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Calabi-Yau algebras and their deformations

by
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Dedicated to Professors Toma Albu and Constantin Năstăsescu on the occasion of their 70th birthdays

Abstract

This is a survey of our joint works on graded Calabi-Yau algebras, Calabi-Yau Hopf algebras and their PBW-deformations.

Key Words: Calabi-Yau algebra, Artin-Schelter regular algebra, Hopf algebra, PBW-deformation.
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Introduction

Calabi-Yau algebras appeared naturally in theoretic physics [KS, Gin1]. They seek wide applications in many branches of mathematics, say, noncommutative geometry [Gin2, Bo1, Bo2, Br, VdB2, BP], and representation theory [Ke1, Ke2, Ke3, ES, BS, IR, CZ, KR, IR]. In this survey, we focus on our works on Koszul Calabi-Yau algebras, Calabi-Yau Hopf algebras and their deformations.

Let $\Bbbk$ be an algebraically closed field with characteristic zero, and let A be a $\Bbbk$-algebra. A is called a *Calabi-Yau* algebra of dimension d [Gin1] if

(i) A is homologically smooth; that is, A has a finite resolution of finitely generated projective A-bimodules;

(ii) $\mathrm{Ext}^i_{A^e}(A, A\otimes A) = 0$ if $i \neq d$ and $\mathrm{Ext}^d_{A^e}(A, A\otimes A) \cong A$ as A-bimodules, where $A^e = A\otimes A^{op}$ is the enveloping algebra of A.

Let $A = \oplus_{n\in\mathbb{Z}} A_n$ be a $\mathbb{Z}$-graded algebra, and $M = \oplus_{i\in\mathbb{Z}} M_i$ be a graded A-bimodule. For any integer l, $M(l)$ is a graded A-bimodule whose degree i component is $M(l)_i = M_{i+l}$. A graded algebra A is called a *graded Calabi-Yau* algebra of dimension d if (i) A has a finite resolution of finitely generated graded projective A-bimodules, and (ii) $\mathrm{Ext}^i_{A^e}(A, A\otimes A) = 0$ when $i \neq d$ and $\mathrm{Ext}^d_{A^e}(A, A\otimes A) \cong A(l)$ as graded A-bimodules for some integer l.

The survey is organized as follows. In Section 1, we discuss the Calabi-Yau property of N-Koszul algebras. It is well known that a positively graded Calabi-Yau algebra A with $A_0 = \Bbbk$ is Artin-Schelter regular. We summarize some criteria for an N-Koszul Artin-Schelter regular algebra to be graded Calabi-Yau, and provide a method to construct graded Calabi-Yau algebras from known Artin-Schelter regular algebras.

In Section 2, we mainly discuss the Calabi-Yau property of pointed Hopf algebras. We give a necessary and sufficient condition for an Artin-Schelter regular Hopf algebra to be Calabi-Yau. It is relatively easy to determine the Calabi-Yau property of cocommutative Hopf algebras since a cocommutative pointed Hopf algebra is isomorphic to a skew group algebra of a universal enveloping algebra with a group algebra. When the pointed Hopf algebra under consideration is noncocommutative, we are only able to determine the Calabi-Yau property of pointed Hopf algebras of finite Cartan type.

In Section 3, we discuss the PBW-deformations of Koszul Calabi-Yau algebras. We summarize some criterion theorems for a PBW-deformation of a Koszul Calabi-Yau algebra to be again Calabi-Yau. In particularly, a PBW-deformation of a polynomial algebra is exactly a Sridharan enveloping algebra of a finite Lie algebra. We provide some equivalent conditions for a Sridharan enveloping algebra to be Calabi-Yau, and we are able to classify all the Calabi-Yau Sridharan enveloping algebras of dimension 3.

1 Koszul Calabi-Yau algebras

In this section, we always assume that $A = \oplus_{n\in\mathbb{Z}} A_n$ is a positively graded algebra with $A_0 = \Bbbk$ and $\dim A_n < \infty$ for all $n > 0$. Let $E(A) = \oplus_{i\geq 0} \operatorname{Ext}^i_A({}_A\Bbbk, {}_A\Bbbk)$ be the space of extensions of the trivial graded module ${}_A\Bbbk$. Endowed with the Yoneda product, $E(A)$ is a positively graded algebra, and is usually called *the Yoneda Ext-algebra* of A.

Recall that A is called an *Artin-Schelter regular* algebra [AS] if A has finite global dimension d, $\operatorname{Ext}^i_A(\Bbbk_A, A) = \operatorname{Ext}^i_A({}_A\Bbbk, A) = 0$ for all $i \neq d$ and $\operatorname{Ext}^d_A(\Bbbk_A, A) = \operatorname{Ext}^d_A({}_A\Bbbk, A) = \Bbbk$. It is well known that a graded Calabi-Yau algebra A is Artin-Schelter regular [BM]. It is an interesting question to find graded Calabi-Yau algebras amongst known Artin-Schelter regular algebras. We do this in the view of Koszul algebras. Given an integer $N \geq 2$, a positively graded algebra A is called an *N-Koszul algebra* [Be1, YZ] if the trivial graded module ${}_A\Bbbk$ has a graded projective resolution

$$\cdots \longrightarrow P^{-i} \longrightarrow P^{-i+1} \longrightarrow \cdots \longrightarrow P^0 \longrightarrow {}_A\Bbbk \longrightarrow 0,$$

such that the graded projective module P^{-i} is generated in degree $\delta(i)$, where

$$\delta(i) = \begin{cases} \frac{i}{2}N, & \text{if } i \text{ is even;} \\ \frac{i-1}{2}N + 1, & \text{if } i \text{ is odd.} \end{cases}$$

