Calabi-Yau algebras and their deformations

by

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Dedicated to Professors Toma Albu and Constantin Năstăsescu
on the occasion of their 70th birthdays

Abstract

This is a survey of our joint works on graded Calabi-Yau algebras, Calabi-Yau Hopf algebras and their PBW-deformations.

Key Words: Calabi-Yau algebra, Artin-Schelter regular algebra, Hopf algebra, PBW-deformation.

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Introduction

Calabi-Yau algebras appeared naturally in theoretic physics [KS, Gin1]. They seek wide applications in many branches of mathematics, say, noncommutative geometry [Gin2, Bo1, Bo2, Br, VdB2, BP], and representation theory [Ke1, Ke2, Ke3, ES, BS, IR, CZ, KR, IR]. In this survey, we focus on our works on Koszul Calabi-Yau algebras, Calabi-Yau Hopf algebras and their deformations.

Let $k$ be an algebraically closed field with characteristic zero, and let $A$ be a $k$-algebra. $A$ is called a Calabi-Yau algebra of dimension $d$ [Gin1] if

(i) $A$ is homologically smooth; that is, $A$ has a finite resolution of finitely generated projective $A$-bimodules;

(ii) $\text{Ext}^i_A(A, A \otimes A) = 0$ if $i \neq d$ and $\text{Ext}^d_A(A, A \otimes A) \cong A$ as $A$-bimodules, where $A^e = A \otimes A^{op}$ is the enveloping algebra of $A$.

Let $A = \bigoplus_{n \in \mathbb{Z}} A_n$ be a $\mathbb{Z}$-graded algebra, and $M = \bigoplus_{i \in \mathbb{Z}} M_i$ be a graded $A$-bimodule. For any integer $l$, $M(l)$ is a graded $A$-bimodule whose degree $i$ component is $M(l)_i = M_{i+l}$. A graded algebra $A$ is called a graded Calabi-Yau algebra of dimension $d$ if (i) $A$ has a finite resolution of finitely generated graded projective $A$-bimodules, and (ii) $\text{Ext}^i_A(A, A \otimes A) = 0$ when $i \neq d$ and $\text{Ext}^d_A(A, A \otimes A) \cong A(l)$ as graded $A$-bimodules for some integer $l$. 
The survey is organized as follows. In Section 1, we discuss the Calabi-Yau property of $N$-Koszul algebras. It is well known that a positively graded Calabi-Yau algebra $A$ with $A_0 = k$ is Artin-Schelter regular. We summarize some criteria for an $N$-Koszul Artin-Schelter regular algebra to be graded Calabi-Yau, and provide a method to construct graded Calabi-Yau algebras from known Artin-Schelter regular algebras.

In Section 2, we mainly discuss the Calabi-Yau property of pointed Hopf algebras. We give a necessary and sufficient condition for an Artin-Schelter regular Hopf algebra to be Calabi-Yau. It is relatively easy to determine the Calabi-Yau property of cocommutative Hopf algebras since a cocommutative pointed Hopf algebra is isomorphic to a skew group algebra of a universal enveloping algebra with a group algebra. When the pointed Hopf algebra under consideration is noncocommutative, we are only able to determine the Calabi-Yau property of pointed Hopf algebras of finite Cartan type.

In Section 3, we discuss the PBW-deformations of Koszul Calabi-Yau algebras. We summarize some criterion theorems for a PBW-deformation of a Koszul Calabi-Yau algebra to be again Calabi-Yau. In particularly, a PBW-deformation of a polynomial algebra is exactly a Sridharan enveloping algebra of a finite Lie algebra. We provide some equivalent conditions for a Sridharan enveloping algebra to be Calabi-Yau, and we are able to classify all the Calabi-Yau Sridharan enveloping algebras of dimension 3.

1 Koszul Calabi-Yau algebras

In this section, we always assume that $A = \oplus_{n \in \mathbb{Z}} A_n$ is a positively graded algebra with $A_0 = k$ and $\dim A_n < \infty$ for all $n > 0$. Let $E(A) = \oplus_{i \geq 0} \text{Ext}^i_A(Ak, Ak)$ be the space of extensions of the trivial graded module $Ak$. Endowed with the Yoneda product, $E(A)$ is a positively graded algebra, and is usually called the Yoneda Ext-algebra of $A$.

