Bull. Math. Soc. Sci. Math. Roumanie
Tome 56(104) No. 3, 2013, 365–368

Depth of some special monomial ideals *

by
Dorin Popescu and Andrei Zarojanu

Dedicated to Professors Toma Albu and Constantin Năstăsescu on the occasion of their 70th birthdays

Abstract

Let $I \supsetneq J$ be two squarefree monomial ideals of a polynomial algebra over a field. Suppose that I is generated by one squarefree monomial of degree $d > 0$, and other squarefree monomials of degrees $\geq d+1$. If the Stanley depth of I/J is $\leq d+1$ then almost always the usual depth of I/J is $\leq d+1$ too.

Key Words: Monomial Ideals, Depth, Stanley depth.
2010 Mathematics Subject Classification: Primary 13C15;
Secondary 13F20, 13F55, 13P10.

Introduction

Let $S = K[x_1, \ldots, x_n]$ be the polynomial algebra in n variables over a field K and $I \supsetneq J$ two squarefree monomial ideals of S. Suppose that I is generated by squarefree monomials of degrees $\geq d$ for some positive integer d. Modulo a multigraded isomorphism we may assume either that $J = 0$, or J is generated in degrees $\geq d+1$. Then $\operatorname{depth}_S I/J \geq d$ (see [1, Proposition 3.1], [3, Lemma 1.1]) and upper bounds are given by Stanley's Conjecture if it holds. Finding such upper bounds is the subject of several papers [2], [3], [4], [7], [5]. We remind below the notion of Stanley depth.

Let $P_{I\setminus J}$ be the poset of all squarefree monomials of $I \setminus J$ with the order given by the divisibility. Let $\mathcal{P}$ be a partition of $P_{I\setminus J}$ in intervals $[u, v] = \{w \in P_{I\setminus J} : u|w, w|v\}$, let us say $P_{I\setminus J} = \cup_i [u_i, v_i]$, the union being disjoint. Define $\operatorname{sdepth} \mathcal{P} = \min_i \deg v_i$ and the *Stanley depth* of I/J given by $\operatorname{sdepth}_S I/J = \max_{\mathcal{P}} \operatorname{sdepth} \mathcal{P}$, where $\mathcal{P}$ runs over the set of all partitions of $P_{I\setminus J}$ (see [1], [8]). Stanley's Conjecture says that $\operatorname{sdepth}_S I/J \geq \operatorname{depth}_S I/J$.

Let r be the number of squarefree monomials of degree d of I and B (resp. C) be the set of squarefree monomials of degrees $d+1$ (resp. $d+2$) of $I \setminus J$. Set $s = |B|$, $q = |C|$. If either

*The support from grant ID-PCE-2011-1023 of Romanian Ministry of Education, Research and Innovation is gratefully acknowledged.

$s > r+q$, or $r > q$, or $s < 2r$ then $\operatorname{sdepth}_S I/J \leq d+1$ and if Stanley's Conjecture holds then any of these numerical conditions would imply $\operatorname{depth}_S I/J \leq d+1$, independently of the characteristic of K. In particular this was proved directly in [4] and [7].

Suppose that $r = 1$. If $d = 1$ we showed in [5, Theorem 1.10] that almost always Stanley's Conjecture holds. It is the purpose of this note to complete the result for $d \geq 1$ in the next form.

