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On a p-Hamiltonian system

by
Ling Ding, Lin Li and Chun Li

Abstract

The aim of this article is to establish the existence of at least three and infinitely many periodic solutions for p-Hamiltonian systems depending of a real parameter, moreover write is based on some very recent critical points theorems.

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1 Introduction and main results

Consider the existence of periodic solutions for the p-Hamiltonian systems

$$\begin{cases} -(|u'|^{p-2}u')' + A(t)|u|^{p-2}u = \lambda \nabla F(t,u), \\ u(T) - u(0) = u'(T) - u'(0) = 0, \end{cases} \tag{$\mathcal{H}$}$$

where $\lambda \in]0,+\infty[$, $p > 1$, $T > 0$, $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is a function satisfies the following assumption:
$F(t,u)$ is measurable with respect to t, for all $u \in \mathbb{R}^N$, continuously differentiable in u, for a.e. $t \in [0,T]$, and there exist $a \in C(\mathbb{R}^+;\mathbb{R}^+)$ and $b \in L^1([0,T];\mathbb{R}^+)$ such that

$$|F(t,u)| \le a(|u|)b(t), \qquad |\nabla F(t,u)| \le a(|u|)b(t)$$

for all $u \in \mathbb{R}^N$ and a.e. $t \in [0,T]$. $A = (a_{ij}(t))_{N\times N}$ is a symmetric matrix valued function with $a_{ij} \in L^\infty[0,T]$ and there exists a positive constant $\underline{\lambda}$ such that $(A(t)|x|^{p-2}x, x) \ge \underline{\lambda}|x|^p$ for all $x \in \mathbb{R}^N$ and $t \in [0,T]$, that is, $A(t)$ is positive definite for all $t \in [0,T]$.

Here and in the sequel, the Sobolev space $W_T^{1,p}$ is defined by

$$W_T^{1,p} = \left\{ u : [0,T] \to \mathbb{R}^N \;\middle|\; \begin{array}{l} u \text{ is absolutely continuous,} \\ u(0) = u(T) \text{ and } u' \in L^p(0,T;\mathbb{R}^N) \end{array} \right\},$$

and endowed with the norm

$$\|u\|_A = \left(\int_0^T |u'(t)|^p \, \mathrm{d}t + \int_0^T (A(t)|u(t)|^{p-2}u(t), u(t)) \, \mathrm{d}t \right)^{1/p}.$$

Observe that

$$\begin{aligned} (A(t)|x|^{p-2}x, x) &= |x|^{p-2} \sum_{i,j=1}^{N} a_{ij}(t) x_i x_j \\ &\le |x|^{p-2} \sum_{i,j=1}^{N} |a_{ij}(t)||x_i||x_j| \\ &\le \left(\sum_{i,j=1}^{N} \|a_{ij}\|_\infty \right) |x|^p, \end{aligned}$$

then there exists a constant $\overline{\lambda} \le \sum\limits_{i,j=1}^{N} \|a_{ij}\|_\infty$ such that $(A(t)|x|^{p-2}x, x) \le \overline{\lambda}|x|^p$ for all $x \in \mathbb{R}^N$. So, we have

$$\min\{1, \underline{\lambda}\}\|u\|^p \le \|u\|_A^p \le \max\{1, \overline{\lambda}\}\|u\|^p, \tag{1.1}$$

where

$$\|u\| = \left(\int_0^T |u(t)|^p \, \mathrm{d}t + \int_0^T |u'(t)|^p \, \mathrm{d}t \right)^{1/p}.$$