In the case that $N = 2$, an N-Koszul algebra is usually called a *Koszul* algebra which was introduced by Priddy about forty years ago in [Pr]. An N-Koszul algebra is generated in degree 1. So, we may write A as a quotient algebra of a tensor algebra, say $A = T(V)/(R)$ where V is a finitely dimensional vector space, $T(V)$ is the tensor algebra of V, $R \subseteq V^{\otimes N}$ is a subspace and (R) is the two-sided ideal of $T(V)$ generated by R. Associated to A, there is a homogeneous dual algebra $A^! = T(V^*)/(R^\perp)$ where V^* is the dual space of V and $R^\perp \subseteq (V^*)^{\otimes N}$

is the orthogonal complement of R in $(V^*)^{\otimes N}$. Clearly, $(A^!)^! \cong A$. As a graded vector space, the Yoneda Ext-algebra $E(A) = \oplus_{i\geq 0} A^!_{\delta(i)}$ [BM, HL]. If A is a Koszul algebra, then $E(A) \cong A^!$ as graded algebras [BGS], and in this case $A^!$ is also a Koszul algebra.

If an N-Koszul algebra A is Artin-Schelter regular, then its Yoneda Ext-algebra $E(A)$ is a graded Frobenius algebra [Sm1, BM]. Recall that a finitely dimensional positively graded algebra E is called a *graded Frobenius* algebra if there is an integer d and a homogeneous non-degenerate bilinear form $\langle -,-\rangle : E \times E \longrightarrow \Bbbk(d)$ such that $\langle ab, c\rangle = \langle a, bc\rangle$ for all homogeneous elements $a, b, c \in E$. For a graded Frobenius algebra E there is a unique graded algebra automorphism $\varphi : E \to E$, called the *Nakayama automorphism of* E, such that $\langle a, b\rangle = \langle \varphi(b), a\rangle$ for all homogeneous elements $a, b \in E$. A graded Frobenius algebra E is called a *graded symmetric* algebra if $\langle a, b\rangle = (-1)^{|a||b|}\langle b, a\rangle$ for all homogeneous elements $a, b \in E$, where $|a|$ and $|b|$ are the degree of a and b respectively.

The Calabi-Yau property of an N-Koszul algebra is equivalent to certain symmetric property on its Yoneda Ext-algebra.

Proposition 1. *[HVZ2] An N-Koszul algebra A is a graded Calabi-Yau algebra if and only if its Yoneda Ext-algebra $E(A)$ is a graded symmetric algebra.*

In view of this property of N-Koszul Calabi-Yau algebras, we may construct new graded Calabi-Yau algebras from known Artin-Schelter regular algebras by the traditional methods, say skew polynomial algebras or more generally Ore extensions.

For an Artin-Schelter regular algebra, we have the following result, which was proved by Van den Bergh in [VdB1] for Koszul algebras and by Berger and Marconnet in [BM, Proof of Theorem 6.3] for general N-Koszul algebras.

Theorem 1. *Let A be an N-Koszul Artin-Schelter regular algebra of global dimension d. Let φ be the Nakayama automorphism of $E(A)$, and ϕ the automorphism of A induced by φ. Then $\mathrm{Ext}^i_{A^e}(A, A\otimes A) = 0$ for $i \neq d$, and*

$$\mathrm{Ext}^d_{A^e}(A, A\otimes A) \cong {}_1A_\xi(\delta(d)),$$

where ξ is the automorphism of A defined by $\xi(a) = (-1)^{|a|(d+1)}\phi^{-1}(a)$ for all homogeneous element $a \in A$, and ${}_1A_\xi$ is the A-bimodule with the regular A-action on the left side and the right A-action defined by $x\cdot a = x\xi(a)$ for all $x, a \in A$.

The automorphism ξ in the theorem above is usually called the *Nakayama automorphism* of A.

Let A be a Koszul algebra, and σ a graded automorphism of A. Let $B = A[z;\sigma]$ be the graded skew polynomial algebra with coefficients in A. Clearly, B is also a Koszul algebra. The Yoneda Ext-algebra of B can be presented as follows. Let $E = \Bbbk \oplus E_1 \oplus E_2 \oplus \cdots$ be a positively graded algebra, and M a graded E-bimodule. The trivial extension of E by M is defined to be the graded algebra $\Gamma(E, M) = E \oplus M$ with the product $(x_1, m_1)*(x_2, m_2) = (x_1x_2, x_1\cdot m_2 + m_1\cdot x_2)$ for $x_i \in E$ and $m_i \in M$. If ψ and τ are two automorphisms of E, then the notion ${}_\psi E_\tau$ is the E-bimodule defined by $a\cdot x\cdot b = \psi(a)x\tau(b)$ for $a, x, b \in E$.

