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Systolic simplicial complexes are collapsible

by
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To the memory of Professor Mircea-Eugen Craioveanu (1942 - 2012).

Abstract

We find necessary and sufficient conditions for the collapsibility of a finite simplicial complex of arbitrary finite dimension. Our main result states that any finite systolic simplicial complex of finite dimension, collapses to a point. A simplicial complex is systolic if it is simply connected, connected and locally 6-large. Local 6-largeness is a simple combinatorial condition defined in terms of links in the complex. Our proof is based on the fact that any cycle in a systolic complex has a van Kampen diagram of minimal area whose disk is itself systolic.

Key Words: Simplicial complex, star of a simplex, link of a simplex, systole of a complex, locally 6-large, systolic, van Kampen diagram, collapsibility.
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1 Introduction

In this paper we give a combinatorial characterization of collapsible simplicial complexes of arbitrary dimension. The proof uses van Kampen diagrams.

The combinatorial condition on simplicial complexes we consider is called local 6-largeness and it was introduced by T. Januszkiewicz and J. Swiatkowski in [9] and independently by F. Haglund in [8]. Local 6-largeness in dimension 2 is called the 6-property of the simplicial complex (see [2], [9], [3], [4], [6], [5], [7], [11], [10]). A 2-dimensional simplicial complex has the 6-property if the link of each of its vertices is a graph of girth at least 6. The *girth* of a graph is defined as the minimum number of edges in a circuit. A weaker condition, called weakly systolicity, and resembling metric non-positive curvature, was introduced by D. Osajda in [13].

J. Corson and B. Trace proved in [3] that finite, simply connected, simplicial 2-complexes with the 6-property, i.e. systolic 2-complexes, collapse to a point. It is the paper's objective is to show that finite dimensional systolic simplicial complexes enjoy the same property. Similar to the proof given in [3], our proof also uses van Kampen diagrams (see [12]).

T. Januszkiewicz and J. Swiatkowski showed in 2006 that the weaker condition of contractibility does characterize finite dimensional systolic simplicial complexes (see [9], chapter 4, page 21). We strengthen this condition by showing that such complexes are even collapsible.

We emphasize that it is known that weakly systolic complexes are collapsible. Namely, V. Chepoi and D. Osajda proved their collapsibility in [1] (see Theorem 5.1) by showing that the underlying graphs of such complexes are dismantable. Our proof, however, is shorter, resembling the one given in [3] in the 2-dimensional case. Namely, we will introduce three elementary operations on edge-paths in a simplicial complex and we will show, using a result regarding van Kampen diagrams (see [9], chapter 1, page 11) that in a locally 6-large complex, one can pass from one edge-path to any homotopic edge-path via a finite sequence of such operations.

2 Preliminaries

We present in this section the notions we shall work with and the results we shall refer to.

Let X be a simplicial complex. A *closed edge* is an edge together with its endpoints. An *oriented edge* of X is an oriented 1-simplex of X, $e = [v_0, v_1]$. We denote by $i(e) = v_0$, the *initial* vertex of e, by $t(e) = v_1$, the *terminus* of e, and by $e^{-1} = [v_1, v_0]$, the *inverse* of e. A finite sequence $\alpha = e_1 e_2 ... e_n$ of oriented closed edges in X such that $t(e_i) = i(e_{i+1})$ for all $1 \leq i \leq n-1$, is called an *edge-path* in X. The *inverse* of α is the edge-path $\alpha^{-1} = e_n^{-1} ... e_2^{-1} e_1^{-1}$. If $t(e_n) = i(e_0)$, then we call α a *closed edge-path* or a *cycle*. We denote by $|\alpha|$ the number of 1-simplices contained in α and we call $|\alpha|$ the *length* of α.

Let α be an i-simplex of X. If β is a k-dimensional face of α but not of any other simplex in X, then we say there is an *elementary collapse* from X to $X \setminus \{\alpha, \beta\}$. If $X = X_0 \supseteq X_1 \supseteq ... \supseteq X_n = L$ are simplicial complexes such that there is an elementary collapse from X_{j-1} to X_j, $1 \leq j \leq n$, then we say that X *simplicially collapses* to L.