Let

$$k_0 = \sup_{u \in W_T^{1,p} \setminus \{0\}} \frac{\|u\|_\infty}{\|u\|_A}, \qquad \|u\|_\infty = \sup_{t \in [0,T]} |u(t)|, \tag{1.2}$$

where $|\cdot|$ is the usual norm in $\mathbb{R}^N$. Since $W_T^{1,p} \hookrightarrow C^0$ is compact, one has $k_0 < +\infty$ and for each $u \in W_T^{1,p}$, there exists $\xi \in [0, T]$ such that $|u(\xi)| = \min\limits_{t \in [0,T]} |u(t)|$. Hence, by Hölder's inequality, one has

$$|u(t)| \le \sqrt[q]{2} \max\{T^{\frac{1}{q}}, T^{-\frac{1}{p}}\}\|u\| \tag{1.3}$$

for each $t \in [0, T]$ and $q = \frac{p}{p-1}$(see [18] for more details). So, by (1.2) and the above expression, we have

$$\|u\|_\infty \le \sqrt[q]{2} \max\{T^{\frac{1}{q}}, T^{-\frac{1}{p}}\}\|u\| \le \sqrt[q]{2} \max\{T^{\frac{1}{q}}, T^{-\frac{1}{p}}\} \left(\min\{1, \underline{\lambda}\}\right)^{-\frac{1}{p}} \|u\|_A,$$

then from this and (1.1) it follows that

$$k_0 \le k := \sqrt[q]{2} \max\{T^{\frac{1}{q}}, T^{-\frac{1}{p}}\} \left(\min\{1, \underline{\lambda}\}\right)^{-\frac{1}{p}}. \tag{1.4}$$

As usual, a weak solution of problem $(\mathcal{H})$ is any $u \in W_T^{1,p}$ such that

$$\int_0^T ((|u'(t)|^{p-2}u'(t), v'(t)) + (A(t)|u(t)|^{p-2}u(t), v(t)))\,\mathrm{d}t = \lambda \int_0^T (\nabla F(t, u(t)), v(t))\,\mathrm{d}t \qquad (1.5)$$

for all $v \in W_T^{1,p}$.

Contrast to Hamiltonian systems, for the general case $p > 1$, the study on the existence and multiplicity of periodic solutions are new, see [19, 20, 21, 22, 25, 26, 27, 28, 30, 29]. Lü et al. [20] deals with the existence of infinitely many periodic solutions for the ordinary p-Laplacian system

$$\begin{cases} -(|u'|^{p-2}u')' = \nabla F(t, u), \\ u(T) - u(0) = u'(T) - u'(0) = 0, \end{cases} \qquad (1.6)$$

here $F\colon [0, T] \times \mathbb{R}^N \to \mathbb{R}$ is a function continuously differentiable with respect to the second variable and satisfying some usual growth conditions. They states that if

$$\limsup_{R\to+\infty} \inf_{|a|=R} \int_0^T F(s, a)\,\mathrm{d}s = +\infty$$

and

$$\liminf_{r\to+\infty} \sup_{|b|=r} \int_0^T F(s, b)\,\mathrm{d}s = -\infty,$$

then

(i) there exists a sequence (u_n) of periodic solutions of (1.6) such that u_n is a critical point of φ and $\lim_{n\to\infty} \varphi(u_n) = +\infty$,

(ii) there exists a sequence (u_n^*) of periodic solutions of (1.6) such that u_n^* is a local minimum point of φ and $\lim_{n\to\infty} \varphi(u_n^*) = -\infty$,

where $\varphi(u) = \frac{1}{p}\int_0^T |u'(t)|^p\,\mathrm{d}t - \int_0^T F(t, u(t))\,\mathrm{d}t$.

The same variational methods as were used there, which are based on the seminal paper of Bonanno and Marano [11], have been applied to obtain multiple solutions (see, for instance, [7, 14, 4, 10, 6, 9, 16, 17, 24]). In particular, Li et al. [18] had proved the existence of at least three solutions for problem $(\mathcal{H})$. The technical approach is based on the three critical points theorem of Averna and Bonanno [1]. Their theorem under novel assumptions ensures the existence of an open interval $\Lambda \subseteq [0, +\infty)$ such that, for each $\lambda \in \Lambda$, problem $(\mathcal{H})$ admits at least three weak solutions. Very recently in [3], presenting a version of the infinitely many critical points theorem of B. Ricceri (see [23, Theorem 2.5]), the existence of an unbounded sequence of weak solutions for a Sturm-Liouville problem, having discontinuous nonlinearities, has been established. In a such approach, an appropriate oscillating behavior of the nonlinear term either at infinity or at zero is required. This type of methodology has been used then in several works in order to obtain existence results for different kinds of problems (see, for instance, [7, 5, 8, 12, 15, 10, 13]). In this paper, we will give out more piecewise results than that in [18]. Now, we state our main results.

