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Some new disconjugacy criteria for second order differential equations with a middle term

by
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Abstract

In this paper, for a second order differential equation with a middle term, we will establish some new criteria for disconjuagcy on an interval I, i.e., any nontrivial solution of the equation has at most one zero on this interval. We, also establish some sufficient conditions for disfocality and obtain a lower bound for an eigenvalue of a boundary value problem. Some examples are considered to illustrate the main results.

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1 Introduction

In this paper, we will consider the second order differential equation with a middle term

$$(r(t)x^{'}(t))^{'} + p(t)x^{'}(t) + q(t)x(t) = 0, \quad \text{for} \ \ t \in I, \tag{1.1}$$

where I is an interval of reals and p, q and r are real valued functions defined on I such that $r(t) > 0$. The main aim in this paper is to establish some criteria for disconjugcy and disfocality in an interval $I = [a, b] \subset \mathbb{R}$. We also find an explicit formula for the lower bound of the first eigenvalue of the eigenvalue problem

$$-\left(x^{'}(t)\right)^{'} - p(t)x^{'}(t) + q(t)x(t) = \lambda x(t), \ \ x(a) = x(b) = 0. \tag{1.2}$$

Equation (1.1) is said to be a disconjugate on the interval $[a,\, b]$, if there is no nontrivial solution of (1.1) has two zeros on $[a,\, b]$. Equation (1.1) is said to be a nonoscillatory on $[t_0,\, \infty)$ if there exists $c \in [t_0,\, \infty)$ such that this equation is disconjugate on $[c,\, d]$ for every $d > c$.

If a nontrivial solution of (1.1) has a zero at a, then the first zero of x to the right of a is called the right conjugate point of a. Successive zeros are isolated and hence yield a counting of conjugate points. If $x(t)$ satisfies $x^{'}(a) = 0$, then the first zero b of $x(t)$ (say $x(b) = 0$) to the right of a is called the first right focal point of a. On other words, we say that (1.1) is right disfocal (left disfocal) on $[a,\ b]$ if the solution $x(t)$ of (1.1) which satisfies $x^{'}(a) = 0$ ($x^{'}(b) = 0$) has no zeros in $[a,\ b]$. The best known existence result in the literature for disconjugacy has been proved by Lyapunov [14]. He proved that if $q(t)$ is a positive continuous on the closed interval $[a, b]$ and if

$$(b-a)\int_a^b q(t)dt \leq 4, \tag{1.3}$$

then $x^{''}(t) + q(t)x(t) = 0$ is disconjugate. Since the appearance of the inequality (1.3) various proofs and generalizations or improvements have appeared in the literature for different types of differential equations, we refer to the papers [2, 5, 6, 7, 8, 9, 10, 12, 15, 16, 17, 18, 19, 21] and the references cited therein. Most of the sufficient conditions for disconjugacy that has been obtained in these mentioned papers are formulated for differential equations without midle terms and few results, we refer to [7, 8], has been obtained for a special case of equation (1.1) with a middle term of the form

$$x^{''}(t) + p(t)x^{'}(t) + q(t)x(t) = 0. \tag{1.4}$$

2 Main Results

In this section we state and prove the main results and give some examples to illustrate the main results. The main results will be proved by making use of Hardy's inequality, Hölder's inequality and some Opial and Wirtinger type inequalities. The Hardy inequality [13] of the differential form that we will need in this paper states that: If x is absolutely continuous on (a, b) with $x(a) = 0$ or $x(b) = 0$, then the following inequality holds

$$\left(\int_a^b q(t)\left|x(t)\right|^n dt\right)^{\frac{1}{n}} \leq C\left(\int_a^b r(t)\left|x^{'}(t)\right|^m dt\right)^{\frac{1}{m}}, \tag{2.1}$$

where q, r, the weighted functions, are measurable in the interval (a, b) and m, n are real parameters satisfy $0 < n \leq \infty$ and $1 \leq m \leq \infty$. The constant C satisfies $C \leq k(m,n)A(a,b)$, for $1 < m \leq n$, where