Let σ be a simplex of X. The *link* of X at σ, denoted $\mathrm{Lk}(\sigma, \mathrm{X})$, is the subcomplex of X consisting of all simplices in X which are disjoint from σ and which, together with σ, span a simplex of X. The (closed) *star* of σ in X, denoted $\mathrm{St}(\sigma, \mathrm{X})$, is the union of all simplices of X that contain σ. The star $\mathrm{St}(\sigma, \mathrm{X})$ is the join of σ and the link $\mathrm{Lk}(\sigma, \mathrm{X})$. We denote the join operation by $*$. A subcomplex L in X is called *full* (in X) if any simplex of X spanned by a set of vertices in L, is a simplex of L. A *full cycle* in X is a cycle that is full as subcomplex of X. We define the *systole* of X by:

$$sys(X) = \min\{|\alpha| : \alpha \text{ is a full cycle in } X\}.$$

We introduce further a curvature condition on simplicial complexes in combinatorial terms.

We call a simplicial complex X 6-*large* if $sys(X) \geq 6$ and $sys(\mathrm{Lk}(\sigma, \mathrm{X})) \geq 6$ for each simplex σ of X. We call X *locally* 6-*large* if the star of every simplex of X is 6-large. We call X 6-*systolic* if it is connected, simply connected and locally 6-large. We abbreviate $6-$systolic to systolic.

A *combinatorial map* $f : X_1 \to X_2$ between two simplicial complexes X_1 and X_2 is a continuous map such that each open simplex of X_1 is mapped homeomorphically onto an open simplex of X_2. We call a combinatorial map *nondegenerate* if it is injective on each simplex of the triangulation. A *combinatorial* 2-*complex* is a simplicial 2-complex such that the 2-cells are attached through continuous maps from S^1 to the 1-skeleton of the complex. S^1 denotes the unit circle in $\mathbf{R}^2$. A *combinatorial disk* is a combinatorial 2-complex homeomorphic to a disk.

We shall study a simplicial complex X by associating to each cycle α in X a diagram in the Euclidean plane, called a van Kampen diagram, which contains all the essential information about α. Van Kampen diagrams are useful tools for showing collapsibility (see [12], [3]).

Let $\alpha = e_0e_1...e_n$ be a cycle in X. A *van Kampen diagram* for α is a pair (D, ϕ). D is a finite, simply connected combinatorial disk embedded in the plane, bounded by the closed edge-path $\beta = f_0f_1...f_n$. $\phi : D \to X$ is a combinatorial map assigning to each edge f_i of β in ∂D an edge $\phi(f_i) = e_i$ of α in X such that $\phi(f_i^{-1}) = \phi(f_i)^{-1}$ for all $0 \leq i \leq n$. The *area* of the diagram is given by the number of 2-simplices of D.

Let v be a vertex of X. The *degree* of v, denoted by deg v, is the number of edges having v as initial vertex.

The following lemmas represent a higher dimensional version of well known results in small cancellation theory which deals only with dimension 2 (see [12], chapter V, page $237 - 242$). They will play a crucial role when showing the main result of the paper.

Theorem 2.1. *Let X be a simply connected simplicial complex and let α be a cycle in X. Then there exists a nondegenerate van Kampen diagram (D, ϕ) for α (i.e. ϕ is a nondegenerate combinatorial map) such that ϕ is a isomorphism from the boundary of D to α.*

For the proof we refer to [9], chapter 1, page 12.

Theorem 2.2. *Let X be a systolic simplicial complex and let α be a cycle in X. Let (D, ϕ) be a nondegenerate van Kampen diagram for α. If D has minimal area, then D is systolic. If moreover α is a full cycle, then D has at least one interior vertex and every boundary vertex of D is contained in at least two 2-simplices.*

For the proof see [9], chapter 1, page 14.

3 Collapsing a systolic simplicial complex

We consider in this section finite systolic simplicial complexes of arbitrary finite dimension and prove that they are collapsible. J. Corson and B. Trace already showed that, in dimension 2, systolicity implies collapsibility (see [3]). Our proof is similar to theirs relying on the following result shown by T. Januszkiewicz and J. Swiatkowski in [9]: there exists, for any cycle in a systolic complex, a van Kampen diagram of minimal area whose disk is itself systolic.

We start by showing an important inequality that holds in any triangulated disks whose interior vertices have degree at least 6. It represents a combinatorial Gauss-Bonnet theorem on triangulated disks. Its proof is also given in [3] (see Lemma 2.1).

Lemma 3.1. *Let D be a triangulated disk whose interior vertices have degree at least* 6*. Then:*

$$\sum_{v \in \partial D} (4 - \text{deg } v) \geq 6,$$

summing over the boundary vertices of D.