Recall that $A$ is called an Artin-Schelter regular algebra $[AS]$ if $A$ has finite global dimension $d$, $\text{Ext}^i_A(kA, A) = 0$ for all $i \neq d$ and $\text{Ext}^d_A(kA, A) = k$. It is well known that a graded Calabi-Yau algebra $A$ is Artin-Schelter regular $[BM]$. It is an interesting question to find graded Calabi-Yau algebras amongst known Artin-Schelter regular algebras. We do this in the view of Koszul algebras. Given an integer $N \geq 2$, a positively graded algebra $A$ is called an $N$-Koszul algebra $[Be1, YZ]$ if the trivial graded module $Ak$ has a graded projective resolution

$$\cdots \rightarrow P^{-i} \rightarrow P^{-i+1} \rightarrow \cdots \rightarrow P^0 \rightarrow Ak \rightarrow 0,$$

such that the graded projective module $P^{-i}$ is generated in degree $\delta(i)$, where

$$\delta(i) = \begin{cases} \frac{N}{2}, & \text{if } i \text{ is even;} \\ \frac{i-1}{2}N + 1, & \text{if } i \text{ is odd.} \end{cases}$$

In the case that $N = 2$, an $N$-Koszul algebra is usually called a Koszul algebra which was introduced by Priddy about forty years ago in $[Pr]$. An $N$-Koszul algebra is generated in degree 1. So, we may write $A$ as a quotient algebra of a tensor algebra, say $A = T(V)/(R)$ where $V$ is a finitely dimensional vector space, $T(V)$ is the tensor algebra of $V$, $R \subseteq V^\otimes N$ is a subspace and $(R)$ is the two-sided ideal of $T(V)$ generated by $R$. Associated to $A$, there is a homogeneous dual algebra $A' = T(V^*)/(R^*)$ where $V^*$ is the dual space of $V$ and $R^* \subseteq (V^*)^\otimes N$.
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is the orthogonal complement of \( R \) in \((V^*)^\otimes N\). Clearly, \((A^!)^1 \cong A\). As a graded vector space, the Yoneda Ext-algebra \( E(A) = \oplus_{i \geq 0} A^i \) [BM, HL]. If \( A \) is a Koszul algebra, then \( E(A) \cong A^1 \) as graded algebras [BGS], and in this case \( A^1 \) is also a Koszul algebra.

If an \( N \)-Koszul algebra \( A \) is Artin-Schelter regular, then its Yoneda Ext-algebra \( E(A) \) is a graded Frobenius algebra [Sm1, BM]. Recall that a finitely dimensional positively graded \( \langle \cdot,\cdot \rangle \) is the Yoneda Ext-algebra of \( E \rightarrow \) degenerate bilinear form \( A \) as graded algebras [BGS], and in this case \( A \) is the orthogonal complement of \( R \). For a graded Frobenius algebra \( E \) there is a unique graded algebra automorphism \( \varphi : E \to E \), called the Calabi-Yau automorphism of \( E \), such that \( \langle a,b \rangle = \langle \varphi(b),a \rangle \) for all homogeneous elements \( a,b \in E \). A graded Frobenius algebra \( E \) is called a graded symmetric algebra if \( \langle a,b \rangle = (\psi^{-1}(b,a)) = (\varphi(a)) = (\varphi(a)) \) for all homogeneous elements \( a,b \in E \), where \( |a| \) and \( |b| \) are the degree of \( a \) and \( b \) respectively.

The Calabi-Yau property of an \( N \)-Koszul algebra is equivalent to certain symmetric property on its Yoneda Ext-algebra.

**Proposition 1.** [HVZ2] An \( N \)-Koszul algebra \( A \) is a graded Calabi-Yau algebra if and only if its Yoneda Ext-algebra \( E(A) \) is a graded symmetric algebra.

In view of this property of \( N \)-Koszul Calabi-Yau algebras, we may construct new graded Calabi-Yau algebras from known Artin-Schelter regular algebras by the traditional methods, say skew polynomial algebras or more generally Ore extensions.

For an Artin-Schelter regular algebra, we have the following result, which was proved by Van den Bergh in [VdB1] for Koszul algebras and by Berger and Marconnet in [BM, Proof of Theorem 6.3] for general \( N \)-Koszul algebras.