$$\begin{aligned}
A(a,b) \quad &: \quad = \sup_{a<t<b}\left(\int_t^b q(t)dt\right)^{\frac{1}{n}}\left(\int_a^t r^{1-m^*}(s)ds\right)^{1/m^*}, \text{ if } x(a)=0,\\
A(a,b) \quad &: \quad = \sup_{a<t<b}\left(\int_a^t q(t)dt\right)^{\frac{1}{n}}\left(\int_t^b r^{1-m^*}(s)ds\right)^{1/m^*}, \text{ if } x(b)=0,
\end{aligned}$$

and $m^* = m/(m-1)$. The constant $k(m,n)$ appears in various forms. For example,

$$k(m,n) := m^{1/m}(m^*)^{1/m^*},\ k(m,n) := n^{1/n}(n^*)^{1/n^*}.$$

Note that the inequality (2.1) has immediate application to the case where $x(a) = x(b) = 0$. In this case, we see that (2.1) is satisfied if and only if

$$A(a,b) = \sup_{(c,d)\subset(a,b)} (\int_c^d q(t)dt)^{\frac{1}{n}} \min((\int_a^c r^{1-m^*}(s)ds)^{1/m^*}, (\int_d^b r^{1-m^*}(s)ds)^{1/m^*}), \tag{2.2}$$

exists and finite. The Opial inequality that we will need in order to prove the main results in this paper is due to Bessack and Das [3]. This inequality states that: If x is absolutely continuous on $[a,\ b]$ with $x(a) = 0$, then the following inequality holds

$$\int_a^b B(t)\left|x(t)\right|^m \left|x^{'}(t)\right|^n dt \leq K_1(m,n)\int_a^b A(t)\left|x^{'}(t)\right|^{m+n} dt, \tag{2.3}$$

where $m,\ n$ are real numbers such that $mn > 0$ and $m + n > 1$, A and B are nonnegative, measurable functions on (a,b) such that $\int_a^t (A^{\frac{-1}{(m+n-1)}}(s)ds < \infty$, and

$$K_1(m,n) := \left(\frac{n}{n+m}\right)^{\frac{n}{n+m}} \left[\int_a^b \frac{B^{\frac{n+m}{m}}(t)}{A^{\frac{n}{m}}(t)} \left(\int_a^t (A^{\frac{-1}{(m+n-1)}}(s)ds\right)^{m+n-1} dt\right]^{\frac{m}{m+n}}. \tag{2.4}$$

If we replace $x(a) = 0$ by $x(b) = 0$, then (2.3) holds where $K_1(m,n)$ is replaced by

$$K_2(m,n) := \left(\frac{n}{n+m}\right)^{\frac{n}{n+m}} \left[\int_a^b \frac{B^{\frac{n+m}{m}}(t)}{A^{\frac{n}{m}}(t)} \left(\int_t^b (A^{\frac{-1}{(m+n-1)}}(s)ds\right)^{m+n-1} dt\right]^{\frac{m}{m+n}}. \tag{2.5}$$

Note that the inequality (2.3) has an immediate application to the case where $x(a) = x(b) = 0$. In this case we will assume that there exists $\tau \in (a,b)$ such that

$$\int_\tau^b (A^{\frac{-1}{(m+n-1)}}(s)ds = \int_a^\tau (A^{\frac{-1}{(m+n-1)}}(s)ds. \tag{2.6}$$

Then the inequality (2.3) holds with a new constant $K(m,n)$ which is given from the equation $K(m,n) = K_1(m,n) = K_2(m,n)$, when (2.6) is satisfied. For more details of different types of Opial inequalities, we refer the reader to the book [1].

Now, we are ready to state and prove the main results for equation (1.1). We will assume that there exists $\tau \in (a,b)$ such that

$$\int_a^\tau \frac{1}{r(t)}dt = \int_\tau^b \frac{1}{r(t)}dt, \tag{2.7}$$

and denote by $R(a,b)$. We introduce the following notations:

$$C(q,r) := \frac{1}{4\pi}A(a,b), \text{ and } K(p,r) := \sqrt{\frac{1}{2}R(a,b)}\left[\int_a^b \frac{\left|p(t)\right|^2}{r(t)}dt\right]^{\frac{1}{2}}, \tag{2.8}$$

where $A(a,b)$ is defined as in (2.2).