Proposition 2. *[HVZ5] Let A be a Koszul algebra, σ a graded algebra automorphism of A and $B = A[z;\sigma]$. Then $E(B) \cong \Gamma(A^!, {}_\epsilon A^!_\psi(-1))$, where $\psi = (\sigma^{-1})^!$ is the automorphism of $A^!$*

induced by σ^{-1} *and* ϵ *is the automorphism of* $A^!$ *defined by* $\epsilon(x) = (-1)^{|x|}x$ *for all homogeneous element* $x \in A^!$.

From Propositions 1 and 2 and Theorem 1, we obtain the following result.

Theorem 2. *[HVZ5] Let* A *be a Koszul Artin-Schelter regular algebra of global dimension* d *with the Nakayama automorphism* ξ*. Then the skew polynomial algebra* $B = A[z;\xi]$ *is a Calabi-Yau algebra of dimension* $d+1$.

We next consider graded Calabi-Yau algebras of lower global dimensions. Let A be a positively graded algebra which is generated in degree 1. If A is an Artin-Schelter regular algebra of global dimension 2, then $A \cong \Bbbk\langle x_1,\dots,x_n\rangle/(f)$, where $\Bbbk\langle x_1,\dots,x_n\rangle$ is the free algebra generated by $x_1,\dots,x_n$, and (f) is the two-sided ideal generated by the element f. The element f is presented as follows: $f = (x_1,\dots,x_n)M(x_1,\dots,x_n)^t$, where M is an $n\times n$ invertible matrix with entries in $\Bbbk$.

Proposition 3. *[HVZ3] Let* $A = \Bbbk\langle x_1,\dots,x_n\rangle/(f)$ *and let* M *be an* $n\times n$ *invertible matrix. Then we have* $\operatorname{Ext}^i_{A^e}(A, A\otimes A) = 0$ *for* $i \neq 2$*, and*

$$\operatorname{Ext}^2_{A^e}(A, A^e) \cong {}_1A_\xi(-2),$$

where ξ *is an automorphism defined by* $\xi(y) = -(x_1,\dots,x_n)M^tM^{-1}\mathbf{k}^t$ *in which* $y = k_1x_1 + \cdots + k_nx_n$ *and* $\mathbf{k} = (k_1,\dots,k_n)$.

As a corollary, we have

Corollary 1. *[Zh, Be2, Bo1] Let* $A = \Bbbk\langle x_1,\dots,x_n\rangle/(f)$ *where* $f = (x_1,\dots,x_n)M(x_1,\dots,x_n)^t$ *and* M *is an* $n\times n$ *matrix. Then* A *is Calabi-Yau of dimension 2 if and only if* M *is invertible and anti-symmetric.*

If A is an Artin-Schelter regular algebra of global dimension 3, then A is an N-Koszul algebra. A is isomorphic to a quotient algebra $\Bbbk\langle x_1,\dots,x_n\rangle/(r_1,\dots,r_n)$ generated by $x_1,\dots,x_n$ subject to the relations $r_1,\dots,r_n \in \Bbbk\langle x_1,\dots,x_n\rangle$ of degree N. Let V be the vector space spanned by $x_1,\dots,x_n$, and R be the vector space spanned by $r_1,\dots,r_n$. That A is of global dimension 3 implies $\dim(V\otimes R\bigcap R\otimes V) = 1$. Fix a basis z of $V\otimes R\bigcap R\otimes V$. As originally suggested in [AS], the element $z\in R\otimes V$ can be written as

$$z = \mathbf{r}Q^{(1)}\mathbf{x}^t,$$

where $Q^{(1)}$ is an $n\times n$ matrix, $\mathbf{x} = (x_1,\dots,x_n)$ and $\mathbf{r} = (r_1,\dots,r_n)$. On the other hand, since $z\in V\otimes R$, there is an $n\times n$ matrix $Q^{(2)}$ with

$$z = \mathbf{x}Q^{(2)}\mathbf{r}^t.$$

Proposition 4. *[HVZ3] With the notions as above. We have*

(i) *the matrices* $Q^{(1)}$ *and* $Q^{(2)}$ *are invertible;*

(ii) $\mathrm{Ext}^3_{A^e}(A, A\otimes A) \cong {}_1A_\xi(N+1)$, *where* ξ *acts on generators of* A *by*

$$\xi(x_1,\ldots,x_1) = (x_1,\ldots,x_n)Q^{(1)^t}Q^{(2)^{-1}}.$$

(iii) A *is Calabi-Yau if and only if* $Q^{(1)} = Q^{(2)^t}$.

Part of graded Calabi-Yau algebras of dimension 3 can be obtained by Ore extensions from Calabi-Yau algebras of dimension 2. Now let A be a graded Calabi-Yau algebra of dimension 2. By Corollary 1, $A \cong \Bbbk\langle x_1,\ldots,x_n\rangle/(f)$ for some $n \geq 2$ and $f = (x_1,\ldots,x_n)M(x_1,\ldots,x_n)^t$, where M is an $n\times n$ invertible anti-symmetric matrix. Note that any invertible anti-symmetric matrix is cogredient to a *standard form*:

$$\Omega = \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 1 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ 0 & \cdots & -1 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ -1 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Since the algebra defined by the relation $f = (x_1,\ldots,x_n)M(x_1,\ldots,x_n)^t$ is isomorphic to the algebra defined by the relation $r = (x_1,\ldots,x_n)\Omega(x_1,\ldots,x_n)^t$, we may assume that M itself is standard. Let δ be a graded derivation of the free algebra $\Bbbk\langle x_1,\ldots,x_n\rangle$ of degree 1. If $\delta(f) = 0$, then δ induces a graded derivation $\overline{\delta}$ on A. Let $B = A[z;\overline{\delta}]$ be the Ore extension of A defined by the graded derivation $\overline{\delta}$.