Theorem *Suppose that $I \subset S$ is minimally generated by a squarefree monomial f of degree d, and a set E of squarefree monomials of degrees $\geq d+1$. Assume that $s \neq q+1$ and $\operatorname{sdepth}_S I/J = d+1$. Then $\operatorname{depth}_S I/J \leq d+1$.*

1 Proof of the Theorem

We may assume that $s < q+1$ because if $s > q+1$ then by [4] we get $\operatorname{depth}_S I/J \leq d+1$. Also we may suppose that $C \subset (f, B)$ by [5, Lemma 1.6]. Induct on $|E|$. Suppose that $E = \emptyset$. If $C = \emptyset$, then $\operatorname{depth}_S I/J \leq d+1$ by [5, Lemma 1.5]. Otherwise, let $c = fx_{n-1}x_n \in C$ and $I' = (B \setminus \{fx_{n-1}, fx_n\})$. In the exact sequence

$$0 \to I'/J \cap I' \to I/J \to I/J + I' \to 0$$

the last term has sdepth $d+2$ since $c \notin I' + J$ and so the first one has sdepth $\leq d+1$ by [6, Lemma 2.2] and even depth $\leq d+1$ by [3, Theorem 4.3]. Then the Depth Lemma gives $\operatorname{depth}_S I/J \leq d+1$.

Set $I_n = (B \setminus \{fx_n\})$, $J_n = I_n \cap J$. In the following exact sequence

$$0 \to I_n/J_n \to I/J \to I/(I_n + J) \to 0$$

the last term has sdepth depth $d+1$ since $[f, fx_n]$ is the whole poset of $(f)/(f) \cap (I_n + J)$ and $x_n \notin ((J + I_n) : f)$. If the first term has sdepth $= d+1$ then by [[3], Theorem 4.3.] we get depth $= d+1$ and applying Depth Lemma the conclusion follows. So we can assume that there exists a partition $\mathcal{P}_n$ of I_n/J_n with sdepth $d+2$. We may suppose that all intervals of $\mathcal{P}_{b_i}$ (as well as of other partitions which we will use) starting with a monomial v of degree $\geq d+2$ have the form $[v, v]$. In $\mathcal{P}_n$ we can't have the interval $[c, c]$, $c = fx_{n-1}x_n$, or the interval $[fx_{n-1}, c]$ because otherwise we can switch it with $[f, c]$ and get a partition of I/J with sdepth $d+2$. Thus we have in $\mathcal{P}_n$ the interval $[b_1, c]$, $b_1 \in E$. Switching the interval $[b_1, c]$ with the interval $[fx_n, c]$ we get a partition $P_{B_{b_1}}$ for $(B_{b_1})/J_{b_1}$ where $B_{b_1} = B \setminus \{b_1\}$ and $J_{b_1} = (B_{b_1}) \cap J$.

In $\mathcal{P}_{B_{b_1}}$ we have an interval $[\bar{c}, \bar{c}]$ because $s < q+1$. Thus there exists $b_2 \in E$ such that $\bar{c} \in (b_2)$. Note that $\bar{c} \notin (b_1)$ because otherwise we may replace in $\mathcal{P}_{B_{b_1}}$ the interval $[\bar{c}, \bar{c}]$ with the interval $[b_1, \bar{c}]$ and get a partition $\mathcal{P}_B$ for $(B)/(B) \cap J$ with sdepth $= d+2$. This leads to a contradiction because we may change in P_B two intervals like $[fx_i, fx_ix_j]$, $[fx_j, c']$ for some c' with $[f, fx_ix_j]$, $[c', c']$ and get a partition $\mathcal{P}$ for I/J with sdepth $d+2$.

Let $I_{b_2} = (f, E \setminus \{b_2\})$, $J_{b_2} = I_{b_2} \cap J$. In the following exact sequence

$$0 \to I_{b_2}/J_{b_2} \to I/J \to I/(I_{b_2} + J) \to 0$$

the last term has depth $\geq d+1$ because it is isomorphic with $(b_2)/(b_2) \cap (I_{b_2} + J)$. If $\operatorname{sdepth} I_{b_2}/J_{b_2} \leq d+1$ then the first term has depth $\leq d+1$ by the induction hypothesis, so by the Depth Lemma we get $\operatorname{depth} I/J \leq d+1$.