Theorem 2.1. *Assume that r, p, q are real valued functions defined on $[a,b]$ such that $r(t)>0$ and $\int_a^t r^{-1}(s)ds<\infty$. If $C(q,r)+K(p,r)<1$. Then (1.1) is disconjugate on $[a,b]$.*

Proof. Suppose that (1.1) is not disconjugate on $[a,b]$. Then there is a nontrivial solution $x(t)$ vanishes twice on $[a,b]$. Without loss of generality we may assume that there exists a solution of (1.1) such that $x(a)=x(b)=0$. Multiplying (1.1) by $x(t)$ and integrating by parts from a to b and using the boundary conditions, we have

$$\int_a^b r(t)\left(x^{'}(t)\right)^2 dt \leq \int_a^b |q(t)|\, x^2(t)dt + \int_a^b |p(t)| \left|x^{'}(t)\right| |x(t)|\, dt. \tag{2.9}$$

Applying the Hardy inequality (2.1) on the term $\int_a^b |q(t)|\, x^2(t)dt$, with $m=n=2$, we see that

$$\int_a^b |q(t)|\, x^2(t)dt < C(q,r)\int_a^b r(t)\left(x^{'}(t)\right)^2 dt, \tag{2.10}$$

where $C(q,r)$ is defined as in (2.8). Applying the inequality (2.3) on the term $\int_a^b |p(t)|\, x^{'}(t)x(t)dt$, with $B(t)=|p(t)|$, $A(t)=r(t)$, $m=n=1$, we have that

$$\int_a^b |p(t)| \left|x^{'}(t)\right| |x(t)|\, dt \leq K(p,r)\int_a^b r(t)\left(x^{'}(t)\right)^2 dt, \tag{2.11}$$

where $K(p,r)$ is defined as in (2.8). Substituting (2.10) and (2.11) into (2.9), we have that $1\leq C(q,r)+K(p,r)$, which is a contradiction with the assumption of theorem. The proof is complete.

Instead of the Hardy inequality, one can apply the inequality of Lin (see [1, Page 72])

$$\int_a^b q(t)x^{\gamma}(t)dt < \frac{1}{2}\left(\frac{b-a}{2}\right)^{\gamma-1}\left(\int_a^b q(t)dt\right)\int_a^b \left(x^{'}(t)\right)^{\gamma} dt, \tag{2.12}$$

with $\gamma=2$, where $x(t)$ is absolutely continuous function and satisfies $x(a)=x(b)=0$ on the term $\int_a^b |q(t)|\, x^2(t)dt$, to get that

$$\int_a^b |q(t)|\, x^2(t)dt < \left(\frac{b-a}{4}\right)\left(\int_a^b |q(t)|\, dt\right)\int_a^b \left(x^{'}(t)\right)^2 dt.$$

Using this inequality, we have the following result.

Theorem 2.2. *Assume that p, q are real valued functions defined on $[a,b]$. If*

$$(b-a)\int_a^b |q(t)|\, dt + 2\left[(b-a)\int_a^b |p(t)|^2\, dt\right]^{\frac{1}{2}} < 4, \tag{2.13}$$

then the equation (1.4) is disconjugate on $[a,b]$.

In Theorem 2.2 if we assume that $\sup|p(t)|=A$ and $\sup|q(t)|\leq B$, then we have the following result of disconjugacy.

Corollary 2.1. *Assume that* p, q *are real valued functions defined on* $[a,b]$ *and* $\sup_{t\in[a,b]}|p(t)| = A$ *and* $\sup_{t\in[a,b]}|q(t)| = B$. *If* $B(b-a)^2+2A(b-a) < 4$, *then the equation (1.4) is disconjugate on* $[a,b]$.

Theorem 2.3. *Assume that* r, p, q *are real valued functions defined on* $[a,b]$ *such that* $r(t) > 0$, *and there exists a differentiable function* Q *such that* $Q^{'}(t) = q(t)$, *and* $\int_a^t r^{-1}(s)ds < \infty$. *If*

$$R(a,b)\int_a^b \frac{|2Q(t)-p(t)|^2}{r(t)}dt < 2, \tag{2.14}$$

Then (1.1) is disconjugate on $[a,b]$.

Proof. We proceed as in the proof of Theorem 2.1 to get (2.9). Integrating by parts (2.9) and the assumption $Q^{'}(t) = q(t)$, and using the boundary conditions $x(a) = x(b) = 0$, we get that

$$\int_a^b r(t)\left(x^{'}(t)\right)^2 dt \le \int_a^b |2Q(t)-p(t)|\left|x^{'}(t)\right||x(t)|\,dt. \tag{2.15}$$

Applying the inequality (2.3) on the term $\int_a^b |2Q(t)-p(t)|\,x^{'}(t)x(t)dt$, with $B(t) = |2Q(t)-p(t)|$, $A(t) = r(t)$, $m = n = 1$ and substituting into (2.15), we get a contradiction with (2.14). The proof is complete.