**Theorem 1.** Let \( A \) be an \( N \)-Koszul Artin-Schelter regular algebra of global dimension \( d \). Let \( \varphi \) be the Nakayama automorphism of \( E(A) \), and \( \psi \) the automorphism of \( A \) induced by \( \varphi \). Then \( \text{Ext}^{i}_{A_{\varphi}}(A, A \otimes A) = 0 \) for \( i \neq d \), and \( \text{Ext}^{d}_{A_{\varphi}}(A, A \otimes A) \cong 1 A_{\xi}(\delta(d)), \)

where \( \xi \) is the automorphism of \( A \) defined by \( \xi(a) = (-1)^{|a|(d+1)} \varphi^{-1}(a) \) for all homogeneous element \( a \in A \), and \( 1 A_{\xi} \) is the \( A \)-bimodule with the regular \( A \)-action on the left side and the right \( A \)-action defined by \( x \cdot a = x \xi(a) \) for all \( x,a \in A \).

The automorphism \( \xi \) in the theorem above is usually called the Calabi-Yau automorphism of \( A \).

Let \( A \) be a Koszul algebra, and \( \sigma \) a graded automorphism of \( A \). Let \( B = A[z;\sigma] \) be the graded skew polynomial algebra with coefficients in \( A \). Clearly, \( B \) is also a Koszul algebra. The Yoneda Ext-algebra of \( B \) can be presented as follows. Let \( E = k \oplus E_1 \oplus E_2 \oplus \cdots \) be a positively graded algebra, and \( M \) a graded \( E \)-bimodule. The trivial extension of \( E \) by \( M \) is defined to be the graded algebra \( \Gamma(E,M) = E \oplus M \) with the product \( (x_1,m_1) \ast (x_2,m_2) = (x_1 x_2, x_1 \cdot m_2 + m_1 \cdot x_2) \) for \( x_i \in E \) and \( m_i \in M \). If \( \psi \) and \( \tau \) are two automorphisms of \( E \), then the notion \( \psi E \psi \) is the \( E \)-bimodule defined by \( a \cdot x \cdot b = \psi(a) x \tau(b) \) for \( a,x,b \in E \).

**Proposition 2.** [HVZ3] Let \( A \) be a Koszul algebra, \( \sigma \) a graded algebra automorphism of \( A \) and \( B = A[z;\sigma] \). Then \( E(B) \cong \Gamma(A^1, A^1 \psi(-1)) \), where \( \psi = (\sigma^{-1})^1 \) is the automorphism of \( A^1 \).
induced by $\sigma^{-1}$ and $\epsilon$ is the automorphism of $A^1$ defined by $\epsilon(x) = (-1)^{|x|}x$ for all homogeneous element $x \in A^1$.

From Propositions 1 and 2 and Theorem 1, we obtain the following result.

**Theorem 2.** [HVZ5] Let $A$ be a Koszul Artin-Schelter regular algebra of global dimension $d$ with the Nakayama automorphism $\xi$. Then the skew polynomial algebra $B = A[z; \xi]$ is a Calabi-Yau algebra of dimension $d + 1$.

We next consider graded Calabi-Yau algebras of lower global dimensions. Let $A$ be a positively graded algebra which is generated in degree 1. If $A$ is an Artin-Schelter regular algebra of global dimension 2, then $A \cong k\langle x_1, \ldots, x_n \rangle/(f)$, where $k\langle x_1, \ldots, x_n \rangle$ is the free algebra generated by $x_1, \ldots, x_n$, and $(f)$ is the two-sided ideal generated by the element $f$. The element $f$ is presented as follows: $f = (x_1, \ldots, x_n)M(x_1, \ldots, x_n)^t$, where $M$ is an $n \times n$ invertible invertible matrix with entries in $k$.

**Proposition 3.** [HVZ3] Let $A = k\langle x_1, \ldots, x_n \rangle/(f)$ and let $M$ be an $n \times n$ invertible matrix. Then we have $\text{Ext}^i_{A^e}(A, A \otimes A) = 0$ for $i \neq 2$, and

$$\text{Ext}_A^2(A, A^e) \cong A_1 \xi(-2),$$

where $\xi$ is an automorphism defined by $\xi(y) = -(x_1, \ldots, x_n)M^tM^{-1}k^t$ in which $y = k_1x_1 + \cdots + k_nx_n$ and $k = (k_1, \ldots, k_n)$.