Theorem 3. *[HVZ4] Let* M *be an* $n\times n$ *standard anti-symmetric matrix for some* $n \geq 2$*. Put* $f = (x_1,\ldots,x_n)M(x_1,\ldots,x_n)^t$ *and* $A = \Bbbk\langle x_1,\ldots,x_n\rangle/(f)$*. With the notions as above.*

(i) B *is a graded Calabi-Yau algebra of dimension 3;*

(ii) *Write* $\delta(x_i) = \sum_{s,t=1}^{n} k^i_{st}x_i\otimes x_j$ *for all* $i = 1,\ldots,n$*. Assume that there is an integer* j *such that* $k^i_{jj} = 0$ *for all* $i = 1,\ldots,n$*. Then* B *is a graded coherent algebra.*

The class of algebras obtained in the theorem above includes the graded Calabi-Yau algebras studied by Smith in [Sm2], which were constructed from the octonions. The theorem above can be viewed as a generalization of [Sm2, Theorem 5.7 and Prop. 9.2].

2 Calabi-Yau pointed Hopf algebras

Let H be a Noetherian Hopf algebra. Similar to the graded case, one can introduce the Artin-Schelter regularity for Hopf algebras. H is said to be *Artin-Schelter regular* [BZ] if (i) the global dimension of H is finite, say, d, (ii) $\dim \mathrm{Ext}^d_H({}_H\Bbbk, {}_HH) = 1$ and $\mathrm{Ext}^i_H({}_H\Bbbk, {}_HH) = 0$ for all $i \neq d$, where ${}_H\Bbbk$ is the trivial H-module defined by the counit of H, and (iii) the right version of (ii) is satisfied. For an Artin-Schelter regular Hopf algebra, a nonzero element in the one-dimensional right H-module $\mathrm{Ext}^d_H({}_H\Bbbk, {}_HH)$ is called a *right homological integral* of H, and a nonzero element in the one-dimensional left H-module $\mathrm{Ext}^d_H(\Bbbk_H, H_H)$ is called a *left homological integral* of H [LWZ]. The homological integral was proved to be a powerful tool to study infinite dimensional Hopf algebras [LWZ, BZ]. If the left one-dimensional H-module $\mathrm{Ext}^d_H(\Bbbk_H, H_H)$ is isomorphic to the trivial module ${}_H\Bbbk$ (or equivalently, the right H-module $\mathrm{Ext}^d_H({}_H\Bbbk, {}_HH)$ is isomorphic to the trivial module $\Bbbk_H$), then H is said to be *unimodular* [LWZ].

For a Noetherian Hopf algebra, we have the following result.

Theorem 4. *[HVZ1] Let H be a Noetherian Hopf algebra with antipode S. Then H is Calabi-Yau of dimension d if and only if*

(i) *H is Artin-Schelter regular of global dimension d and unimodular,*

(ii) *S^2 is an inner automorphism of H.*

With the help of the above theorem, we may find out Calabi-Yau Hopf algebras from Noetherian pointed Hopf algebras.

Let us firstly consider the cocommutative pointed Hopf algebras. It is well known that a cocommutative Hopf algebra (note that $\Bbbk$ is algebraically closed) is isomorphic to a smash product of a universal enveloping algebra of a Lie algebra with a group algebra. We have the following result for cocommutative pointed Hopf algebra.

Theorem 5. *[HVZ1] Let $\mathfrak{g}$ be a finite dimensional Lie algebra, and $G \subseteq Aut_{Lie}(\mathfrak{g})$ a finite group. Then the skew group algebra $U(\mathfrak{g})\#\Bbbk G$ is a Calabi-Yau Hopf algebra if and only if $G \subseteq SL(\mathfrak{g})$ and the Lie algebra $\mathfrak{g}$ is unimodular, that is, for any $x \in \mathfrak{g}$, $\mathrm{tr}(\mathrm{ad}_{\mathfrak{g}}(x)) = 0$.*

For the cocommutative Calabi-Yau Hopf algebra of lower dimensions, we have the following results.