Proof: We denote the set of interior vertices of D by $int(D)$. We denote by V, V_{int}, V_{ext}, E, E_{ext} and F the number of vertices, interior vertices, exterior vertices, edges, exterior edges and 2-cells of D, respectively. The following relations hold in any triangulated disk: $1 = V - E + F$ (Euler's characteristic), $2E - E_{ext} = 3F$, $V_{ext} = E_{ext}$, $\sum_v \text{deg } v = 2E$. Using these relations, we obtain:

$$\begin{aligned}
6 &= 6(V - E + F) = \\
&= 6V - 6E + 6(\tfrac{2}{3}E - \tfrac{1}{3}E_{ext}) = \\
&= 6V - 2E_{ext} - 2E = \\
&= 6V_{int} + 4V_{ext} - (\sum_{v \in int(D)} \deg\, v + \sum_{v \in \partial D} \deg\, v) = \\
&= (6V_{int} - \sum_{v \in int(D)} \deg\, v) + (4V_{ext} - \sum_{v \in \partial D} \deg\, v).
\end{aligned}$$

Thus

$$6 = \sum_{v \in int(D)} (6 - \deg\, v) + \sum_{v \in \partial D} (4 - \deg\, v).$$

Because D is a disk whose interior vertices have degree at least 6, the above relation implies that

$$\sum_{v \in \partial D} (4 - \deg\, v) \geq 6.$$

□

Let X be a simplicial $n-$complex, n arbitrary. We introduce the following elementary operations on edge-paths in X.

1. *Free reduction*:

   Let α be an edge-path containing a subpath of the form ee^{-1} and let β be the edge-path obtained by deleting this subpath. We call the passage from α to β a *free reduction*.

2. *Short-cut*:

   Let v_0 and v_k be two of the $k+1$ vertices of a k-simplex σ of X, $2 \leq k \leq n$. Let $\alpha = e_0 e_1 ... e_m$ and $\beta = f_0 f_1 ... f_p$ be two edge-paths of X such that $|\alpha| \neq |\beta|$, $e_0, ..., e_m, f_0, ..., f_p$ being distinct edges of σ with $i(e_0) = i(f_0) = v_0, t(e_m) = t(f_p) = v_k, e_i \cap f_j = \varnothing, 1 \leq i \leq m-1, 1 \leq j \leq p-1$. The passage from α to β, is called a *short-cut*.

3. *Elementary exchange*:

   Let τ be a $(k-1)$-simplex in X spanned by the vertices $v^0, ..., v^{k-1}$. Let v_0 and v_1 be two vertices of X such that $\tau * \{v_0\}$ and $\tau * \{v_1\}$ are two distinct k-simplices of X, $2 \leq k \leq n$. The passage from one edge-path to another, obtained by replacing a subpath $[v_0, v^i][v^i, v_1], 0 \leq i \leq k-1$, by the subpath $[v_0, v^j][v^j, v_1], 0 \leq j \leq k-1, i \neq j$, is called an *elementary exchange*.

We notice that free reductions and short-cuts change the length of edge-paths, whereas elementary exchanges do not. This will be important. Besides, we notice that none of the elementary operations alter the endpoints of edge-paths.

If we can pass from one edge-path α to another edge-path β via a finite sequence of elementary exchanges, then we call the edge-paths α and β *exchangeable* and we write $\alpha \equiv \beta$. An edge-path α is *strongly reduced* if for any edge-path β, exchangeable with α, β does not admit a free reduction or short-cut.

If there exists a finite sequence of elementary operations passing from an edge-path α to another edge-path β, then α and β are path-homotopic. The converse affirmation remains true in locally 6-large simplicial complexes of arbitrary dimension. The proof of the following theorem is similar to the one given by J. Corson and B. Trace for 2-complexes (see [3], Theorem 3.1).

Theorem 3.2. *Let X be a locally 6-large simplicial complex of finite dimension. Let β be a strongly reduced edge-path in X. Let α be any edge-path, path-homotopic to β such that the cycle $\alpha\beta^{-1}$ is a full subcomplex in X. Then there exists a finite sequence of elementary operations passing from α to β.*

Proof: We apply elementary operations on α until it becomes strongly reduced. We must prove the existence of a finite sequence of elementary exchanges from α to β.

Since the paths α and β are path-homotopic, the cycle $\alpha\beta^{-1}$ is null-homotopic. The proof is by induction on the area of $\alpha\beta^{-1}$. There exists a van Kampen diagram (D, ϕ) for $\alpha\beta^{-1}$. We choose D to be of minimal area. Because X is locally 6-large, and $\alpha\beta^{-1}$ is a full cycle in X, Theorem 2.2 guarantees that D is itself locally 6-large, it has at least one interior vertex and every boundary vertex of D is contained in at least two 2-simplices.

