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An analytic study on the multi-pantograph delay equations with variable coefficients*

by
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Abstract

In this paper, the homotopy perturbation method (HPM) is applied to solve the multi-pantograph delay equations with variable coefficients. The sufficient conditions are given to assure the convergence of this method. Several examples are presented to demonstrate the efficiency and reliability of the HPM and numerical results are discussed. Compared with other related methods in references, the results of the HPM show its better performance than others.

Key Words: Multi-pantograph equations, homotopy perturbation method, homotopy analysis method, delay differential equations, convergence analysis.
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1 Introduction

In this paper, we consider the following multi-pantograph equations

$$u'(t) = a(t)u(t) + \sum_{i=1}^{l} b_i(t)u(q_i t) + f(t), \quad 0 < t \le T, \tag{1}$$

$$u(0) = u_0, \tag{2}$$

where $a(t)$, $b_i(t)$ and $f(t)$ are analytical functions, $q_i \in (0,1)$, $i = 1, 2, \cdots, l$.

The pantograph equations is a kind of delay differential equations and arise in many applications such as electrodynamics, astrophysics, nonlinear dynamical systems, probability theory on algebraic structures, quantum mechanics and cell growth, etc.

In recent years, the multi-pantograph equations were studied by many authors numerically and analytically. For instance, Muroya et al. [15] used the collocation method to solve the

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multi-pantograph delay equation numerically. In 2004, Liu and Li [13] showed some properties of the analytic solution and numerical solution of the multi-pantograph equations, respectively. On the other hand, Li and Liu [11] applied the Runge-Kutta methods to the multi-pantograph delay equation. And Liu et al. [14] used the modified Runge-Kutta methods for solving the pantograph equation. In 2005, Evans and Raslan [2] used the Adomian decomposition method for solving the delay differential equation. In 2007, Keskin et al. [10] applied the differential transform method to obtain the approximate solution. At the same time, Sezer and Dascioglu [16] developed and applied the Taylor method to the generalized pantograph equation with retarded case or advanced case. In addition, Sezer et al. [17] applied the Taylor method to the nonhomogenous multi-pantograph equation with variable coefficients and obtained the approximate solution. Lately, Brunner [1] used the collocation methods for pantograph-type volterra functional equations with multiple delays and Yu [18] applied the variational iteration method to the multi-pantograph delay equation, respectivley. The analytical results have been obtained in terms of convergent series, but their computational process is complex.

Since the homotopy perturbation method (HPM) was proposed and developed by He [6-9], this method has been successfully applied to solve many types of nonlinear problems [3-5]. In practice, HPM is a powerful analytic tool and does not need small parameters in the equations. Moreover, HPM yields rapidly convergent series solutions. On the other hand, the homotopy analysis method (HAM) was presented and developed by Liao [12], this method also has been successfully applied to solve many types of nonlinear problems. It is pointed out that HAM involve the presence of an auxiliary parameter $\hbar$ which is determined from the so-called $\hbar$-curves in order to ensure a fast convergence of the method. But in general case, it is not easy to get the optimal values of $\hbar$ of HAM. In the iterative methods by HAM for solving the multi-pantograph delay equations, if the convergence-control parameter $\hbar$ equates -1, then we get the HPM, which is a special case of the late HAM.

In this paper, we intend to effectively employ the homotopy analysis method to solve the multi-pantograph delay equations with variable coefficients. The approximate analytical results can be obtained with only a few iterations. When the convergence-control parameter equates -1 in the iterative methods, the convergence region and rate of solution of HPM series are independent of the parameter. The sufficient conditions to guarantee the convergence of series solution of the considered problems are presented. Finally, numerical results show that the numerical solution approximates the exact solution with a high degree of accuracy. And the procedure is more simple.

An outline of the paper is as follows. In Section 2, the homotopy analysis method will be presented as it applies to the multi-pantograph delay equations. When the convergence-control parameter equates -1, the sufficient conditions are given to assure the convergence of HPM. In Section 3, the algorithm of HPM is implemented for some numerical examples. The conclusions are made in the last section.

2 Construction of iterative methods by HAM

In the following we consider two cases of the source term $f(t)$ for solving the multi-pantograph delay equations.