As a corollary, we have

**Corollary 1.** [Zh, Be2, Bo1] Let $A = k\langle x_1, \ldots, x_n \rangle/(f)$ where $f = (x_1, \ldots, x_n)M(x_1, \ldots, x_n)^t$ and $M$ is an $n \times n$ matrix. Then $A$ is Calabi-Yau of dimension 2 if and only if $M$ is invertible and anti-symmetric.

If $A$ is an Artin-Schelter regular algebra of global dimension 3, then $A$ is an $N$-Koszul algebra. $A$ is isomorphic to a quotient algebra $k\langle x_1, \ldots, x_n \rangle/(r_1, \ldots, r_n)$ generated by $x_1, \ldots, x_n$ subject to the relations $r_1, \ldots, r_n \in k\langle x_1, \ldots, x_n \rangle$ of degree $N$. Let $V$ be the vector space spanned by $x_1, \ldots, x_n$, and $R$ be the vector space spanned by $r_1, \ldots, r_n$. That $A$ is of global dimension 3 implies $\dim(V \otimes R \cap R \otimes V) = 1$. Fix a basis $z$ of $V \otimes R \cap R \otimes V$. As originally suggested in [AS], the element $z \in R \otimes V$ can be written as

$$z = rQ^{(1)}x^t,$$

where $Q^{(1)}$ is an $n \times n$ matrix, $x = (x_1, \ldots, x_n)$ and $r = (r_1, \ldots, r_n)$. On the other hand, since $z \in V \otimes R$, there is an $n \times n$ matrix $Q^{(2)}$ with

$$z = xQ^{(2)}r^t.$$

**Proposition 4.** [HVZ3] With the notions as above. We have

(i) the matrices $Q^{(1)}$ and $Q^{(2)}$ are invertible;
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(ii) \( \text{Ext}^3_A(A, A \otimes A) \cong \text{Ext}_1^A(N + 1) \), where \( \xi \) acts on generators of \( A \) by

\[
\xi(x_1, \ldots, x_1) = (x_1, \ldots, x_n)Q^{(1)}Q^{(2)}^{-1}.
\]

(iii) \( A \) is Calabi-Yau if and only if \( Q^{(1)} = Q^{(2)} \).

Part of graded Calabi-Yau algebras of dimension 3 can be obtained by Ore extensions from Calabi-Yau algebras of dimension 2. Now let \( A \) be a graded Calabi-Yau algebra of dimension 2. By Corollary 1, \( A \cong k\langle x_1, \ldots, x_n \rangle / (f) \) for some \( n \geq 2 \) and \( f = (x_1, \ldots, x_n)M(x_1, \ldots, x_n)^t \), where \( M \) is an \( n \times n \) invertible anti-symmetric matrix. Note that any invertible anti-symmetric matrix is cogredient to a standard form:

\[
\Omega = \begin{pmatrix}
0 & \cdots & 0 & 1 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 1 & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
1 & \cdots & 0 & 0 & \cdots & 0
\end{pmatrix}.
\]

Since the algebra defined by the relation \( f = (x_1, \ldots, x_n)M(x_1, \ldots, x_n)^t \) is isomorphic to the algebra defined by the relation \( r = (x_1, \ldots, x_n)\Omega(x_1, \ldots, x_n)^t \), we may assume that \( M \) itself is standard. Let \( \delta \) be a graded derivation of the free algebra \( k\langle x_1, \ldots, x_n \rangle \) of degree 1. If \( \delta(f) = 0 \), then \( \delta \) induces a graded derivation \( \overline{\delta} \) on \( A \). Let \( B = A[z; \overline{\delta}] \) be the Ore extension of \( A \) defined by the graded derivation \( \overline{\delta} \).

**Theorem 3.** [HVZ4] Let \( M \) be an \( n \times n \) standard anti-symmetric matrix for some \( n \geq 2 \). Put \( f = (x_1, \ldots, x_n)M(x_1, \ldots, x_n)^t \) and \( A = k\langle x_1, \ldots, x_n \rangle / (f) \). With the notions as above.

(i) \( B \) is a graded Calabi-Yau algebra of dimension 3;

(ii) Write \( \delta(x_i) = \sum_{s,t=1}^n k^i_{st}x_i \otimes x_j \) for all \( i = 1, \ldots, n \). Assume that there is an integer \( j \) such that \( k^i_{jj} = 0 \) for all \( i = 1, \ldots, n \). Then \( B \) is a graded coherent algebra.