In the following, we will establish a new condition for disconjugacy depends on the sign of the quadratic functional formula associated with (1.1).

Theorem 2.4. *Assume that* r, p, q *are real valued functions defined on* $[a,b]$ *such that* $r(t) > 0$ and $\int_a^t r^{-1}(s)ds < \infty$. *If*

$$F(p,q,r) := \int_a^b \left[r(t)(x^{'}(t))^2 - \lambda_1|q(t)|\,x^2(t)\right]dt > 0, \tag{2.16}$$

where $\lambda_1 = \left[1-\left(\frac{R(a,b)}{2}\int_a^b \frac{|p(t)|^2}{r(t)}dt\right)^{\frac{1}{2}}\right]^{-1} \neq 0$, *then (1.1) is disconjugate on* $[a,b]$.

Proof. Suppose that (1.1) is not disconjugate on $[a,b]$. Then there is a nontrivial solution $x(t)$ vanishes twice on $[a,b]$. Without loss of generality, we may assume that there exists a solution of (1.1) such that $x(a) = x(b) = 0$. Multiplying (1.1) by $x(t)$ and integrating by parts from a to b, and using the boundary conditions $x(a) = x(b) = 0$, we have

$$\int_a^b r(t)\left(x^{'}(t)\right)^2 dt \le \int_a^b |q(t)|\,x^2(t)dt + \int_a^b |p(t)|\left|x^{'}(t)\right||x(t)|\,dt. \tag{2.17}$$

Applying the inequality (2.3) on the term $\int_a^b |p(t)|\,x^{'}(t)x(t)dt$, with $B(t) = |p(t)|$, $A(t) = r(t)$, $m = n = 1$, we have that

$$\int_a^b |p(t)|\left|x^{'}(t)\right||x(t)|\,dt \le \left[\frac{R(a,b)}{2}\int_a^b \frac{|p(t)|^2}{r(t)}dt\right]^{\frac{1}{2}}\int_a^b r(t)\left(x^{'}(t)\right)^2 dt. \tag{2.18}$$

Substituting (2.18) into (2.17), we have that

$$\left[1-\left[\frac{R(a,b)}{2}\int_a^b\frac{|p(t)|^2}{r(t)}dt\right]^{\frac{1}{2}}\right]\int_a^b r(t)\left(x^{'}(t)\right)^2dt-\int_a^b|q(t)|\,x^2(t)dt\leq 0,$$

so that $\int_a^b\left[r(t)\left(x^{'}(t)\right)^2dt-\lambda_1\,|q(t)|\,x^2(t)\right]dt\leq 0$, which is a contradiction with (2.16). This completes the proof.

From Theorem 2.4, we can obtain the following result.

Corollary 2.2. *Assume that* p, q *are real valued functions defined on* $[a,b]$. *If*

$$(b-a)\int_a^b|p(t)|^2\,dt>4, \tag{2.19}$$

then (1.4) is disconjugate on $[a,b]$.

Now, we apply the Yang inequality [20] to obtain a new form of the quadratic functional $F(p,q,r)$. The Yang inequality states that: If $p(t)$ is a positive bounded function and y is an absolutely continuous on $[a,b]$ with $y(a)=y(b)=0$, $m\geq 0$, $n\geq 1$, then

$$\int_a^b p(t)\,|y(t)|^m\left|y^{'}(t)\right|^n dt\leq\frac{n}{m+n}(\frac{b-a}{2})^m\int_a^b p(t)\left|y^{'}(t)\right|^{m+n}dt. \tag{2.20}$$

Applying this inequality on the term $\int_a^b|p(t)|\,x^{'}(t)x(t)dt$, with $m=n=1$, and using in (2.17), and proceeding as in the proof of Theorem 2.4, we have the following result.