Theorem 6. *[HVZ1] Let H be a cocommutative Hopf algebra such that it has finite group-like elements and the subspace of its primitive elements is finite dimensional. Then*

(i) *H is Calabi-Yau of dimension 2 if and only if there is a finite group G and a group map $\nu : G \to SL(2, \Bbbk)$ such that $H \cong \Bbbk[x, y]\#\Bbbk G$, where the G-action on $\Bbbk[x, y]$ is given by ν.*

(ii) *H is Calabi-Yau of dimension 3 if and only if $H \cong U(\mathfrak{g})\#\Bbbk G$, where $\mathfrak{g}$ is one of the 3-dimensional Lie algebras listed below and G is a finite group with a group morphism $\nu : G \to Aut_{Lie}(\mathfrak{g})$ such that $\mathrm{im}(\nu)$ is also a subgroup of $SL(\mathfrak{g})$:*

   (a) *The 3-dimensional simple Lie algebra $\mathfrak{sl}(2, \Bbbk)$;*

   (b) *$\mathfrak{g}$ has a basis $\{x, y, z\}$ such that $[x, y] = y$, $[x, z] = -z$ and $[y, z] = 0$.*

   (c) *The Heisenberg algebra, that is; $\mathfrak{g}$ has a basis $\{x, y, z\}$ such that $[x, y] = z$ and $[x, z] = [y, z] = 0$;*

   (d) *The 3-dimensional abelian Lie algebra.*

We next consider the Calabi-Yau property of the noncocommutative pointed Hopf algebras. We restrain ourselves to pointed Hopf algebras of finite Cartan type. We recall some notions and terminology from [AnS].

- Γ is a free abelian group of finite rank s;
- $(a_{ij}) \in \mathbb{Z}^{n\times n}$ is a Cartan matrix of finite type. $diag(d_1, \cdots, d_n)$ is a diagonal matrix of positive integers such that $d_i a_{ij} = d_j a_{ji}$, which is minimal with this property;
- $\mathcal{X}$ is the set of connected components of the Dynkin diagram corresponding to the Cartan matrix (a_{ij}). If $1 \leq i, j \leq n$, then $i \sim j$ means that they belong to the same connected component;
- $(q_I)_{I\in\mathcal{X}}$ is a family of elements in $\Bbbk$ which are not roots of 1;
- Choose elements $g_1, \cdots, g_n \in \Gamma$ and characters $\chi_1, \cdots, \chi_n \in \hat{\Gamma}$ such that

$$\langle \chi_j, g_i\rangle\langle \chi_i, g_j\rangle = q_I^{d_i a_{ij}}, \langle \chi_i, g_i\rangle = q_I^{d_i}, \text{ for all } 1 \leq i < j \leq n,\ i \in I.$$

Set $\mathcal{D} = \mathcal{D}(\Gamma, (a_{ij})_{1\leq i,j\leq n}, (g_i)_{1\leq i\leq n}, (\chi_i)_{1\leq i\leq n})$. A *linking datum* $\lambda = (\lambda_{ij})$ for $\mathcal{D}$ is a collection $(\lambda_{ij})_{1\leq i<j\leq n, i\nsim j} \in \{0, 1\}$ such that $\lambda_{ij} = 0$ if $g_i g_j = 1$ or $\chi_i\chi_j \neq \varepsilon$. Write the datum $\lambda = 0$, if $\lambda_{ij} = 0$ for all $1 \leq i < j \leq n$.

The datum $(\mathcal{D}, \lambda) = (\Gamma, (a_{ij}), (g_i), (\chi_i), (\lambda_{ij}))$ is called a *generic datum of finite Cartan type* for the group Γ.

Given a generic datum $(\mathcal{D}, \lambda)$ of finite Cartan type. Denote by $U(\mathcal{D}, \lambda)$ the algebra with generators $x_1, \cdots, x_n; z_1^{\pm 1}, \cdots, z_s^{\pm 1}$ and relations

$$\begin{array}{ll}
z_k^{\pm1} z_l^{\pm1} = z_l^{\pm1} z_k^{\pm1}, \quad z_k^{\pm1} z_k^{\mp1} = 1, & 1 \leq k, l \leq s,\\
z_k x_i = \chi_i(z_k) x_i z_k, & 1 \leq i \leq n,\ 1 \leq k \leq s,\\
(\mathrm{ad}_c x_i)^{1-a_{ij}}(x_j) = 0, & 1 \leq i \neq j \leq n,\ i \sim j,\\
x_i x_j - \chi_j(g_i) x_j x_i = \lambda_{ij}(1 - g_i g_j), & 1 \leq i < j \leq n,\ i \nsim j,
\end{array}$$

where ad_c is the braided adjoint representation (for details, see [AnS, Sect. 1]).

Theorem 7. *[AnS] Let $(\mathcal{D}, \lambda) = (\Gamma, (a_{ij}), (g_i), (\chi_i), (\lambda_{ij}))$ be a generic datum of finite Cartan type. The algebra $U(\mathcal{D}, \lambda)$ defined as above is a pointed Hopf algebra with comultiplication defined by*

$$\Delta(g_k) = g_k \otimes g_k,\ \ \Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i,\ \ 1 \leq k \leq s, 1 \leq i \leq n.$$

The Hopf algebra $U(\mathcal{D}, \lambda)$ is Noetherian [YZ], and the Calabi-Yau property of $U(\mathcal{D}, \lambda)$ is shown in the following theorem.