The class of algebras obtained in the theorem above includes the graded Calabi-Yau algebras studied by Smith in [Sm2], which were constructed from the octonions. The theorem above can be viewed as a generalization of [Sm2, Theorem 5.7 and Prop. 9.2].

2 Calabi-Yau pointed Hopf algebras

Let \( H \) be a Noetherian Hopf algebra. Similar to the graded case, one can introduce the Artin-Schelter regularity for Hopf algebras. \( H \) is said to be Artin-Schelter regular [BZ] if (i) the global dimension of \( H \) is finite, say, \( d \), (ii) \( \dim \text{Ext}^d_H(Hk, H) = 1 \) and \( \text{Ext}^i_H(Hk, H) = 0 \) for all \( i \neq d \), where \( Hk \) is the trivial \( H \)-module defined by the counit of \( H \), and (iii) the right version
of (ii) is satisfied. For an Artin-Schelter regular Hopf algebra, a nonzero element in the one-dimensional right $H$-module $\text{Ext}^d_H(Hk, HH)$ is called a right homological integral of $H$, and a nonzero element in the one-dimensional left $H$-module $\text{Ext}^d_H(k_H, H)$ is called a left homological integral of $H$ [LWZ]. The homological integral was proved to be a powerful tool to study infinite dimensional Hopf algebras [LWZ, BZ]. If the left one-dimensional $H$-module $\text{Ext}^d_H(Hk, HH)$ is isomorphic to the trivial module $Hk$ (or equivalently, the right $H$-module $\text{Ext}^d_H(Hk, H)$ is isomorphic to the trivial module $k_H$), then $H$ is said to be unimodular [LWZ].

For a Noetherian Hopf algebra, we have the following result.

Theorem 4. [HVZ1] Let $H$ be a Noetherian Hopf algebra with antipode $S$. Then $H$ is Calabi-Yau of dimension $d$ if and only if

(i) $H$ is Artin-Schelter regular of global dimension $d$ and unimodular,

(ii) $S^2$ is an inner automorphism of $H$.

With the help of the above theorem, we may find out Calabi-Yau Hopf algebras from Noetherian pointed Hopf algebras.

Let us firstly consider the cocommutative pointed Hopf algebras. It is well known that a cocommutative Hopf algebra (note that $k$ is algebraically closed) is isomorphic to a smash product of a universal enveloping algebra of a Lie algebra with a group algebra. We have the following result for cocommutative pointed Hopf algebra.

Theorem 5. [HVZ1] Let $g$ be a finite dimensional Lie algebra, and $G \subseteq \text{Aut}_{\text{Lie}}(g)$ a finite group. Then the skew group algebra $U(g)\#kG$ is a Calabi-Yau Hopf algebra if and only if $G \subseteq \text{SL}(g)$ and the Lie algebra $g$ is unimodular, that is, for any $x \in g$, $\text{tr}(\text{ad}_g(x)) = 0$.

For the cocommutative Calabi-Yau Hopf algebra of lower dimensions, we have the following results.

Theorem 6. [HVZ1] Let $H$ be a cocommutative Hopf algebra such that it has finite group-like elements and the subspace of its primitive elements is finite dimensional. Then

(i) $H$ is Calabi-Yau of dimension 2 if and only if there is a finite group $G$ and a group map $\nu : G \to \text{SL}(2, k)$ such that $H \cong k[x, y]\#kG$, where the $G$-action on $k[x, y]$ is given by $\nu$.

(ii) $H$ is Calabi-Yau of dimension 3 if and only if $H \cong U(g)\#kG$, where $g$ is one of the 3-dimensional Lie algebras listed below and $G$ is a finite group with a group morphism $\nu : G \to \text{Aut}_{\text{Lie}}(g)$ such that $\text{im}(\nu)$ is also a subgroup of $\text{SL}(g)$:

(a) The 3-dimensional simple Lie algebra $\mathfrak{sl}(2, k)$;

(b) $g$ has a basis $\{x, y, z\}$ such that $[x, y] = y, [x, z] = -z$ and $[y, z] = 0$.

(c) The Heisenberg algebra, that is; $g$ has a basis $\{x, y, z\}$ such that $[x, y] = z$ and $[x, z] = [y, z] = 0$;

(d) The 3-dimensional abelian Lie algebra.
We next consider the Calabi-Yau property of the noncocommutative pointed Hopf algebras. We restrain ourselves to pointed Hopf algebras of finite Cartan type. We recall some notions and terminology from [AnS].