Theorem 2.5. *Assume that* r, p, q *are real valued functions defined on* $[a,b]$ *such that* $r(t)>0$ *and* $|p(t)|$ *is bounded and* $R(t)=\left[r(t)-\frac{b-a}{4}\,|p(t)|\right]>0$. *If*

$$F(p,q,r):=\int_a^b\left[R(t)(x^{'}(t))^2-|q(t)|\,x^2(t)\right]dt>0, \tag{2.21}$$

then (1.1) is disconjugate on $[a,b]$.

In the following, we will establish some new sufficient conditions for disfocality of the equation (1.1), i.e., sufficient conditions so that there does not exist a nontrivial solution x of (1.1) satisfying $x(a)=x^{'}(b)=0$ or $x^{'}(a)=x(b)=0$. We introduce the following notations:

$$K_1(p,Q,r):=\sqrt{\frac{1}{2}}\left[\int_a^b\frac{(2\,|Q(t)|+|p(t)|)^2}{r(t)}\left(\int_a^t\frac{ds}{r(s)}\right)dt\right]^{\frac{1}{2}},$$

where $Q(t)=\int_t^b|q(s)|\,ds$, and

$$K_2(p,Q,r):=\sqrt{\frac{1}{2}}\left[\int_a^b\frac{(2\,|Q(t)|+|p(t)|)^2}{r(t)}\left(\int_t^b\frac{ds}{r(s)}\right)dt\right]^{\frac{1}{2}},$$

where $Q(t)=\int_a^t|q(s)|\,ds$.

Theorem 2.6. *Assume that r, p, q are real valued functions defined on $[a,b]$ such that $r(t) > 0$ and $\int_a^t r^{-1}(s)ds < \infty$. Suppose that x is a nontrivial solution of (1.1). If $x(a) = x^{'}(b) = 0$, then $K_1(p,Q,r) \geq 1$, where $Q(t) = \int_t^b |q(s)|\, ds$. If $x^{'}(a) = x(b) = 0$, then $K_2(p,Q,r) \geq 1$, where $Q(t) = \int_a^t |q(s)|\, ds$.*

Proof. We prove that $K_1(p,Q,r) \geq 1$. Multiplying (1.1) by x and integrating by parts and using the boundary conditions $x(a) = x^{'}(b) = 0$ and $Q(t) = \int_t^b |q(s)|\, ds$, we get that

$$\int_a^b r(t) \left|x^{'}(t)\right|^2 dt \leq \int_a^b \left(2\left|Q(t)\right| + |p(t)|\right) |x(t)| \left|x^{'}(t)\right| dt. \tag{2.22}$$

Applying the inequality (2.3) with $B(t) = \left(2\left|Q(t)\right| + |p(t)|\right)$, $A(t) = r(t)$, $m = 1$ and $n = 1$, we have that

$$\int_a^b r(t) \left|x^{'}(t)\right|^2 dt \leq K_1(p,Q,r) \int_a^b r(t) \left|x^{'}(t)\right|^2 dt.$$

Dividing both sides by $\int_a^b r(t) \left|x^{'}(t)\right|^2 dt$, we have that $K_1(p,Q,r) \geq 1$, which is the first desired inequality. The proof of $K_2(p,Q,r) \geq 1$ is similar using the integration by parts and (2.5) instead of (2.4). The proof is complete.

Note that the term $rxx^{'}$ in Theorem 2.4 vanishes if $x(a) = x^{'}(b) = 0$ and $x^{'}(a) = x(b) = 0$. So that we have the following results for disfocality.

Theorem 2.7. *Assume that r, p, q are real valued functions defined on $[a,b]$ such that $r(t) > 0$ and $\int_a^t r^{-1}(s)ds < \infty$. If $\int_a^b \left[r(t)(x^{'}(t))^2 - \lambda_* \left|q(t)\right| x^2(t)\right] dt > 0$, where*

$$\lambda_* = \left[1 - \left(\frac{1}{2}\int_a^b \frac{|p(t)|^2}{r(t)} \left(\int_a^t \frac{ds}{r(s)}\right) dt\right)^{\frac{1}{2}}\right]^{-1},$$

then there does not exist a solution $x(t)$ of (1.1) satisfies $x(a) = x^{'}(b) = 0$. If

$$\int_a^b \left[r(t)(x^{'}(t))^2 - \lambda_{**} \left|q(t)\right| x^2(t)\right] dt > 0,$$

where

$$\lambda_{**} = \left[1 - \left(\frac{1}{2}\int_a^b \frac{|p(t)|^2}{r(t)} \left(\int_t^b \frac{ds}{r(s)}\right) dt\right)^{\frac{1}{2}}\right]^{-1},$$

then there does not exist a solution $x(t)$ of (1.1) satisfies $x^{'}(a) = x(b) = 0$.