Theorem 1. *Assume that there exist a constant vector* $d = (d_1, \cdots, d_N) \in \mathbb{R}^N$*, a positive constant* c *with* $c < k|d|(\underline{\lambda}T)^{1/p}$*, such that*

(H1) $\frac{\int_0^T \max_{x\in B} F(t,x)\,\mathrm{d}t}{c^p} < \frac{\int_0^T F(t,d)\,\mathrm{d}t}{\overline{\lambda}T(k|d|)^p}$, *where* $B = \{x \in \mathbb{R}^N \mid 0 \le |x| \le c\}$;

(H2) $\limsup_{|x|\to+\infty} \frac{F(t,x)}{|x|^p} < \frac{\int_0^T \max_{x\in B} F(t,x)\,\mathrm{d}t}{c^p}$ *for a.e.* $t \in [0,T]$ *and all* $x \in \mathbb{R}^N$;

(H3) $F(t,0) = 0$ *for all* $t \in [0,T]$.

Then, there exist a non-empty open interval

$$\Lambda := \left]\frac{\overline{\lambda}T(k|d|)^p}{pk^p\int_0^T F(t,d)\,\mathrm{d}t}, \frac{c^p}{pk^p\int_0^T \max_{x\in B} F(t,x)\,\mathrm{d}t}\right[$$

with the following property: for each $\lambda \in \Lambda$ *problem* $(\mathcal{H})$ *has at least three solutions.*

Remark 1. Under the more weak assumptions as for Theorem 2 of [18], Theorem 1 ensures a more precise conclusion. In fact, our condition (H2) is more weak than condition (i_3) in Theorem 2 of [18]. For example, let $F(s) = \frac{s^p}{\ln(2+s^2)}$. Clearly, function F satisfies our condition (H2) but doesn't satisfies (i_3) in Theorem 2 of [18]. Furthermore, Theorem 1 give out a larger interval Λ than Theorem 2 of [18].

Now, assume F is a C^1 function on $\mathbb{R}^N$, with $F(0) = 0$, we can get the following result.

Corollary 1. *Assume that there exist a positive constant* c *and a vector* $d \in \mathbb{R}^N$ *with* $c < |d|$*, such that*

(J1) $\frac{\max_{|x|\le c} F(x)}{c^p} < \frac{1}{\overline{\lambda}k^p}\frac{F(d)}{|d|^p}$;

(J2) $\limsup_{|x|\to\infty} \frac{F(x)}{|x|^p} < \frac{\max_{|x|\le c} F(x)}{c^p}$.

Then, for every function $b \in L^1([0,T]) \setminus \{0\}$ *that is a.e. nonnegative and for every* $\lambda \in \left]\frac{\overline{\lambda}k^p}{p\|b\|_1}\frac{|d|^p}{F(d)}, \frac{1}{p\|b\|_1 k^p}\frac{c^p}{\max_{|x|\le c} F(x)}\right[$*, problem*

$$\begin{cases} -(|u'|^{p-2}u')' + A(t)|u|^{p-2}u = \lambda b(t)\nabla F(u), \\ u(T) - u(0) = u'(T) - u'(0) = 0, \end{cases} \tag{$\mathcal{H}_a$}$$

admits at least three solutions.

Remark 2. Corollary 1 improves Theorem 3 of [18] since conditions (J1) in our results are more general than conditions (j_1). We also obtain a larger interval of parameter than the interval insured by Theorem 3 in [18].

Now, we want to get infinitely many periodic solutions for the perturbed Hamiltonian system. Put

$$\mathcal{A} := \liminf_{|x|\to+\infty} \frac{\int_0^T \max_{|\xi|\le x} F(t,\xi)\,\mathrm{d}t}{|x|^p},$$

$$\mathcal{B} := \limsup_{|x|\to+\infty} \frac{\int_0^T F(t,x)\,\mathrm{d}t}{|x|^p},$$

$$\lambda_1 := \frac{\overline{\lambda} T}{pk^p \mathcal{B}},$$

$$\lambda_2 := \frac{1}{pk^p \mathcal{A}}.$$

Theorem 2. *Assume that,*

(H4) *F is non-negative in $[0,T] \times \mathbb{R}^N$;*

(H5) *$\mathcal{A} < \overline{\lambda} T \mathcal{B}$.*

Then, for every $\lambda \in \Lambda :=]\lambda_1, \lambda_2[$, the problem $(\mathcal{H})$ admits an unbounded sequence of periodic solutions which is unbounded in $W_T^{1,p}$.