- \( \Gamma \) is a free abelian group of finite rank \( s \);
- \((a_{ij}) \in \mathbb{Z}^{n \times n} \) is a Cartan matrix of finite type. \( \text{diag}(d_1, \cdots, d_n) \) is a diagonal matrix of positive integers such that \( d_ia_{ij} = d_ja_{ji}, \) which is minimal with this property;
- \( \mathcal{X} \) is the set of connected components of the Dynkin diagram corresponding to the Cartan matrix \((a_{ij})\). If \( 1 \leq i, j \leq n \), then \( i \sim j \) means that they belong to the same connected component;
- \((q_I)_{I \in \mathcal{X}}\) is a family of elements in \( k \) which are not roots of \( 1 \);
- Choose elements \( g_1, \cdots, g_n \in \Gamma \) and characters \( \chi_1, \cdots, \chi_n \in \hat{\Gamma} \) such that
  \[
  \langle \chi_j, g_i \rangle \langle \chi_i, g_j \rangle = q_i^{d_{ij}}, \langle \chi_i, g_i \rangle = q_i^1,
  \]
  for all \( 1 \leq i < j \leq n, i \in I \).

Set \( \mathcal{D} = \mathcal{D}(\Gamma, (a_{ij})_{1 \leq i, j \leq n}, (g_i)_{1 \leq i \leq n}, (\chi_i)_{1 \leq i \leq n}) \). A linking datum \( \lambda = (\lambda_{ij}) \) for \( \mathcal{D} \) is a collection \((\lambda_{ij})_{1 \leq i < j \leq n, i \sim j \in \{0, 1\}}\) such that \( \lambda_{ij} = 0 \) if \( g_ig_j = 1 \) or \( \chi_i\chi_j \neq \varepsilon \). Write the datum \( \lambda = 0 \), if \( \lambda_{ij} = 0 \) for all \( 1 \leq i < j \leq n \).

The datum \((\mathcal{D}, \lambda) = (\Gamma, (a_{ij}), (g_i), (\chi_i), (\lambda_{ij}))\) is called a generic datum of finite Cartan type for the group \( \Gamma \).

Given a generic datum \((\mathcal{D}, \lambda)\) of finite Cartan type. Denote by \( U(\mathcal{D}, \lambda) \) the algebra with generators \( x_1, \cdots, x_n; z_1^{\pm 1}, \cdots, z_s^{\pm 1} \) and relations
\[
  z_k^{\pm 1} z_i^{\pm 1} = z_i^{\pm 1} z_k^{\pm 1}, \quad z_k^{\pm 1} z_i = 1, \quad 1 \leq k, l \leq s,
\]
\[
  z_k x_i = \chi_i(z_k) x_i z_k, \quad 1 \leq i \leq n, 1 \leq k \leq s,
\]
\[
  (\text{ad}_x(x)^{-1} a_i)(x_j) = 0, \quad 1 \leq i \neq j \leq n, i \sim j,
\]
\[
  x_i x_j - \chi_j(g_i) x_j x_i = \lambda_{ij}(1 - g_i g_j), \quad 1 \leq i < j \leq n, i \sim j.
\]
where \( \text{ad}_x \) is the braided adjoint representation (for details, see [AnS, Sect. 1]).

**Theorem 7.** [AnS] Let \((\mathcal{D}, \lambda) = (\Gamma, (a_{ij}), (g_i), (\chi_i), (\lambda_{ij}))\) be a generic datum of finite Cartan type. The algebra \( U(\mathcal{D}, \lambda) \) defined as above is a pointed Hopf algebra with comultiplication defined by
\[
  \Delta(g_k) = g_k \otimes g_k, \quad \Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i, \quad 1 \leq k \leq s, 1 \leq i \leq n.
\]

The Hopf algebra \( U(\mathcal{D}, \lambda) \) is Noetherian [YZ], and the Calabi-Yau property of \( U(\mathcal{D}, \lambda) \) is shown in the following theorem.

**Theorem 8.** [YZ] Let \((\mathcal{D}, \lambda)\) be a generic datum of finite Cartan type. The pointed Hopf algebra \( U(\mathcal{D}, \lambda) \) is Calabi-Yau if and only if \( \prod_{i=1}^p \chi_i = \varepsilon \) and \( S^2 \) is inner, where \( p \) is the number of the positive roots of the Cartan matrix, \( \chi_i \)'s are the positive roots and \( S \) is the antipode of \( U(\mathcal{D}, \lambda) \).