(I). If f is independent of q_i and u, then we use the homotopy analysis technique [7] and directly introduce a parameter $p \in [0,1]$ in the original equations in the form

$$
\begin{aligned}
(1-p)L(u(t)) &= hp(L(u(t)) - \sum_{i=1}^{l} b_i(t)u(q_i t)), \quad 0 < t \leq T, &(3)\\
u(0) &= u_0, &(4)
\end{aligned}
$$

where $L(u(t)) = u'(t) - a(t)u(t) - f(t)$ and h is the convergence-control parameter. Of course, we can use other linear operators. For example, $L(u(t)) = u'(t) - a(t)u(t)$ or $L(u(t)) = u'(t)$, accordingly we need change the nonlinear part.

Assume that the solution of Eqs. (3)-(4) can be expressed as a power series in p :

$$
u(t) = \sum_{j=0}^{\infty} u_j(t)p^j. \tag{5}
$$

Substituting (5) into (3)-(4) and obtaining following equations

$$
\begin{aligned}
&p^0 : L(u_0(t)) = 0, \quad u_0(0) = u_0,\\
&p^n : L(u_n(t)) = (1+h)L(u_{n-1}(t)) - h \sum_{i=1}^{l} b_i(t)u_{n-1}(q_i t), \quad u_n(0) = 0, \quad n = 1, 2 \cdots ,
\end{aligned}
$$

From the above equations, we suppose that

$$
L(u_n(t)) = -h \sum_{j=1}^{n} (1+h)^{n-j} \sum_{i=1}^{l} b_i(t)u_{j-1}(q_i t), \quad n = 1, 2 \cdots ,
$$

According to the mathematical induction, we have

$$
\begin{aligned}
L(u_{n+1}(t)) &= (1+h)L(u_n(t)) - h \sum_{i=1}^{l} b_i(t)u_n(q_i t)\\
&= -h(1+h) \sum_{j=1}^{n} (1+h)^{n-j} \sum_{i=1}^{l} b_i(t)u_{j-1}(q_i t) - h \sum_{i=1}^{l} b_i(t)u_n(q_i t)\\
&= -h \sum_{j=1}^{n+1} (1+h)^{n+1-j} \sum_{i=1}^{l} b_i(t)u_{j-1}(q_i t), \quad n = 1, 2 \cdots .
\end{aligned}
$$

So we obtain

$$
u_{n+1}(t) = -h \sum_{j=1}^{n+1} (1+h)^{n+1-j} \sum_{i=1}^{l} \int_0^t b_i(\eta)u_{j-1}(q_i \eta)e^{\int_\eta^t a(s)ds} d\eta.
$$

Setting $h = -1$ in above equations, we have the algorithm of HPM

$$
\begin{aligned}
u_0(t) &= u_0 e^{\int_0^t a(s)ds} + \int_0^t f(\eta)e^{\int_\eta^t a(s)ds} d\eta, &(6)\\
u_n(t) &= \sum_{i=1}^{l} \int_0^t b_i(\eta)u_{n-1}(q_i \eta)e^{\int_\eta^t a(s)ds} d\eta, \quad n = 1, 2, \cdots . &(7)
\end{aligned}
$$

Therefore, we get the n-order approximate solution

$$\widehat{u}_n(t) = \sum_{j=0}^{n} u_j(t) = u_0(t) + \sum_{i=1}^{l} \int_0^t b_i(\eta)\widehat{u}_{n-1}(q_i\eta)e^{\int_\eta^t a(s)ds}d\eta. \tag{8}$$

(II). If f is dependent on q_i and u, then we adopt following perturbation equation

$$\begin{aligned}(1-p)\widetilde{L}(\widetilde{u}(t)) &= hp(\widetilde{L}\widetilde{u} - f(t, q_1, \cdots, q_l) - \sum_{i=1}^{l} b_i(t)\widetilde{u}(q_i t)),\ 0 < t \le T, && (9)\\ \widetilde{u}(0) &= u_0, && (10)\end{aligned}$$

where $\widetilde{L}(\widetilde{u}(t)) = \widetilde{u}'(t) - a(t)\widetilde{u}(t)$. Taking $h = -1$ and using the same technique and denotations, we substitute (5) into (9)-(10) and obtain the following equations

$$\begin{aligned}\widetilde{u}_0(t) &= u_0 e^{\int_0^t a(s)ds}, && (11)\\ \widetilde{u}_1(t) &= \int_0^t (f(\eta, q_1, \cdots, q_l) + \sum_{i=1}^{l} b_i(\eta)\widetilde{u}_0(q_i\eta))e^{\int_\eta^t a(s)ds}d\eta, && (12)\\ \widetilde{u}_n(t) &= \sum_{i=1}^{l} \int_0^t b_i(\eta)\widetilde{u}_{n-1}(q_i\eta)e^{\int_\eta^t a(s)ds}d\eta,\ n = 2, 3, \cdots. && (13)\end{aligned}$$