Theorem 8. *[YZ] Let $(\mathcal{D}, \lambda)$ be a generic datum of finite Cartan type. The pointed Hopf algebra $U(\mathcal{D}, \lambda)$ is Calabi-Yau if and only if $\prod_{i=1}^{p} \chi_{\upsilon_i} = \varepsilon$ and S^2 is inner, where p is the number of the positive roots of the Cartan matrix, υ_i's are the positive roots and S is the antipode of $U(\mathcal{D}, \lambda)$.*

Remark. (i) *The Calabi-Yau property of a quantum enveloping algebra was already shown in [Ch].*

(ii) *Calabi-Yau pointed Hopf algebras of finite Cartan type of dimensions less than 5 were classified in [YZ].*

3 PBW deformations

Let $A = \oplus_{i\geq 0} A_i$ be a graded algebra with $A_0 = \Bbbk$. A *PBW-deformation* of A is a filtered algebra U with an ascending filtration $0 \subseteq F_0U \subseteq F_1U \subseteq F_2U \subseteq \cdots$ such that the associated graded algebra $gr(U)$ is isomorphic to A.

In this section, we only consider PBW-deformations of Koszul algebras. Let $A = T(V)/(R)$ be a Koszul algebra. A PBW-deformation U of A is determined by two linear maps $\nu : R \to V$ and $\theta : R \to \Bbbk$ in sense that $U \cong T(V)/(r - \nu(r) - \theta(r) : r \in R)$, where the linear maps ν and θ satisfy *Jacobian type conditions* (see [BG, PP]):

$$\begin{aligned} (\nu\otimes 1 - 1\otimes\nu)(R\otimes V\cap V\otimes R) &\subseteq R \\ [\nu(\nu\otimes 1 - 1\otimes\nu) - (\theta\otimes 1 - 1\otimes\theta)](R\otimes V\cap V\otimes R) &= 0 \\ \theta(\nu\otimes 1 - 1\otimes\nu)(R\otimes V\cap V\otimes R) &= 0. \end{aligned}$$

If $\theta = 0$, then U is called an *augmented* PBW-deformation of A.

Since A is a Koszul algebra, the Yoneda Ext-algebra of A is also a Koszul algebra. Moreover, $E(A) \cong A^!$. Henceforth, we identify $E(A)$ with $A^!$. Recall $A^! = T(V^*)/(R^\perp)$. Hence $A^!_1 = V^*$ and $A^!_2 = R^*$. So, we may view θ as an element in $A^!_2$. By the Jacobian type conditions above, the dual map $\nu^* : V^* \to R^*$ induces a graded derivation ∂ on $A^!$, so that the triple $(A^!, \partial, \theta)$ is a *curved differential graded (DG) algebra* [PP], that is, the identity $\partial^2(x) = [\theta, x]$ holds for all $x \in A^!$. We call $(A^!, \partial, \theta)$ the curved DG algebra *dual* to the PBW-deformation U of A. If U is an augmented PBW-deformation of A, then $(A^!, \partial)$ is a usual DG algebra.

Proposition 5. *[HVZ3] Let $A = T(V)/(R)$ be a Koszul Artin-Schelter regular algebra of global dimension d, and let ξ be the Nakayama automorphism of A (see Sect. 1). Assume that $\{x_1, \dots, x_n\}$ is a basis of V, and $\{x_1^*, \dots, x_n^*\}$ is the dual basis of V^*.*

Let $U = T(V)/(r - \nu(r) - \theta(r) : r \in R)$ be a PBW-deformation of A, and let $(A^!, \partial, \theta)$ be the curved DG algebra dual to U. Choose a basis ϖ of $A^!_d$, and assume that $\{\omega_1, \dots, \omega_n\}$ is the basis of $A^!_{d-1}$ such that $x_i^*\omega_j = \delta^i_j\varpi$. Assume further $\partial(\omega_i) = \lambda_i\varpi$ for all $i = 1, \dots, n$. Then $\mathrm{Ext}^i_{U^e}(U, U\otimes U) = 0$ for $i \neq d$, and

$$\mathrm{Ext}^d_{U^e}(U, U\otimes U) \cong {}_1U_\zeta,$$

where the automorphism ζ acts on the generator by $\zeta(x_i) = \xi(x_i) + \lambda_i$.

From the above proposition, we have the following result.

Theorem 9. *[HVZ3] Let $A = T(V)/(R)$ be a Koszul Calabi-Yau algebra of dimension d. Assume that U is an augmented PBW-deformation of A, and that $(A^!, \partial)$ is the DG algebra dual to U. Then the following are equivalent:*

(i) *U is a Calabi-Yau algebra;*

(ii) *$E(U) = \oplus_{i=1}^d \mathrm{Ext}^i_U(\Bbbk, \Bbbk)$ is a graded symmetric algebra;*

(iii) $\partial(A^!_{d-1}) = 0$.

Remark 1. A similar result also appeared in [WZ] under the hypothesis that A is Noetherian.

The Calabi-Yau property of a nonaugmented PBW-deformation is sometimes equivalent to that of an augmented one.

Theorem 10. *[HVZ3, HZ] Let $A = T(V)/(R)$ be a Koszul Calabi-Yau algebra of dimension d. Assume that both $U = T(V)/(r - \nu(r) - \theta(r) : r \in R)$ and $U' = T(V)/(r - \nu(r) : r \in R)$ are PBW-deformations of A. If U' is Calabi-Yau, then so is U.*

Conversely, if U is Calabi-Yau and A is a domain, then U' is Calabi-Yau.