**Remark.** (i) The Calabi-Yau property of a quantum enveloping algebra was already shown in [Ch].

(ii) Calabi-Yau pointed Hopf algebras of finite Cartan type of dimensions less than 5 were classified in [YZ].
3 PBW deformations

Let $A = \oplus_{i \geq 0} A_i$ be a graded algebra with $A_0 = k$. A PBW-deformation of $A$ is a filtered algebra $U$ with an ascending filtration $0 \subseteq F_0 U \subseteq F_1 U \subseteq F_2 U \subseteq \cdots$ such that the associated graded algebra $gr(U)$ is isomorphic to $A$.

In this section, we only consider PBW-deformations of Koszul algebras. Let $A = T(V)/(R)$ be a Koszul algebra. A PBW-deformation $U$ of $A$ is determined by two linear maps $\nu : R \to V$ and $\theta : R \to k$ in sense that $U \cong T(V)/(r - \nu(r) - \theta(r) : r \in R)$, where the linear maps $\nu$ and $\theta$ satisfy Jacobian type conditions (see [BG, PP]):

\[
\begin{align*}
(\nu \otimes 1 - 1 \otimes \nu)(R \otimes V \cap V \otimes R) & \subseteq R \\
[\nu(\nu \otimes 1 - 1 \otimes \nu) - (\theta \otimes 1 - 1 \otimes \theta)](R \otimes V \cap V \otimes R) & = 0 \\
\theta(\nu \otimes 1 - 1 \otimes \nu)(R \otimes V \cap V \otimes R) & = 0.
\end{align*}
\]

If $\theta = 0$, then $U$ is called an augmented PBW-deformation of $A$.

Since $A$ is a Koszul algebra, the Yoneda Ext-algebra of $A$ is also a Koszul algebra. Moreover, $E(A) \cong A^!$. Henceforth, we identify $E(A)$ with $A^!$. Recall $A^! = T(V^*)/(R^\perp)$. Hence $A^! = V^*$ and $A_{\nu}^! = R^\perp$. So, we may view $\theta$ as an element in $A_{\nu}^!$. By the Jacobian type conditions above, the dual map $\nu^* : V^* \to R^\perp$ induces a graded derivation $\partial$ on $A^!$, so that the triple $(A^!,\partial,\theta)$ is a curved differential graded (DG) algebra [PP], that is, the identity $\partial^2(x) = [\theta,x]$ holds for all $x \in A^!$. We call $(A^!,\partial,\theta)$ the curved DG algebra dual to the PBW-deformation $U$ of $A$. If $U$ is an augmented PBW-deformation of $A$, then $(A^!,\partial)$ is a usual DG algebra.

**Proposition 5.** [HVZ3] Let $A = T(V)/(R)$ be a Koszul Artin-Schelter regular algebra of global dimension $d$, and let $\xi$ be the Nakayama automorphism of $A$ (see Sect. 1). Assume that \{x_1, \ldots, x_n\} is a basis of $V$, and \{x_1^*, \ldots, x_n^*\} is the dual basis of $V^*$.

Let $U = T(V)/(r - \nu(r) - \theta(r) : r \in R)$ be a PBW-deformation of $A$, and let $(A^!,\partial,\theta)$ be the curved DG algebra dual to $U$. Choose a basis $\mathcal{w}$ of $A_{\nu}^!$, and assume that \{x_1, \ldots, x_n\} is the basis of $A_{d-1}$ such that $x_i^* x_j = \delta_{ij} \mathcal{w}$. Assume further $\partial(\omega_i) = \lambda_i \mathcal{w}$ for all $i = 1, \ldots, n$. Then $\Ext_{U^!}(U, U \otimes U) = 0$ for $i \neq d$, and

\[\Ext_{U^!}(U, U \otimes U) \cong 1 U_{\xi},\]

where the automorphism $\zeta$ acts on the generator by $\zeta(x_i) = \xi(x_i) + \lambda_i$.

From the above proposition, we have the following result.

