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External Homogenization for Hopf Algebras: a Coring Point of View

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Dedicated to Professors Toma Albu and Constantin Năstăsescu
on the occasion of their 70th birthdays

Abstract

We give a coring version for the external homogenization for Hopf algebras, which is a generalization of a construction from graded rings, called the group ring of a graded ting. We also provide a coring version of a Maschke-type theorem.

Key Words: Hopf algebra, Galois coring, crossed product, coseparable coring, separable functor.
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1 Introduction

C. Năstăsescu constructed in [7] the group ring of a graded ring. If G is a group and R is a G-graded ring, then the group ring of R, denoted by $R[G]$, is constructed on the free left R-module with basis indexed by the elements of G, and turns out to be a strongly G-graded ring, and even more than that, a crossed product. The inspiration for this construction was provided by the operation of external homogenization for rings graded by the integers [10].

The construction was extended by C. Năstăsescu, F. Panaite and F. Van Oystaeyen in [8] to the case of Hopf algebra coactions: if H is a Hopf algebra and A is a right H-comodule algebra, they define the external homogenization $A[H]$ as a certain right H-comodule algebra structure on $A \otimes H$. It is proved in [8] that the subalgebra of coinvariants of $A[H]$ is isomorphic, as an algebra, to A, and $A[H]^{co(H)} \subseteq A[H]$ is a cleft extension, hence it is Galois and has the normal basis property [1], [5]. This last fact about cleft extensions, as well as any other notions or results concerning Hopf algebras that are mentioned here and not explained in detail may be found in [4].

In the first section we will give a different proof for the fact that the external homogenization $A[H]$ is a crossed product. The point we are trying to make is that a big part of the external homogenization construction may be recovered by dealing almost exclusively with corings. We start by recalling the definition of a coring:

Definition 1. *Let A be a ring. An A-bimodule $\mathcal{C}$ is called a* coring *if there exist A-bimodule maps $\Delta : \mathcal{C} \longrightarrow \mathcal{C} \otimes_A \mathcal{C}$ and $\varepsilon : \mathcal{C} \longrightarrow A$, such that Δ is coassociative and ε is a counit.*

The most basic example of a coring is a coalgebra over a commutative ring A, but we have to point out that, even when the base ring A is commutative, the left and right A-module structures need not be the same. Another fundamental example is the canonical coring associated to the ring homomorphism $i : B \longrightarrow A$:

$$\mathcal{C} = A \otimes_B A,$$

$$\Delta : \mathcal{C} \longrightarrow \mathcal{C} \otimes_A \mathcal{C} \simeq A \otimes_B A \otimes_B A, \quad \Delta(a \otimes b) = a \otimes 1 \otimes b,$$

and

$$\varepsilon : \mathcal{C} \longrightarrow A, \quad \varepsilon(a \otimes b) = ab.$$

Example 1. *Let H be a k-Hopf algebra, and A a right H-comodule algebra via $a \mapsto a_{[0]} \otimes a_{[1]}$. Then $A \otimes H$ becomes an A-coring as follows: the left A-module structure is given by multiplication on the first component, and the right A module structure is given by $(a \otimes h)b = ab_{[0]} \otimes hb_{[1]}$. The comultiplication is $\Delta : A \otimes H \longrightarrow (A \otimes H) \otimes_A (A \otimes H) \simeq A \otimes H \otimes H$, $\Delta(a \otimes h) = a \otimes h_{(1)} \otimes h_{(2)}$, and the counit is $\varepsilon : A \otimes H \longrightarrow A$, $\varepsilon(a \otimes h) = \varepsilon(h)a$.*

Recall that if A is a right H-comodule algebra, and if $B = A^{coH}$, then A/B is a Hopf-Galois extension if and only if the coring $A \otimes H$ from Example 1 is isomorphic to the canonical coring $A \otimes_B A$ via the map sending $a \otimes b \in A \otimes_B A$ to $ab_{[0]} \otimes b_{[1]} \in A \otimes H$.

Corings were introduced by Sweedler in [13], and were given a lot of attention beginning in the late 1990's, after Takeuchi remarked that many examples of (generalized) Hopf modules are in fact just comodules over some corings. For example, if A is a right H-comodule algebra, the the category of right relative (A, H)-Hopf modules is equivalent to the category of right comdules over the coring $A \otimes H$ from Example 1. Moreover, if $B = A^{coH}$, then A/B is a Hopf-Galois extension if and only if the coring $A \otimes H$ from Example 1 is Galois with respect to the group-like element $1 \otimes 1$ [2, Example 5.4]. (Recall that if $(\mathcal{C}, x)$ is an A-coring with fixed grouplike element x, and $B = A^{co\mathcal{C}} = \{a \in A \mid \rho^A(a) = a \otimes x\}$, then $(\mathcal{C}, x)$ is a Galois coring if the canonical coring morphism $can : A \otimes_B A \longrightarrow \mathcal{C}$, $can(a \otimes b) = axb$ is an isomorphism.)