One can apply an inequality due to Boyd [4] and the Hölder inequality to obtain some results about the spacing between conjugate points of the solution. The Boyd inequality states that: If $x \in C^1[a,\, b]$ with $x(a) = x(b) = 0$, then

$$\int_a^b |x(t)|^{\nu} \left|x^{'}(t)\right|^{\eta} dt \leq L(\nu,\eta)(\frac{b-a}{2})^{\nu} \left(\int_a^b \left|x^{'}(t)\right|^{\eta} dt\right)^{\frac{\nu+\eta}{\eta}}, \tag{2.23}$$

where $L(\nu,\eta)$ is defined by $L(\nu,\eta) := \frac{\eta\nu^{\eta}}{\nu+\eta}\left(\frac{\nu}{\nu+\eta}\right)^{\frac{\nu}{\eta}}\left(\dfrac{\Gamma((\eta+1)/\eta+\frac{1}{\nu})}{\Gamma\left((\eta+1)/\eta\right)\Gamma\left(\frac{1}{\nu}\right)}\right)^{\nu}$, and Γ is the Gamma function. Applying the inequality (2.23) on the term $\left(\int_a^b |x(t)|^2 \left|x^{'}(t)\right|^2 dt\right)^{\frac{1}{2}}$, with $\nu=2$, and $\eta=2$, we see that

$$\left(\int_a^b |x(t)|^2 \left|x^{'}(t)\right|^2 dt\right)^{\frac{1}{2}} \leq \frac{4}{\pi^2}(\frac{b-a}{2})^2\left(\int_a^b \left|x^{'}(t)\right|^2 dt\right)^2. \tag{2.24}$$

Theorem 2.8. *Assume that $r(t)$ is a nonincreasing function and $Q^{'}(t)=q(t)$ on $[a,b]$. Suppose that x is a nontrivial solution of (1.1). If $x(a)=x(b)=0$, then*

$$2\left(\int_a^b Q^2(t)dt\right)^{\frac{1}{2}}+\left(\int_a^b |p(t)|^2 dt\right)^{1/2} \geq \frac{\pi r(b)}{2\,(b-a)}. \tag{2.25}$$

Proof. As in the proof of Theorem 2.6, we have that

$$\int_a^b r(t)\left|x^{'}(t)\right|^2 dt \leq 2\int_a^b Q(t)\,|x(t)|\left|x^{'}(t)\right| dt+\int_\alpha^\beta |p(t)|\left|x^{'}(t)\right||x(t)|\,dt. \tag{2.26}$$

Applying the Hölder inequality with $m=n=2$, and using the fact that $r(t)$ is a nonincreasing and using (2.24), we have

$$\int_a^b |Q(t)|\,|x(t)|\left|x^{'}(t)\right| dt \leq \frac{(\frac{4}{\pi^2})^{\frac{1}{2}}}{r(b)}(\frac{b-a}{2})\left(\int_a^b Q^2(t)dt\right)^{\frac{1}{2}} \times \left(\int_a^b r(t)\left|x^{'}(t)\right|^2 dt\right). \tag{2.27}$$

Applying the Hölder inequality and using the fact that $r(t)$ is a nonincreasing, we have also that

$$\int_a^b p(t)\,|x(t)|\left|x^{'}(t)\right| dt \leq \frac{(\frac{4}{\pi^2})^{\frac{1}{2}}}{r(b)}(\frac{b-a}{2})\left(\int_a^b |p(t)|^2 dt\right)^{1/2} \times \left(\int_a^b r(t)\left|x^{'}(t)\right|^2 dt\right). \tag{2.28}$$

Substituting (2.27) and (2.28) into (2.26) and cancelling the term $\int_a^b r(t)\left|x^{'}(t)\right|^2 dt$, we get the desired inequality (2.25). The proof is complete.

As a special case, we have the following result for the equation (1.4).