We next consider PBW-deformations of some specific graded Calabi-Yau algebra. Let A be a Koszul Artin-Schelter regular algebra of global dimension d. Let ξ be the Nakayama automorphism of A (see Sect. 1). Then the skew polynomial algebra $A[z;\xi]$ is a Koszul Calabi-Yau algebra of global dimension $d+1$ (see, Theorem 2). Let $U = T(V)/(r-\nu(r)-\theta(r) : r \in R)$ be a PBW-deformation of A, and let ζ be the automorphism defined in Proposition 5. Then we have

Theorem 11. *[HVZ5] Keep the notation as above.*

(i) *$U[z;\zeta]$ is a PBW-deformation of $A[z;\xi]$;*

(ii) *If, further, A is Calabi-Yau, then $U[z;\zeta]$ is also Calabi-Yau.*

If $A = \Bbbk[x_1, \ldots, x_n]$ is the polynomial algebra generated by variables $x_1, \ldots, x_n$, then a PBW-deformation of A is equivalent to a Sridharan enveloping algebra of an n-dimensional Lie algebra. We recall from [Sr] the definition of Sridharan enveloping algebra. Let $\mathfrak{g}$ be a finite dimensional Lie algebra, and let $f \in Z^2(\mathfrak{g}, \Bbbk)$ be a 2-cocycle, that is; $f : \mathfrak{g} \times \mathfrak{g} \to \Bbbk$ such that

$$f(x,x) = 0 \text{ and } f(x,[y,z]) + f(y,[z,x]) + f(z,[x,y]) = 0$$

for all $x, y, z \in \mathfrak{g}$. The *Sridharan enveloping algebra* of $\mathfrak{g}$ is defined to be the quotient algebra $U_f(\mathfrak{g}) = T(\mathfrak{g})/I$, where I is the two-sided ideal of $T(\mathfrak{g})$ generated by the elements

$$x \otimes y - y \otimes x - [x,y] - f(x,y), \qquad \text{for all } x, y \in \mathfrak{g}.$$

The following result says that the Calabi-Yau property of a Sridharan enveloping algebra is independent of the choice of the 2-cocycle f.

Theorem 12. *[HVZ1] Let $\mathfrak{g}$ be a finite dimensional Lie algebra. For an arbitrary 2-cocycle $f \in Z^2(\mathfrak{g}, \Bbbk)$, the following are equivalent.*

(i) *The Sridharan enveloping algebra $U_f(\mathfrak{g})$ is Calabi-Yau of dimension d.*

(ii) *The universal enveloping algebra $U(\mathfrak{g})$ is Calabi-Yau of dimension d.*

(iii) *$\dim \mathfrak{g} = d$ and $\mathfrak{g}$ is unimodular.*

By the theorem above, we are able to classify all the Calabi-Yau Sridharan enveloping algebras of dimension 3.

Theorem 13. *[HVZ1] A Sridharan enveloping algebra $U_f(\mathfrak{g})$ is Calabi-Yau of dimension 3 if and only if $U_f(\mathfrak{g})$ is isomorphic to $\Bbbk\langle x, y, z\rangle/(R)$ with the commuting relations R listed in the following table:*

Case	$\{x, y\}$	$\{x, z\}$	$\{y, z\}$
1	z	$-2x$	$2y$
2	y	$-z$	0
3	z	0	0
4	0	0	0
5	y	$-z$	1
6	z	1	0
7	1	0	0

where $\{x, y\} = xy - yx$.

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References

[AnS] N. Andruskiewitsch and H.-J. Schneider, A characterization of quantum groups, *J.* Reine Angew. Math., 577 (2004), 81–104.

[AS] M. Artin, W.F. Schelter, Graded algebras of global dimension 3, *A*dv. Math. **66** (1987), 171–216.

[BGS] A. Beilinson, V. Ginzburg, and W. Soergel, Koszul duality patterns in representation theory, *J.* Amer. Math. Soc. **9** (1996), 473–527.

[Be1] R. Berger, Koszulity for nonquadratic algebras,*J.* Algebra **239** (2001), 705–734.

[Be2] R. Berger, Gerasimov's theorem and N-Koszul algebras, *J.* London Math. Soc. **79** (2009), 631–648.

[BM] R. Berger, N. Marconnet, Koszul and Gorenstein properties for homogeneous algebras, *A*lgebras Repr. Theory **9** (2006), 67–97.

[BP] R. Berger, A. Pichereau, Calabi-Yau algebras viewed as deformations of Poisson algebras, *a*rXiv:1107.4472.

[BS] J. Bialkowski, and A. Skowroński, Calabi-Yau stable module categories of finite types, *C*olloq. Math. **109** (2007), 257–259.

[Bo1] R. Bocklandt, Graded Calabi-Yau algebras of dimension 3, with an appendix "The signs of Serre functor" by M. Van den Bergh. *J.* Pure Appl. Algebra **212** (2008),14–32.

[Bo2] R. Bocklandt, Calabi-Yau algebras and weighted quiver polyhedra, *Math. Z.* **273** (2013), 311–329.

[BG] A. Braverman, D. Gaitsgory, Poincaré-Birkhoff-Witt theorem for quadratic algebras of Koszul type, *J. Algebra* **181** (1996), 315–328.

[Br] N. Broomhead, Dimer models and Calabi-Yau algebras, *Memiors Amer. Math. Soc.* **215**, 2012.

[BZ] K.A. Brown, J.J. Zhang, Dualising complexes and twisted Hochschild (co)homology for Noetherian Hopf algebras, *J. Algebra* **320** (2008), 1814–1850.