Therefore, we have the following n-order approximate solution

$$\widehat{u}_n(t) = \sum_{j=0}^{n} \widetilde{u}_j(t) = \widetilde{u}_0(t) + \int_0^t (f(\eta, q_1, \cdots, q_l) + \sum_{i=1}^{l} b_i(\eta)\widehat{u}_{n-1}(q_i\eta))e^{\int_\eta^t a(s)ds}d\eta. \tag{14}$$

It is pointed out that Eq. (14) is same with Eq. (8) as $n \ge 1$, when f is independent of u. The iteration formula (6)-(7) and (11)-(13) makes a recurrence sequence $\widehat{u}_n(t)$ respectively. Obviously, the limit of the sequence $\widehat{u}_n(t)$ will be the solution of Eqs. (1)-(2).

Theorem 2.1. Assume that $a(t)$, $b_i(t)$, $(i = 1, \cdots, l)$ and $f(t)$ are continuous functions on the closed interval $[0, T]$, then the series of (8) is convergent as $n \to \infty$ for $t \in [0, T]$.

Proof. According to (6), we have $||u_0(t)||_\infty \le M$, where M is constant and $||u_0(t)||_\infty = \max\limits_{t\in[0,T]} |u_0(t)|$.

From (7), we obtain

$$|u_1(t)| = |\sum_{i=1}^{l} \int_0^t b_i(\eta)u_0(q_i\eta)e^{\int_\eta^t a(s)ds}d\eta| \le Me^{||a(t)||_\infty T} \sum_{i=1}^{l} ||b_i(t)||_\infty t \le Ct,$$

where $C = Me^{||a(t)||_\infty T} \sum\limits_{i=1}^{l} ||b_i(t)||_\infty$ and $||a(t)||_\infty = \max\limits_{t\in[0,T]} |a(t)|$.

From (7)-(8), it follows that

$$\begin{aligned} |u_2(t)| &= |\sum_{i=1}^{l}\int_0^t b_i(\eta)u_1(q_i\eta)e^{\int_\eta^t a(s)ds}d\eta| \\ &\le Me^{||a(t)||_\infty T}\sum_{i=1}^{l}||b_i(t)||_\infty \int_0^t Cq_i\eta d\eta \le q\frac{C^2}{2!}t^2, \end{aligned}$$

where $q = \max\limits_{1\le i\le l} q_i < 1$. Furthermore, we get

$$\begin{aligned} |u_3(t)| &= |\sum_{i=1}^{l}\int_0^t b_i(\eta)u_2(q_i\eta)e^{\int_\eta^t a(s)ds}d\eta| \\ &\le Me^{||a(t)||_\infty T}\sum_{i=1}^{l}||b_i(t)||_\infty \int_0^t q\frac{C^2}{2!}(q_i\eta)^2 d\eta \le q^3\frac{C^3}{3!}t^3. \end{aligned}$$

Suppose that $|u_n(t)| \le q^{\frac{(n-1)n}{2}}\dfrac{C^n}{n!}t^n$. According to the mathematical induction, we have

$$\begin{aligned} |u_{n+1}(t)| &\le |\sum_{i=1}^{l}\int_0^t b_i(\eta)u_n(q_i\eta)e^{\int_\eta^t a(s)ds}d\eta| \\ &\le Me^{||a(t)||_\infty T}\sum_{i=1}^{l}||b_i(t)||_\infty \int_0^t q^{\frac{(n-1)n}{2}}\frac{C^n}{n!}(q_i t)^n d\eta \\ &\le q^{\frac{n(n+1)}{2}}\tfrac{C^{n+1}}{(n+1)!}t^{n+1}, \quad (0<q<1). \end{aligned}$$

As we know the series of $\sum\limits_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}\frac{C^{n+1}}{(n+1)!}t^{n+1}$ is convergent for the whole solution domain $t \in (-\infty, +\infty)$, hence the series of (8) is absolute convergence, i.e., the sequence $\widehat{u}_n(t)$ is convergent for all $t \in [0, T]$ as $n \to \infty$.

Remark: If we choose other values of the convergence-control parameter h of HAM, it is not easy to obtain the rigorous convergence results for the general n-order approximate solution. It is pointed out that we can get the different results for other values of h and the h-curves of a fixed example for demonstrating the convergence of HAM. But we have to compute many series terms for the higher order analytical approximate solutions. Since HPM does not need small parameters and it is easy to use, we present some examples which only use zero-order approximation to get the exact solution for HPM in the next section. So the values of parameter h are optimal in these cases. For other examples, the determination of optimum values of the parameters h and the speed of convergence of these iterative methods need further study.