Let v_0, v_1 and v_2 be boundary vertices of D. Because D has at least one interior vertex, at least one of deg v_0, deg v_1 or deg v_2, say deg v_0, must be three or greater. Because the edge-paths α and β in X are strongly reduced, no boundary vertex of D has degree smaller than 2. Hence $(4 - \deg v_0) \leq 1$ and $(4 - \deg v_i) \leq 2$ for $i \in \{1, 2\}$. Lemma 3.1 further implies

$$\begin{aligned} 6 &\leq (4 - \deg v_0) + (4 - \deg v_1) + (4 - \deg v_2) + \\ &+ \sum_{v \in \partial D, v \notin \{v_0, v_1, v_2\}} (4 - \deg v) \leq \\ &\leq 5 + \sum_{v \in \partial D, v \notin \{v_0, v_1, v_2\}} (4 - \deg v). \end{aligned}$$

So D has an exterior vertex $v \notin \{v_0, v_1, v_2\}$ such that deg $v \leq 3$. So deg $v = 2$ or deg $v = 3$. The open star of v in D contains therefore one or two 2-simplices.

By deleting the open star of v in D, we perform either a short-cut or an elementary exchange on, say α, and we reach another edge-path γ. Because the edge-paths α and β in X are strongly reduced, we can not perform short-cuts on them. So there is an elementary exchange passing from α to γ such that the area of $\gamma\beta^{-1}$ is at least two less than that of $\alpha\beta^{-1}$. The induction hypothesis therefore implies that $\alpha \equiv \gamma \equiv \beta$. □

Let X be a simplicial n-complex. Let e, e' be two directed edges in X such that $i(e) = i(e') = v$. We denote by $\rho(e, e')$ the length of a shortest edge-path in $\mathrm{Lk(v, K)}$ joining $t(e)$ and $t(e')$. We define $\rho(e, e')$ to be infinite, if there does not exist an edge-path joining $t(e)$ and

$t(e')$ in X. We call an edge-path $e_1 ... e_m$ in X a *locally geodesic* if $\rho(e_i^{-1}, e_{i+1}) \geq n+1$ for all $1 \leq i < m$. The name 'locally' does not have its traditional meaning. Instead, it suggests that on a locally geodesic one can not perform any elementary operations. So, no matter the elementary operations one performs on X, there certainly exists in the subcomplex obtained by performing an elementary operation on X, a locally geodesic joining those pairs of points which are joined in X by a locally geodesic.

Remark 3.3. Let X be a finite n-dimensional locally 6-large simplicial complex and let α be a closed edge-path in X. Consider the edge-paths obtained by performing an elementary operation on α and put them in a list L. Note that there is a finite number of such edge-paths. Consider further those edge-paths obtained by performing an elementary operation on some edge-path in L. Put these edge-paths in another list. X being finite, continuing in this manner, after a finite number of steps, we will either reach the trivial path, in which case α is null homotopic, or every edge-path in the final list will have already occurred in a previous list. So, in the second case, there is no finite sequence of elementary operations passing from α to the trivial path. Hence, because X is locally 6-large, Theorem 3.2 ensures that α is not path homotopic to the trivial path, i.e. it is not null homotopic. The edge-path α is therefore essential.

The following result is an application of Theorem 3.2 showing the collapsibility of finite systolic simplicial complexes of arbitrary dimension.

Corollary 3.4. *Any finite systolic $n-$dimensional simplicial complex X is collapsible.*

Proof: Let L be a connected, full subcomplex of X that has more than one vertex and that does not admit any elementary collapses. So the dimension of L is at least 1. In case L is 1-dimensional, since it is not collapsible, it is not contractible either and its fundamental group is therefore nontrivial.

Another case is when L is k-dimensional, $2 \leq k \leq n$, and every m-simplex of L is a proper face of at least two simplices of L, $0 \leq m \leq k-1$. Because L is finite, and it does not admit any elementary operations, by choosing edges in succesion, we can construct in L a locally geodesic α. As L is finite, some subpath of α, say β, is a closed locally geodesic in L. Being a full subcomplex in a locally 6-large simplicial complex, L is itself a locally 6-large simplicial complex (see [9], page 9, Fact 1.2(2)). Remark 3.3 hence guarantees that β is an essential edge-path in L. The fundamental group of L is therefore nontrivial.

But any simply connected connected subcomplex of X has a trivial fundamental group. Any simply connected subcomplex of X containing more than one vertex, admits therefore an elementary collapse. So X can be collapsed to a point by a random sequence of elementary collapses. $\square$

Because collapsible spaces are contractible, the following holds.

Corollary 3.5. *Any finite systolic simplicial complex of finite dimension is contractible.*

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