Remark 3. If

$$\liminf_{|x|\to+\infty} \frac{\int_0^T \max_{|\xi|\le x} F(t,\xi)\,\mathrm{d}t}{|x|^p} = 0 \text{ and } \limsup_{|x|\to+\infty} \frac{\int_0^T F(t,x)\,\mathrm{d}t}{|x|^p} = +\infty,$$

clearly, hypothesis (H5) is verified and Theorem 2 guarantees the existence of infinitely many weak solutions for problem $(\mathcal{H})$, for every $\lambda \in]0,+\infty[$.

Replacing the conditions at infinity of the potential by a similar at zero, the same result holds and, in addition, the sequence of pairwise distinct solutions uniformly converges to zero. Precisely, set

$$\lambda_1^* := \frac{\overline{\lambda} T}{pk^p \limsup_{|x|\to 0} \frac{\int_0^T F(t,x)\,\mathrm{d}t}{|x|^p}}, \quad \lambda_2^* := \frac{1}{pk^p \liminf_{|x|\to 0} \frac{\int_0^T \max_{|\xi|\le x} F(t,\xi)\,\mathrm{d}t}{|x|^p}}.$$

Arguing as in the proof of Theorem 2, we obtain the following result.

Theorem 3. *Assume that F is non-negative in $[0,T]\times\mathbb{R}^N$ and satisfies the following conditions,*

$$\liminf_{|x|\to 0} \frac{\int_0^T \max_{|\xi|\le x} F(t,\xi)\,\mathrm{d}t}{|x|^p} < \overline{\lambda} T \limsup_{|x|\to 0} \frac{\int_0^T F(t,x)\,\mathrm{d}t}{|x|^p}. \tag{1.7}$$

Then, for every $\lambda \in]\lambda_1^*, \lambda_2^*[$, the problem $(\mathcal{H})$ admits an unbounded sequence of non-zero weak solutions which strongly converges to 0 in $W_T^{1,p}$.

2 Preliminary

For the reader's convenience, here we recall the three critical points theorem (Theorem 3.6 in [11]) and infinitely many critical points theorem (Theorem 2.1 in [3]).

Theorem 4. *[11, Theorem 3.6] Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, and $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

$$\Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r > 0$ and $\bar{x} \in X$, with $r < \Phi(\bar{x})$, such that:

(a_1) $\frac{\sup_{\Phi(x)\le r} \Psi(x)}{r} < \frac{\Psi(\bar{x})}{\Phi(\bar{x})}$,

(a_2) *for each* $\lambda \in \Lambda_r := \left] \frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x)\le r} \Psi(x)} \right[$ *the functional $\Phi - \lambda\Psi$ is coercive.*

Then, for each $\lambda \in \Lambda_r$, the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

Theorem 5. *[3, Theorem 2.1] Let X be a reflexive real Banach space, let Φ, $\Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is strongly continuous, sequentially weakly lower semicontinuous and coercive and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, put*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\left(\sup_{v\in\Phi^{-1}(]-\infty,r[)} \Psi(v)\right) - \Psi(u)}{r - \Phi(u)},$$

and

$$\gamma := \liminf_{r\to+\infty} \varphi(r), \qquad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Then, one has

(a) *for every $r > \inf_X \Phi$ and every $\lambda \in \left]0, \frac{1}{\varphi(r)}\right[$, the restriction of the functional $I_\lambda = \Phi - \lambda\Psi$ to $\Phi^{-1}(] - \infty, r[)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.*

(b) *If $\gamma < +\infty$, then, for each $\lambda \in \left]0, \frac{1}{\gamma}\right[$, the following alternative holds:*

either
(b_1) *I_λ possesses a global minimum,,*
or
(b_2) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that* $\lim_{n\to+\infty} \Phi(u_n) = +\infty$.