Now assume that sdepth $I_{b_2}/J_{b_2} \geq d+2$ and let P_{b_2} be a partition on I_{b_2}/J_{b_2} with sdepth $d+2$. In $\mathcal{P}_{b_2}$ we have the interval $[f, fx_ix_j], i, j \in [n] \setminus \operatorname{supp} f$. We have in $\mathcal{P}_{b_2}$ for all $b \in B \setminus \{b_2, fx_i, fx_j\}$ an interval $[b, c_b]$. We define $h_2 : (B \setminus \{b_2\}) \to C$ by $b \to c_b$ and $h_2(fx_i) = h_2(fx_j) = fx_ix_j$ and let $g_2 : \operatorname{Im} h_2 \to (B \setminus \{b_2\})$ defined by $c_b \to b$, $g_2(fx_ix_j) = fx_i$. Similarly we define h_1, g_1 for $\mathcal{P}_{B_{b_1}}$, that is h_1 is given by $b' \to c'$ if $\mathcal{P}_{B_{b_1}}$ has the interval $[b', c']$.

We want to show that we can build a partition $\mathcal{P}$ with sdepth $= d+2$ for I/J. Consider $a_0 = b_1$ and $c_{i-1} = h_2(a_{i-1}), a_i = g_1(c_{i-1}), i > 0$. The construction stops at step e if

1) $a_e = b_2$,

2) $c_e \notin \operatorname{Im} h_1$,

3)$a_e = fx_j$ after $a_u = fx_i, u < e$ already appeared. Note that here we have fixed $a_u = fx_i$.

In the first case we set $c_e = \bar{c}$ and we see that h_1 gives a bijection between $\{a_1, \ldots, a_e\}$ and $\{c_0, \ldots, c_{e-1}\}$. But h_1 also gives a bijection between $B \setminus \{b_1, a_1, \ldots, a_e\}$ and $C \setminus \{\bar{c}, c_0, \ldots, c_{e-1}\}$. Then the intervals $[a_p, c_p]$, $0 \leq p \leq e$ and the intervals $[g_1(\tilde{c}), \tilde{c}]$, $\tilde{c} \in C \setminus \{\bar{c}, c_0, \ldots, c_{e-1}\}$ and some other intervals starting with monomials of degree $\geq d+2$ give a partition $\mathcal{P}_B$ of $P_{B/B\cap J}$ with sdepth $\geq d+2$. As before this is a contradiction with sdepth $I/J = d+1$.

In the second case, as above we see that the intervals $[a_p, c_p]$, $0 \leq p \leq e$ and the intervals $[g_1(\tilde{c}), \tilde{c}]$, $\tilde{c} \in C \setminus \{\bar{c}, c_0, \ldots, c_{e-1}\}$ and some other intervals starting with monomials of degree $\geq d+2$ give a partition $\mathcal{P}_B$ of $P_{B/B\cap J}$ with sdepth $\geq d+2$. Contradiction.

In the last case we see as usual, that h_1 gives a bijection between $\{a_1, \ldots, a_e\}$ and $\{c_0, \ldots, c_{e-1}\}$. But h_1 also gives a bijection between $B \setminus \{b_1, a_1, \ldots, a_e\}$ and $C \setminus \{c_0, \ldots, c_{e-1}\}$. Then the intervals $[a_p, c_p]$, $0 \leq p \leq e-1, p \neq u$ and the intervals $[f, c_u], [g_1(\tilde{c}), \tilde{c}]$, $\tilde{c} \in C \setminus \{c_0, \ldots, c_{e-1}\}$ and some other intervals starting with monomials of degree $\geq d+2$ give the partition $\mathcal{P}$ of $P_{I/J}$ with sdepth $\geq d+2$. Contradiction.

Example 1.1 Let $n = 5$, $I = (x_1x_2, x_3x_4x_5)$ and $J = (x_1x_2x_3x_5, x_1x_2x_4x_5)$. We see that we have sdepth $I/J = d+1 = 3$ and $B = \{x_1x_2x_3, x_1x_2x_4, x_1x_2x_5, x_3x_4x_5\}$ and $C = \{x_1x_2x_3x_4, x_1x_3x_4x_5, x_2x_3x_4x_5\}$ so we are in the case $s = q+1$. We can get depth $I/J \leq 3$ by using [5, Lemma 1.5] for $u = x_1x_2x_5$.