For all unexplained facts about corings the reader is referred to [3].

2 External Homogenization

For the remainder of this note H will denote a Hopf algebra over the field k, and A will be a right H-comodule algebra. As shown in [8, Propositions 3.1 and 3.2], $A \otimes H$ becomes a right H-comodule algebra via:

$$(a \otimes h)(b \otimes g) = ab_{[0]} \otimes S(b_{[1]})hb_{[2]}g \tag{2.1}$$

and

$$\rho : A \otimes H \longrightarrow A \otimes H \otimes H, \quad \rho(a \otimes h) = a_{[0]} \otimes h_{(1)} \otimes a_{[1]}h_{(2)} \tag{2.2}$$

The subalgebra of coinvariants of $A \otimes H$, denoted by $(A \otimes H)^{co(H)}$, is the image of the injective algebra map $\varphi : A \longrightarrow A \otimes H$, defined by

$$\varphi(a) = a_{[0]} \otimes S(a_{[1]})$$

As in Example 1, with A replaced by $A \otimes H$ with the right H-comdule algebra structure described above, $A \otimes H \otimes H$ becomes a $A \otimes H$-coring via

$$\Delta : A \otimes H \otimes H \longrightarrow A \otimes H \otimes H \otimes_{A \otimes H} A \otimes H \otimes H,$$

$$a \otimes h \otimes g \longmapsto a \otimes h \otimes g_{(1)} \otimes 1 \otimes 1 \otimes g_{(2)},$$

and

$$\varepsilon(a \otimes h \otimes g) = a \otimes h\varepsilon(g).$$

The left $A \otimes H$-module structure on $A \otimes H \otimes H$ is given by

$$(a \otimes h)(b \otimes g \otimes e) = (a \otimes h)(b \otimes g) \otimes e = ab_{[0]} \otimes S(b_{[1]})hb_{[2]}g \otimes e \tag{2.3}$$

and the right $A \otimes H$-module structure on $A \otimes H \otimes H$ is given by

$$(b \otimes g \otimes e)(a \otimes h) = ba_{[0]} \otimes S(a_{[1]})ga_{[2]}h_{(1)} \otimes ea_{[3]}h_{(2)} \tag{2.4}$$

Since we are trying to deal with corings only, we can take the above as the definition of the coring $A \otimes H \otimes H$, and check that it is a coring directly, without using Example 1. This is extremely easy. The only condition that deserves to be checked is the fact that the comultiplication is right $A \otimes H$-linear, but even that is very easy, as we see below.

$$\begin{aligned}
\Delta((b \otimes g \otimes e)(a \otimes h)) &= \Delta(ba_{[0]} \otimes S(a_{[1]})ga_{[2]}h_{(1)} \otimes ea_{[3]}h_{(2)}) \\
&= ba_{[0]} \otimes S(a_{[1]})ga_{[2]}h_{(1)} \otimes e_{(1)}a_{[3]}h_{(2)} \otimes \\
&\otimes 1 \otimes 1 \otimes e_{(2)}a_{[4]}h_{(3)} \\
&= (b \otimes g \otimes e_{(1)})(a_{[0]} \otimes h_{(1)}) \otimes 1 \otimes 1 \otimes e_{(2)}a_{[1]}h_{(2)} \\
&= b \otimes g \otimes e_{(1)} \otimes (a_{[0]} \otimes h_{(1)})(1 \otimes 1) \otimes e_{(2)}a_{[1]}h_{(2)} \\
&= b \otimes g \otimes e_{(1)} \otimes (1 \otimes 1 \otimes e_{(2)})(a \otimes h) \\
&= (\Delta(b \otimes g \otimes e))(a \otimes h)
\end{aligned}$$

Proposition 1. *[8, Corollary 3.11.(a)]* $(A \otimes H \otimes H, 1 \otimes 1 \otimes 1)$ *is a Galois coring.*

Proof: The canonical coring associated to $\varphi : A \longrightarrow A \otimes H$ is $A \otimes H \otimes_A A \otimes H$ has comultiplication

$$\Delta : A \otimes H \otimes_A A \otimes H \longrightarrow A \otimes H \otimes_A A \otimes H \otimes_A A \otimes H$$

$$(\text{because } A \otimes H \otimes_A A \otimes H \otimes_{A \otimes H} A \otimes H \otimes_A A \otimes H \simeq A \otimes H \otimes_A A \otimes H \otimes_A A \otimes H)$$

$$a \otimes h \otimes b \otimes g \longmapsto a \otimes h \otimes 1 \otimes 1 \otimes b \otimes g,$$

and counit

$$\varepsilon(a \otimes h \otimes b \otimes g) = (a \otimes h)(b \otimes g) = ab_{[0]} \otimes S(b_{[1]})hb_{[2]}g$$