(c) *if $\delta < +\infty$, for each $\lambda \in \left]0, \frac{1}{\delta}\right[$, the following alternative holds:*

either
(c_1) *there is a global minimum of Φ which is a local minimum of I_λ,*
or
(c_2) *there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ which weakly converges to a global minimum of Φ, with $\lim_{n\to+\infty} \Phi(u_n) = \inf_X \Phi$.*

3 Proof the main results

Proof: [Proof of Theorem 1] For each $u \in X$, let

$$\Phi(u) = \frac{\|u\|_A^p}{p}, \qquad \Psi(u) = \int_0^T F(t,u)\,\mathrm{d}t.$$

Under the condition of Theorem 1, Φ is a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional. Moreover, the Gâteaux derivative of Φ admits a continuous inverse on X^* (we can get this result due to Proposition 2.4. in [2]). Ψ is continuously Gâteaux differential functional whose Gâteaux derivative is compact. Obviously, Φ is bounded on each bounded subset of X.

In particular, for each $u, \xi \in X$,

$$\langle \Phi'(u), \xi\rangle = \int_0^T \left((|u'(t)|^{p-2}u'(t), \xi'(t)) + (A(t)|u(t)|^{p-2}u(t), \xi(t))\right)\mathrm{d}t,$$

$$\langle \Psi'(u), \xi\rangle = \int_0^T (\nabla F(t,u), \xi)\,\mathrm{d}t.$$

Hence, it follows from (1.5) that the weak solutions of equation $(\mathcal{H})$ are exactly the solutions of the equation

$$\Phi'(u) - \lambda\Psi'(u) = 0.$$

Furthermore from (H2) there exist two constants γ, $\tau \in \mathbb{R}$ with

$$\limsup_{|x|\to+\infty} \frac{F(t,x)}{|x|^p} < \gamma < \frac{\int_0^T \max_{x\in B} F(t,x)\,\mathrm{d}t}{c^p}$$

such that

$$F(t,x) \le \gamma\,|x|^p + \tau$$

for a.e. $t \in [0,T]$ and all $x \in \mathbb{R}$. Fix $u \in X$. Then

$$F(t,u(t)) \le \gamma\,|u(t)|^p + \tau$$

for all $t \in [0,T]$. Then, for any fixed $\lambda \in \left]\frac{\overline{\lambda}(k|d|)^p T}{pk^p \int_0^T F(t,d)\,\mathrm{d}t}, \frac{c^p}{pk^p \int_0^T \max_{x\in B} F(t,x)\,\mathrm{d}t}\right[$, since $\|u\|_\infty \leq k\,\|u\|_A$, we have

$$\begin{aligned}\Phi(u) - \lambda\Psi(u) =& \frac{\|u\|_A^p}{p} - \lambda \int_0^T F(t,u(t))\,\mathrm{d}t \\ \geq& \frac{\|u\|_A^p}{p} - \lambda \int_0^T (\gamma\,|u(t)|^p + \tau)\,\mathrm{d}t \\ \geq& \frac{\|u\|_A^p}{p} - \lambda\gamma\,\|u\|_\infty^p\,T - \lambda T\tau \\ \geq& \left(\frac{1}{p} - \lambda\gamma k^p T\right)\|u\|_A^p - \lambda T\tau \\ >& \frac{1}{p}\left(1 - \gamma\frac{c^p}{\int_0^T \max_{x\in B} F(t,x)\,\mathrm{d}t}\right)\|u\|_A^p - \lambda T\tau,\end{aligned}$$

and so,

$$\lim_{\|u\|\to+\infty}(\Phi(u) - \lambda\Psi(u)) = +\infty.$$

So, hypothesis (a_2) of Theorem 4 is satisfied.