**Theorem 9.** [HVZ3] Let $A = T(V)/(R)$ be a Koszul Calabi-Yau algebra of dimension $d$. Assume that $U$ is an augmented PBW-deformation of $A$, and that $(A^!,\partial)$ is the DG algebra dual to $U$. Then the following are equivalent:

(i) $U$ is a Calabi-Yau algebra;

(ii) $E(U) = \oplus_{i=1}^d \Ext_{U^!}(k,k)$ is a graded symmetric algebra;

(iii) $\partial(A_{d-1}^!) = 0$. 
Remark 1. A similar result also appeared in [WZ] under the hypothesis that \( A \) is Noetherian.

The Calabi-Yau property of a nonaugmented PBW-deformation is sometimes equivalent to that of an augmented one.

Theorem 10. [HVZ3, HZ] Let \( A = T(V)/(R) \) be a Koszul Calabi-Yau algebra of dimension \( d \). Assume that both \( U = T(V)/(r - \nu(r) - \theta(r) : r \in R) \) and \( U' = T(V)/(r - \nu(r) : r \in R) \) are PBW-deformations of \( A \). If \( U' \) is Calabi-Yau, then so is \( U \).

Conversely, if \( U \) is Calabi-Yau and \( A \) is a domain, then \( U' \) is Calabi-Yau.

We next consider PBW-deformations of some specific graded Calabi-Yau algebra. Let \( A \) be a Koszul Artin-Schelter regular algebra of global dimension \( d \). Let \( \xi \) be the Nakayama automorphism of \( A \) (see Sect. 1). Then the skew polynomial algebra \( A[z; \xi] \) is a Koszul Calabi-Yau algebra of global dimension \( d + 1 \) (see, Theorem 2). Let \( U = T(V)/(r - \nu(r) - \theta(r) : r \in R) \) be a PBW-deformation of \( A \), and let \( \zeta \) be the automorphism defined in Proposition 5. Then we have

Theorem 11. [HVZ5] Keep the notation as above.

(i) \( U[z; \zeta] \) is a PBW-deformation of \( A[z; \xi] \);

(ii) If, further, \( A \) is Calabi-Yau and \( A \) is a domain, then \( U[z; \zeta] \) is also Calabi-Yau.

If \( A = k[x_1, \ldots, x_n] \) is the polynomial algebra generated by variables \( x_1, \ldots, x_n \), then a PBW-deformation of \( A \) is equivalent to a Sridharan enveloping algebra of an \( n \)-dimensional Lie algebra. We recall from [Sr] the definition of Sridharan enveloping algebra. Let \( g \) be a finite dimensional Lie algebra, and let \( f \in Z^2(g, k) \) be a 2-cocycle, that is; \( f : g \times g \to k \) such that

\[
f(x, x) = 0 \text{ and } f(x, [y, z]) + f(y, [z, x]) + f(z, [x, y]) = 0
\]

for all \( x, y, z \in g \). The Sridharan enveloping algebra of \( g \) is defined to be the quotient algebra \( U_f(g) = T(g)/I \), where \( I \) is the two-sided ideal of \( T(g) \) generated by the elements

\[
x \otimes y - y \otimes x - [x, y] - f(x, y), \quad \text{for all } x, y \in g.
\]

The following result says that the Calabi-Yau property of a Sridharan enveloping algebra is independent of the choice of the 2-cocycle \( f \).

Theorem 12. [HVZ1] Let \( g \) be a finite dimensional Lie algebra. For an arbitrary 2-cocycle \( f \in Z^2(g, k) \), the following are equivalent.

(i) The Sridharan enveloping algebra \( U_f(g) \) is Calabi-Yau of dimension \( d \).

(ii) The universal enveloping algebra \( U(g) \) is Calabi-Yau of dimension \( d \).

(iii) \( \dim g = d \) and \( g \) is unimodular.

By the theorem above, we are able to classify all the Calabi-Yau Sridharan enveloping algebras of dimension 3.
Theorem 13. [HVZ1] A Sridharan enveloping algebra $U_f(g)$ is Calabi-Yau of dimension 3 if and only if $U_f(g)$ is isomorphic to $k\langle x, y, z \rangle/(R)$ with the commuting relations $R$ listed in the following table:

<table>
<thead>
<tr>
<th>Case</th>
<th>${x, y}$</th>
<th>${x, z}$</th>
<th>${y, z}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$z$</td>
<td>$-2x$</td>
<td>$2y$</td>
</tr>
<tr>
<td>2</td>
<td>$y$</td>
<td>$-z$</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>$z$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>$y$</td>
<td>$-z$</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>$z$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

where $\{x, y\} = xy - yx$.

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References


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