The left and right A-module structures on $A \otimes H$ are given by

$$a(b \otimes h) = (a_{[0]} \otimes S(a_{[1]}))(b \otimes h) = a_{[0]}b_{[0]} \otimes S(a_{[1]}b_{[1]})b_{[2]}h \tag{2.5}$$

$$(b \otimes h)a = (b \otimes h)(a_{[0]} \otimes S(a_{[1]})) = ba_{[0]} \otimes S(a_{[1]})h \tag{2.6}$$

We need to prove that the map

$$can : A \otimes H \otimes_A A \otimes H \longrightarrow A \otimes H \otimes H$$

defined by

$$can(a \otimes h \otimes b \otimes g) = ab_{[0]} \otimes S(b_{[1]})hb_{[2]}g_{(1)} \otimes b_{[3]}g_{(2)}$$

is an isomorphism of $A \otimes H$-corings. It is known in general that *can* is a morphism of corings, but this is again very easy to check directly. For example, the check that *can* commutes with comultiplication goes like this:

$$\begin{aligned}
\Delta(can(a \otimes h \otimes b \otimes g)) &= ab_{[0]} \otimes S(b_{[1]})hb_{[2]}g_{(1)} \otimes b_{[3]}g_{(2)} \otimes 1 \otimes 1 \otimes b_{[4]}g_{(3)} \\
&= (a \otimes h)(b_{[0]} \otimes g_{(1)}) \otimes b_{[1]}g_{(2)} \otimes (1 \otimes 1 \otimes b_{[2]}g_{(3)}) \\
&= (a \otimes h)(b_{[0]} \otimes g_{(1)})_{[0]} \otimes (b_{[0]} \otimes g_{(1)})_{[1]} \otimes \\
&\otimes (1 \otimes 1 \otimes b_{[1]}g_{(3)}) \\
&= (a \otimes h \otimes 1)(b_{[0]} \otimes g_{(1)}) \otimes (1 \otimes 1 \otimes b_{[1]}g_{(2)}) \\
&= (a \otimes h \otimes 1) \otimes (b_{[0]} \otimes g_{(1)} \otimes b_{[1]}g_{(2)}) \\
&= (can \otimes can)\Delta(a \otimes h \otimes b \otimes g)
\end{aligned}$$

(the second equality uses (2.1), the third one uses (2.2), the fourth uses (2.4) and the fifth uses (2.3)).

The fact that *can* preserves the counit is equally easy:

$$\varepsilon(can(a \otimes h \otimes b \otimes g)) = \varepsilon(a \otimes h \otimes b \otimes g) = ab_{[0]} \otimes S(b_{[1]})hb_{[2]}g,$$

so the only thing left to check is that *can* has an inverse. The inverse, which is given explicitly in [8, Corollary 3.11], is

$$\alpha : A \otimes H \otimes H \longrightarrow A \otimes H \otimes_A A \otimes H$$
$$\alpha(a \otimes h \otimes g) = a \otimes hS(g_{(1)}) \otimes 1 \otimes g_{(2)}.$$

Checking that α and *can* are inverse to each other is again immediate:

$$can(\alpha(a \otimes h \otimes g)) = a \otimes hS(g_{(1)})g_{(2)} \otimes g_{(3)} = a \otimes h \otimes g$$

and

$$\begin{aligned}
\alpha(can(a \otimes h \otimes b \otimes g)) &= ab_{[0]} \otimes S(b_{[1]})hb_{[2]}g_{(1)}S(b_{[3]}g_{(2)}) \otimes 1 \otimes b_{[4]}g_{(3)} \\
&= ab_{[0]} \otimes S(b_{[1]})h \otimes 1 \otimes b_{[2]}g \\
&= (a \otimes h)b_{[0]} \otimes 1 \otimes b_{[1]}g \text{ (by (2.6))} \\
&= a \otimes h \otimes b_{[0]}(1 \otimes b_{[1]}g) \\
&= a \otimes h \otimes b_{[0]} \otimes S(b_{[1]})b_{[2]}g \text{ (by (2.5))} \\
&= a \otimes h \otimes b \otimes g
\end{aligned}$$

∎

Corollary 1. *[8, Corollary 3.11.(c)] $A \otimes H$ is equivalent to a crossed product of A and H with invertible cocycle.*