Remark 1.2 If in the above example change just one monomial from the generators of J, namely take $J = (x_1x_2x_4x_5, x_2x_3x_4x_5)$ then we have $\operatorname{sdepth}_S I/J = 4$ because the partition induced by the intervals $[x_1x_2, x_1x_2x_3x_4]$, $[x_3x_4x_5, x_1x_3x_4x_5]$, $[x_1x_2x_5, x_1x_2x_3x_5]$ has sdepth $d+2 = 4$. Also we have $\operatorname{depth}_S I/J = 4$.

A question is hinted by the following example.

Example 1.3 Let $n = 5$, $I = (x_1, x_2x_3, x_2x_4, x_2x_5, x_3x_4)$ and J the ideal generated by all squarefree monomials of I of degrees 4. Then $E = \{x_2x_3, x_2x_4, x_2x_5, x_3x_4\}$, $f = x_1$, $B = \{x_1x_2, x_1x_3, x_1x_4, x_1x_5, E\}$, $C = \{x_1x_2x_3, x_1x_2x_4, x_1x_2x_5, x_1x_3x_4, x_2x_3x_4, x_2x_3x_5, x_2x_4x_5\}$. Thus $s = 8 = q+1$. We see that $\operatorname{sdepth}_S I/J = d+2 = 3$ but $\operatorname{depth}_S I/J = d+1 = 2$. Note that here

$$C \subset (\cup_{a \in E} C \cap (f) \cap (a)) \cup (\cup_{a,a' \in E, a \neq a'} C \cap (a) \cap (a'),$$

a condition which might imply always $\operatorname{depth}_S I/J \leq d+1$, the inequality being not true for sdepth.

References

[1] J. Herzog, M. Vladoiu, X. Zheng, *How to compute the Stanley depth of a monomial ideal,* J. Algebra, **322** (2009), 3151-3169.

[2] D. Popescu, *Depth and minimal number of generators of square free monomial ideals*, An. St. Univ. Ovidius, Constanta, 19 (2), (2011), 187-194.

[3] D. Popescu, *Depth of factors of square free monomial ideals*, to appear in Proceedings AMS, arXiv:AC/1110.1963.

[4] D. Popescu, *Upper bounds of depth of monomial ideals*, Journal of Commutative Algebra, **5**, 2013, arXiv:AC/1206.3977.

[5] D. Popescu, A. Zarojanu, *Depth of some square free monomial ideals*, Bull. Math. Soc. Sci. Math. Roumanie, **56(104)**, 2013,117-124.

[6] A. Rauf, *Depth and Stanley depth of multigraded modules*, Comm. Algebra, **38** (2010),773-784.

[7] Y.H. Shen, *Lexsegment ideals of Hilbert depth 1*, (2012), arxiv:AC/1208.1822v1.

[8] R. P. Stanley, *Linear Diophantine equations and local cohomology*, Invent. Math. **68** (1982) 175-193.

Received: 28.01.2013,

Accepted: 24.04.2013.

Dorin Popescu, Simion Stoilow Institute of Mathematics of Romanian Academy, Research unit 5,
P.O.Box 1-764, Bucharest 014700, Romania
E-mail: `dorin.popescu@imar.ro`

Andrei Zarojanu, Faculty of Mathematics and Computer Sciences, University of Bucharest, Str.
Academiei 14, Bucharest, Romania,
and
Simion Stoilow Institute of Mathematics of Romanian Academy, Research group of the project
ID-PCE-2011-1023, P.O.Box 1-764, Bucharest 014700, Romania
E-mail: `andrei_zarojanu@yahoo.com`