

and Theorem 2 follows.

***Remark* 2.** *Remark 1, associated with Theorem 1, is true for Theorem 2 also.*

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Mathematics Department, Indian Institute of Technology, Kharagpur - 721302, India
E-mail: vkj@maths.iitkgp.ernet.in