Corollary 2.3. *Assume that $Q^{'}(t)=q(t)$ on $[a,b]$. Suppose that x is a nontrivial solution of (1.4). If $x(a)=x(b)=0$, then*

$$2\left(\int_a^b Q^2(t)dt\right)^{\frac{1}{2}}+\left(\int_a^b |p(t)|^2 dt\right)^{\frac{1}{2}} \geq \frac{\pi}{2(b-a)}. \tag{2.29}$$

In the following, we will show how Opial and Wirtinger type inequalities may be used to find the lower bound for the eigenvalue of a boundary value problem. In particular, we will apply the Wirtinger inequality

$$\int_0^\pi (x^{'}(t))^{k+1}dt \geq \frac{2\Gamma\left(k+2\right)}{\pi^{k+1}\Gamma^2\left(\left(k+2\right)/2\right)}\int_0^\pi x^{k+1}(t)dt, \text{ for } k \geq 1, \tag{2.30}$$

where $x(t) \in C^1[0,\ \pi]$ and $x(0) = x(\pi) = 0$, due to Agarwal and Pang and the Wirtinger type inequality (see [1])

$$\int_a^b \lambda(t)\left|x(t)\right|^{\gamma+1} dt \leq \frac{1}{2}\left(\int_a^b (t(b-t))^{\frac{\gamma}{2}}\lambda(t)dt\right)\int_a^b \left|x^{'}(t)\right|^{\gamma+1} dt,\ \gamma \geq 1, \tag{2.31}$$

where $x(a) = x(b) = 0$ and $\lambda(t) > 0$ is a continuous function on $[a, b]$, $x(t)$ is an absolutely continuous function on $[a, b]$ to establish a new explicit formula for the lower bounds of the eigenvalue of the eigenvalue problem

$$-\left(x^{'}(t)\right)^{'} - p(t)x^{'}(t) + q(t)x(t) = \lambda x(t),\ \ x(0) = x(\pi) = 0, \tag{2.32}$$

and assume that λ_0 is first eigenvalue of (2.32). The main aim now is to find the lower bound for λ_0.

Theorem 2.9. *Assume that λ_0 is the first positive eigenvalue of (2.32) and $Q^{'}(t) = q(t)+\mu$, where $0 < \mu < \lambda_0$. Then*

$$\frac{\pi^3}{4}\left(\lambda_0-\mu\right) \geq 1-\sqrt{2}\left(\int_0^\pi (t(\pi-t))^{\frac{1}{2}}Q^2(t)dt\right)^{\frac{1}{2}} - \sqrt{\frac{1}{2}}\left(\int_0^\pi (t(\pi-t))^{\frac{1}{2}}\left|p(t)\right|^2 dt\right)^{\frac{1}{2}}. \tag{2.33}$$

Proof. Let $x(t)$ be the eigenfunction of (2.32) corresponding to λ_0. Multiplying (2.32) by $x(t)$ and proceeding as in the proof of Theorem 2.3 to get that

$$-\int_0^\pi p(t)x^{'}(t)x(t)dt + \int_0^\pi q(t)x^2(t)dt = \lambda_0\int_0^\pi x^2(t)dt + \int_0^\pi \left(x^{'}(t)\right)^{'} x(t)dt.$$

This implies, after integrating by parts and using the fact that $x(0) = x(\pi) = 0$, that

$$\begin{aligned}(\lambda_0-\mu)\int_0^\pi x^2(t)dt &= \int_0^\pi \left(x^{'}(t)\right)^2 dt + \int_0^\pi Q^{'}(t)x^2(t)dt - \int_0^\pi p(t)x^{'}(t)x(t)dt \\ \geq\ & \int_0^\pi \left|x^{'}(t)\right|^2 dt - 2\int_0^\pi Q(t)\left|x(t)\right|\left|x^{'}(t)\right|dt - \int_0^\pi \left|p(t)\right|\left|x^{'}(t)\right|x(t)dt.\end{aligned}$$

Proceeding as in the proof of Theorem 2.6 by applying the inequality (2.31), with $\gamma = 1$, and the Wirtinger inequality (2.30), we get that

$$(\lambda_0-\mu)\frac{\pi^3}{4} \geq 1-\sqrt{2}\left(\int_0^\pi (t(\pi-t))^{\frac{1}{2}}Q^2(t)dt\right)^{\frac{1}{2}} - \sqrt{\frac{1}{2}}\left(\int_0^\pi (t(\pi-t))^{\frac{1}{2}}\left|p(t)\right|^2 dt\right)^{\frac{1}{2}}.$$

From this we obtain the lower bound of λ_0 as given in (2.33). The proof is complete.