Proof: The only thing left to prove is that $A \otimes H$ has the normal basis property, i.e. $A \otimes H \simeq \varphi(A) \otimes H$, with the natural left $\varphi(A)$-module and right H-comodule structures. The isomorphism is given by

$$\Phi : A \otimes H \longrightarrow \varphi(A) \otimes H, \quad \Phi(a \otimes h) = a_{[0]} \otimes S(a_{[1]}) \otimes a_{[2]}h$$

and the immediately checked inverse (which is not given explicitly in [8]) is

$$\Psi : \varphi(A) \otimes H \longrightarrow A \otimes H, \quad \Psi(a_{[0]} \otimes S(a_{[1]}) \otimes h) = a_{[0]} \otimes S(a_{[1]})h$$

□

3 A Maschke-type theorem

Recal from [6] that an A-coring $\mathcal{C}$ is *coseparable* if there exists a $\mathcal{C}$-bicomodule splitting of the coproduct, i.e. there exists an A-bimodule map $\pi : \mathcal{C} \otimes_A \mathcal{C} \longrightarrow \mathcal{C}$ such that

$$(\mathcal{C} \otimes_A \pi) \circ (\Delta_{\mathcal{C}} \otimes \mathcal{C}) = \Delta_{\mathcal{C}} \circ \pi = (\pi \otimes_A \mathcal{C}) \circ (\mathcal{C} \otimes_A \Delta_{\mathcal{C}})$$

and

$$\pi \circ \Delta_{\mathcal{C}} = \mathcal{C}.$$

Also recall from [9] that a covariant functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ is *separable* if the natural transformation $Hom_{\mathcal{A}}(-,-) \longrightarrow Hom_{\mathcal{B}}(F(-), F(-))$ splits functorially.

The main result of this section is the following

Proposition 2. *(i) The coring $A \otimes H \otimes H$ from the previous section is coseparable.*
(ii) The forgetful functor $F : \mathcal{M}^{A\otimes H\otimes H} \longrightarrow \mathcal{M}_{A\otimes H}$ is separable.

Proof: By [2, Corollary 3.6.] (i) and (ii) are equivaleent, so we only need to prove that the forgetful functor $F : \mathcal{M}^{A\otimes H\otimes H} \longrightarrow \mathcal{M}_{A\otimes H}$ is separable. Since $A \otimes H \otimes H$ is isomorphic to the canonical coring $A \otimes H \otimes_A A \otimes H$, we need to prove that the forgetful functor $F : \mathcal{M}^{A\otimes H\otimes_A A\otimes H} \longrightarrow \mathcal{M}_{A\otimes H}$ is separable. Since $A \otimes H$ is faithfully flat over $A \simeq \varphi(A)$ (it is a crossed product), we can apply [2, Corollary 3.7.] and it is enough to prove that the extension $\varphi(A) \longrightarrow A\otimes H$ is split, which means that there exists a $\varphi(A)$-bimodule map $E : A\otimes H \longrightarrow \varphi(A)$ such that $E(1 \otimes 1) = 1 \otimes 1$ [11]. Now define

$$E(a \otimes h) = a_{[0]} \otimes S(a_{[1]})\varepsilon(h)$$

We have $E(1 \otimes 1) = 1 \otimes 1$,

$$E((a_{[0]} \otimes S(a_{[1]}))(b \otimes h)) = (a_{[0]} \otimes S(a_{[1]}))E(b \otimes h) = a_{[0]}b_{[0]} \otimes S(a_{[1]}b_{[1]})\varepsilon(h),$$

and this is also equal to

$$E(a \otimes h)(b_{[0]} \otimes S(b_{[1]})) = E((a \otimes h)(b_{[0]} \otimes S(b_{[1]})))$$

and the proof is complete. □

We now get the following Maschke-type theorem:

Corollary 2. *Every right $A \otimes H \otimes H$-comodule which is semisimple (resp. projective, injective) as a right $A \otimes H$-module is a semisimple (resp. projective, injective) $A \otimes H \otimes H$-comodule.*

Proof: This is [2, Corollary 3.8.]. □

In the language of relative Hopf modules this becomes:

Corollary 3. *If $M \in \mathcal{M}^H_{A\otimes H}$ is semisimple (resp. projective, injective) as a right $A \otimes H$-module (i.e. as an object in $\mathcal{M}_{A\otimes H}$) is semisimple (resp. projective, injective) in $\mathcal{M}^H_{A\otimes H}$.*

Please note that this last corollary is different from [8, Proposition 4.7.], which requires H to be semisimple, and replaces "$A \otimes H$-module" (i.e. the structure on $A \otimes H$ is the algebra structure defined at the beginning of Section 1) by "$A \otimes H$-comodule" (i.e. the structure on $A \otimes H$ is the coring structure defined in Example 1).

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