Let $u^*(t) = d$. Then, we have $(u^*(t))' = 0$. It is easy to verify that $u^* \in W_T^{1,p}$, and in particular, one has

$$\|u^*\|^p = T|d|^p. \tag{3.1}$$

Hence, we obtain from $\underline{\lambda}|d|^p \leq (A(t)|d|^{p-2}d, d) \leq \overline{\lambda}|d|^p$ and (3.1) that

$$(\underline{\lambda}T)^{1/p}|d| \leq \|u^*\|_A \leq (\overline{\lambda}T)^{1/p}|d|. \tag{3.2}$$

By $c < k|d|(\underline{\lambda}T)^{1/p}$, it follows from (3.2) that

$$\|u^*\|_A > \frac{c}{k}. \tag{3.3}$$

Hence, thanks to the condition (H1) and (3.2), we have

$$\begin{aligned}\int_0^T \max_{x\in B} F(t,x)\,\mathrm{d}t \quad &< \quad \frac{1}{\overline{\lambda}T}\left(\frac{c}{k|d|}\right)^p \int_0^T F(t,d)dt \\ &\leq \quad \left(\frac{c}{k\|u^*\|_A}\right)^p \int_0^T F(t,u^*)dt.\end{aligned} \tag{3.4}$$

Now, put

$$r = \frac{1}{p}\left(\frac{c}{k}\right)^p.$$

Thanks to (3.3), there exists $u^* \in X$ such that

$$\Phi(u^*) = \frac{\|u^*\|_A^p}{p} > r > 0 = \Phi(0). \tag{3.5}$$

We obtain from (1.2) and $k_0 \leq k$ that

$$\sup_{t\in[0,T]} |u(t)| \leq k\|u\|_A \tag{3.6}$$

for each $u \in X$. Hence, for each $u \in X$ such that

$$\Phi(u) = \frac{\|u\|_A^p}{p} \leq r,$$

by (3.6) one has

$$\sup_{t\in[0,T]} |u(t)| \leq c. \tag{3.7}$$

It follows from (3.4) and (3.7) that

$$\begin{array}{rcl}
\displaystyle\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u) & = & \displaystyle\sup_{\{u|\ \Phi(u)\leq r\}} \int_0^T F(t,u)\,\mathrm{d}t \\
& = & \displaystyle\sup_{\left\{u|\ \|u\|_A^p\leq pr\right\}} \int_0^T F(t,u)\,\mathrm{d}t \\
& \leq & \displaystyle\int_0^T \sup_{\{x|\ 0\leq|x|\leq c\}} F(t,x)\,\mathrm{d}t \\
& = & \displaystyle\int_0^T \max_{x\in B} F(t,x)\,\mathrm{d}t \\
& < & \displaystyle\frac{1}{p}\left(\frac{c}{k}\right)^p \frac{p}{\|u^*\|_A^p} \int_0^T F(t,u^*(t))\,\mathrm{d}t \\
& = & \displaystyle r\frac{p}{\|u^*\|_A^p} \int_0^T F(t,u^*(t))\,\mathrm{d}t \\
& = & \displaystyle r\frac{\Psi(u^*)}{\Phi(u^*)}.
\end{array}$$

So, one has

$$\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u) < r\frac{\Psi(u^*)}{\Phi(u^*)}. \tag{3.8}$$

and (a_1) of Theorem 4 is satisfied.

From Theorem 4, for each $\lambda \in \Lambda_r := \left]\frac{\Phi(u^*)}{\Psi(u^*)}, \frac{r}{\sup_{\Phi(x)\leq r}\Psi(x)}\right[$, the functional $\Phi - \lambda\Psi$ has at least three distinct critical points which are the periodic solutions of $(\mathcal{H})$ and the conclusion is achieved. ∎

Proof: [Proof of Theorem 2] Our goal is to apply Theorem 5. Let $\{c_n\}$ be a real sequence such that $\lim_{n\to+\infty} c_n = +\infty$ and

$$\liminf_{n\to+\infty} \frac{\int_0^T \max_{|\xi|\leq c_n} F(t,\xi)\,\mathrm{d}t}{c_n^p} = \mathcal{A}. \tag{3.9}$$