[Ch] S. Chemla, Rigid dualizing complex for quantum enveloping algebras and algebras of generalized differential operators, *J. Algebra*, **276** (2004), no. 1, 80–102.

[CZ] C. Cibils, P. Zhang, Calabi-Yau objects in triangulated categories, *Trans. Amer. Math. Soc.* **361** (2009), 6501–6519.

[ES] K. Erdmann, and A. Skowroński, The stable Calabi-Yau dimension of tame symmetric algebras, *J. Math. Soc. Japan* **58** (2006), 97–128.

[Gin1] V. Ginzburg, Calabi-Yau algebras, *a*rXiv:math. AG/0612139v2.

[Gin2] V. Ginzburg, Lectures on noncommutative geometry, *a*rXiv:math.AG/0506603.

[HL] J.-W. He, D.-M. Lu, Higher Koszul algebras and A-infinity algebras, *J. Algebra* **293** (2005), 335–362.

[HVZ1] J.-W. He, F. Van Oystaeyen, Y. Zhang, Cocommutative Calabi-Yau Hopf algebras and deformations, *J. Algebra* **324** (2010), 1921–1939.

[HVZ2] J.-W. He, F. Van Oystaeyen, Y. Zhang, Hopf algebra actions on differential graded algebras and applications, Bulletin of Belgian *Math. Soc.* **18** (2011), 99–111.

[HVZ3] J.-W. He, F. Van Oysteayen, Y. Zhang, Deformations of Koszul Artin-Schelter Gorenstein algebras, *Manuscripta Math.*, Doi: 10.1007/s00229-012-0580-z, 2012.

[HVZ4] J.-W. He, F. Van Oysteayen, Y. Zhang, Graded 3-Calabi-Yau algebras as Ore extensions of 2-Calabi-Yau algebras, *a*rXiv:1303.5293.

[HVZ5] J.-W. He, F. Van Oysteayen, Y. Zhang, Skew polynomial algebras with coefficients in Koszul Artin-Schelter regular algebras, *a*rXiv:1304.3799.

[HZ] J.-W. He, S. Zhang, Deformations of quadratic algebras with anti-symmetric relations, *Comm. Algebra* **39** (2011), 2137–2149.

[IR] O. Iyama, and I. Reiten, Fomin-Zelevinsky mutation and tilting modules over Calabi-Yau algebras, Amer. J. Math. 130 (2008), 1087–1149.

[Ke1] B. Keller, On triangulated orbit categories, Documenta Math. **10** (2005), 551–581.

[Ke2] B. Keller, Calabi-Yau triangulated categories, *T*rends in Representation Theory of Algebras, European Mathematical Society, Zurich, 2008.

[Ke3] B. Keller, Deformed Calabi-Yau completions, with an appendix by M. Van den Bergh, J. Reine Ange. Math. (Crelle's Journal) **654** (2011), 125–180.

[KR] B. Keller, and I. Reiten, Cluster-tilted algebras are Gorenstein and stably Calabi-Yau, *A*dv. Math. **211**(2007), 123–151.

[KS] M. Kontsevich, Y. Soibelman, Notes on A-infinity algebras, A-infinity categories and non-commutative geometry. I. *L*ect. Notes in Physics **757** (2009), 153–219, arXiv:math.RA/0606241.

[LWZ] D.-M. Lu, Q.-S. Wu and J.J. Zhang, Homological integral of Hopf algebras, *T*rans. Amer. Math. Soc. **359** (2007), 4945–4975.

[Pr] S. Priddy, Koszul resolutions, *T*rans. Amer. Math. Soc. **152** (1970), 39–60.

[PP] A. Polishchuk, C. Positselski, Quadratic Algebras, *U*niv. Lecture Ser. 37, Amer. Math. Soc., Providence, RI, 2005.

[Sm1] S. P. Smith, Some graded algebras related to ellipitic curves, *C*MS Conf. Proc. **19** (1996), *A*mer. Math. Soc., Providence, RI, 315–348.

[Sm2] S. P. Smith, A 3-Calabi-Yau algebra with G_2 symmetry constructed from the Octonions, *a*rXiv:1104.3824.

[Sr] R. Sridharan, Filtered algebras and representations of Lie algebras, *T*rans. Amer. Math. Soc. **100** (1961), 530–550.

[VdB1] M. Van den Bergh, Existence theorems for dualizing complexes over non-commutative graded and filtered rings, *J*. Algebra **195** (1997), 662–679.

[VdB2] M. Van den Bergh, Calabi-Yau algebras and superpotentials, *t*o appear in Selecta Math.

[WZ] Q.-S. Wu, C. Zhu, Poincaré-Birkhoff-Witt deformations of Koszul Calabi-Yau algebras, *A*lgebra Repr. Theory, 2011, DOI: 10.1007/s10468-011-9312-4.

[YZ] X. YU, Y. Zhang, Calabi-Yau pointed Hopf algebras of finite Cartan type, *t*o appear in J. Noncommut. Geom., arXiv:1111.3342.

[Zh] J. J. Zhang, Non-Noetherian regular rings of dimension 2, *P*roc. Amer. Math. Soc. **126** (1998), 1645–1653.

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