In the following, we give some examples to illustrate the main results. we begin with an example to illustrate the result in Theorem 2.2.

Example 1. *Consider the differential equation*

$$x^{''}(t)+\frac{2}{t}x^{'}(t)+\frac{1}{4t^2}x(t)=0, \quad t\in[1,2]. \tag{2.34}$$

The condition (2.13) now reads

$$\begin{aligned}&\left(\frac{b-a}{4}\right)\int_a^b |q(t)|\,dt+\frac{1}{2}\left[(b-a)\int_a^b |p(t)|^2\,dt\right]^{\frac{1}{2}}\\ =\;&\left(\frac{1}{16}\right)\int_1^2 \frac{1}{t^2}dt+\frac{1}{2}\left[\int_1^2 \frac{1}{t^2}dt\right]^{\frac{1}{2}}=0.3848<1.\end{aligned}$$

Then by Theorem 2.2, we see that the equation (2.34) is disconjugate on $[1,2]$. *Note that the solution of the equation (2.34) is* $x(t)=1/\sqrt{t}$, *which is nowhere equal to zero on* $[1,2]$.

Example 2. *Consider the differential equation*

$$x^{''}(t)+A\cos(\alpha t)x^{'}(t)+B\cos(\beta t)x(t)=0,\ t\in[0,b], \tag{2.35}$$

where A *and* B *are positive constants. Then by Corollary 2.1, we see that (2.35) is disconjugate on* $[0,b]$ *if*

$$b^2B+2Ab<4. \tag{2.36}$$

Note that the condition (2.36), which does not depend on the frequencies α *and* β, *is different from the condition*

$$Ab+2b\,(B/\beta)<2, \tag{2.37}$$

that has been obtained by Clark and Hinton [8] which depends on the frequency β *and does not contain the frequency* α.

Example 3. *Consider the equation*

$$x^{''}(t)+tx^{'}(t)+(\frac{t^2}{4}+\frac{1}{2})x(t)=0, \quad t\in[1,3]. \tag{2.38}$$

The condition (2.19) of Corollary 2.2 on the interval $[1,3]$ *reads* $(b-a)\int_a^b |p(t)|^2\,dt/4=\frac{2}{4}\int_1^3 t^2dt=4.333\,3>1$. *Then the equation (2.38) is disconjugate on the interval* $[1,3]$. *One such solution of (2.38) is* $x(t)=e^{\frac{-t^2}{4}}$, *which is disconjugate on the interval* $[1,3]$. *Also one can consider*

$$x^{''}(t)+\frac{1}{t}x'(t)+\frac{1}{t^2}x(t)=0, \qquad t\in[1,4]. \tag{2.39}$$

and see that the condition (2.19) on the interval $[1,4]$ *reads* $(b-a)\int_a^b |p(t)|^2\,dt/4=\frac{3}{4}\int_1^4 \frac{1}{t^2}dt=0.562\,5<1$. *This means that the condition (2.19) is not satisfied. One such solution of (2.39) is* $x(t)=\sin(\ln t)$ *which is not disconjugate.*

The following example illustrates the result in Corollary 2.3.

Example 4. *Consider the equation*

$$x^{''}(t) + \mu \sin^2(kt)x^{'}(t) + \lambda \cos(kt)\, x(t) = 0, \quad t \in I, \tag{2.40}$$

where $p(t) = \left(\mu \sin^2(kt)\right)$, $q(t) = \lambda \cos(kt)$ *and* λ, μ, k *are positive constants. Let* $x(t)$ *be a solution of (2.40) with* $x(a) = x(b) = 0$ *where* $[a, b] \subseteq I$. *The condition (2.29) in Corollary 2.3 reads*

$$\frac{\pi}{2(b-a)} \leq \frac{2\lambda}{k}\left(\int_a^b \sin^2(kt)dt\right)^{\frac{1}{2}} + \mu\left(\int_a^b \sin^2(kt)dt\right)^{\frac{1}{2}}. \tag{2.41}$$

This implies that $(b-a) \geq \left(\frac{\pi^2 k^2}{4(2\lambda+\mu k)^2}\right)^{\frac{1}{3}}$.

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