Taking into account (4) for every $u \in X$ one has

$$\sup_{t\in[0,T]} |u(t)| \leq k \|u\|_A .$$

Hence, an easy computation ensures that $|u| \leq c_n$ when ever $u \in \Phi^{-1}(]-\infty, r_n[)$, where

$$r_n = \frac{1}{p}\left(\frac{c_n}{k}\right)^p, \qquad \forall n \in \mathbb{N}.$$

Taking into account that $\left\|u^0\right\|_A = 0$ (where $u^0(t) = 0$ for every $t \in [0,T]$) and that $\int_0^T F(t,0)\,\mathrm{d}t \geq 0$ for all $t \in [0,T]$, for every n large enough, one has

$$\begin{aligned}
\varphi(r_n) &= \inf_{u\in\Phi^{-1}(]-\infty,r_n[)} \frac{\left(\sup_{v\in\Phi^{-1}(]-\infty,r_n[)} \Psi(v)\right) - \Psi(u)}{r_n - \Phi(u)} \\
&= \inf_{\|u\|_A^p/p<r_n} \frac{\sup_{\|v\|_A^p/p<r_n} \int_0^T F(t,v)\,\mathrm{d}t - \int_0^T F(t,u)\,\mathrm{d}t}{r_n - \|u\|_A^p/p} \\
&\leq \frac{\sup_{\|v\|_A^p/p<r_n} \int_0^T F(t,v)\,\mathrm{d}t}{r_n} \\
&\leq pk^p \frac{\int_0^T \max_{|x|\leq c_n} F(t,x)\,\mathrm{d}t}{c_n^p}.
\end{aligned}$$

Therefore, since from assumption (H5) one has $\mathcal{A} < +\infty$, we obtain

$$\gamma = \liminf_{n\to+\infty} \varphi(r_n) \leq pk^p\mathcal{A} < +\infty. \tag{3.10}$$

Moreover, we can also observe that, owing to (3.9) and (3.10),

$$\Lambda \subseteq \,]0, 1/\gamma[\,.$$

Now, fix $\lambda \in]\lambda_1, \lambda_2[$ and let us verify that the functional $\Phi(u) - \lambda\Psi(u)$ is unbounded from below. Indeed, we can consider a N positive real sequence $\{d_{i,n}\}$ such that $|d_{i,n}| \to +\infty$ as $n \to \infty$. For each $n \in \mathbb{N}$ define

$$w_{i,n}(t) = d_{i,n}$$

and put $w_n := (w_{1,n}, \cdots, w_{N,n})$. Since $1/\lambda < \frac{pk^p}{\overline{\lambda}T}\mathcal{B}$, we can get that there exists a positive constant $\eta > 0$ and

$$\frac{1}{\lambda} < \eta < \frac{pk^p}{\overline{\lambda}T} \frac{\int_0^T F(t,d_n)\,\mathrm{d}t}{|d_n|^p}. \tag{3.11}$$

Fix $n \in \mathbb{N}$, a simple computation shows that

$$(\underline{\lambda}T)^{1/p}|d_n| \leq \|w_n\|_A \leq (\overline{\lambda}T)^{1/p}|d_n|. \tag{3.12}$$

On the other hand, thanks to (H4), (3.11) and (3.12), we achieve

$$\Phi(w_n) - \lambda\Psi(w_n) \leq \frac{\overline{\lambda}T|d_n|^p}{p} - \lambda\int_0^T F(t,d_n)\,\mathrm{d}t < \frac{|d_n|^p\,\overline{\lambda}T}{pk^p}(1-\lambda\eta)$$

for every $n \in \mathbb{N}$ large enough. Hence, $\Phi(u) - \lambda\Psi(u)$ is unbounded from below.

Applying Theorem 5 we deduce that the functional $\Phi(u) - \lambda\Psi(u)$ admits a sequence of critical points which is unbounded in X. Hence, our claim is proved and the conclusion is achieved. □

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School of Mathematics and Statistics,
Southwest University Chongqing 400715, P. R. China
E-mail: `lilin420@gmail.com`

School of Mathematics and Statistics,
Southwest University Chongqing 400715, P. R. China
E-mail: `lichun2003_1999@163.com`

School of Mathematics and Computer Science,
Hubei University of Arts and Science, Hubei 441053, P. R. China
E-mail: `591517149@qq.com`