3 Numerical experiments

In this section we present some numerical experiments to check the numerical theory developed in the previous section, and the performances of HPM will be shown by the problems with the analytical solutions.

Example 1. **[11]** Now we consider the following multi-pantograph delay equation

$$\begin{cases} u'(t) = -u(t) + b_1(t)u(\frac{1}{2}t) + b_2(t)u(\frac{1}{4}t), \quad 0 < t \le 1, \\ u(0) = 1. \end{cases}$$

where $b_1(t) = -e^{-\frac{1}{2}t}\sin(\frac{1}{2}t)$, $b_2(t) = -2e^{-\frac{3}{4}t}\sin(\frac{1}{4}t)\cos(\frac{1}{2}t)$.

According to Eqs. (6)-(7), we obtain

$$\begin{aligned} u_0(t) = &\ e^{-t}, \\ u_1(t) = &\ (2\cos(\tfrac{1}{2}t) + \tfrac{4}{3}\cos(\tfrac{3}{4}t) - 4\cos(\tfrac{1}{4}t) + \tfrac{2}{3})e^{-t}, \\ u_2(t) = &\ (\tfrac{32}{27}\cos(\tfrac{9}{16}t) + \tfrac{32}{5}\cos(\tfrac{5}{16}t) + \tfrac{4}{3}\cos(\tfrac{1}{2}t) + \tfrac{4}{3}\cos(\tfrac{1}{4}t) - \tfrac{32}{3}\cos(\tfrac{1}{16}t) - \tfrac{32}{21}\cos(\tfrac{7}{16}t) \\ &+ \tfrac{32}{3}\cos(\tfrac{3}{16}t) - 8\cos(\tfrac{3}{8}t) - \tfrac{8}{5}\cos(\tfrac{5}{8}t) + \tfrac{40}{21}\cos(\tfrac{7}{8}t) + \tfrac{20}{9}\cos(\tfrac{3}{4}t) + \tfrac{32}{45}\cos(\tfrac{15}{16}t) \\ &- \tfrac{32}{11}\cos(\tfrac{11}{16}t) - \tfrac{32}{13}\cos(\tfrac{13}{16}t) - \tfrac{8}{3}\cos(\tfrac{1}{8}t) + \tfrac{550156}{135135})e^{-t}. \end{aligned}$$

Comparison of the approximate solution $\widehat{u}_2(t)$ with the exact solution $u(t) = e^{-t}\cos(t)$ is illustrated in Table 1. Numerical results is the same as the results by the variational iteration method in [18]. But the computational process of HPM is easy. Moreover, if we consider the following perturbation equation

$$u'(t) = -u(t) - e^{-t}\sin t + p(b_1(t)u(\frac{1}{2}t) + b_2(t)u(\frac{1}{4}t) + e^{-t}\sin t).$$

Then we can get $u_0(t) = e^{-t}\cos(t)$, and $u_n(t) = 0$, $\forall n \ge 1$. Namely $\widehat{u}_n(t) = u(t)$.

Table 1. Numerical comparison of $\widehat{u}_n(t)$ of Example 1 with $u(t)$

t	0	0.2	0.4	0.6	0.8	1.0
$\widehat{u}_2(t)$	1.00000	0.80241	0.61741	0.45296	0.31306	0.19880
$u(t)$	1.00000	0.80241	0.61741	0.45295	0.31305	0.19877
$\frac{\lvert\widehat{u}_2(t)-u(t)\rvert}{\lvert u(t)\rvert}$	0	7.05E-9	4.77E-7	6.01E-6	3.95E-5	1.91E-4

Example 2. **[2]** Solve the following pantograph delay equation

$$\begin{cases} u'(t) = \frac{1}{2}u(t) + \frac{1}{2}e^{\frac{1}{2}t}u(\frac{1}{2}t), \quad 0 < t \le 1, \\ u(0) = 1. \end{cases}$$

According to Eqs. (6)-(7), we obtain

$$\begin{aligned} &u_0(t) = e^{\frac{1}{2}t}, \\ &u_1(t) = 2(e^{\frac{1}{4}t} - 1)e^{\frac{1}{2}t}, \\ &u_2(t) = (\tfrac{8}{3}e^{\frac{3}{8}t} - 4e^{\frac{1}{4}t} + \tfrac{4}{3})e^{\frac{1}{2}t}, \\ &u_3(t) = (\tfrac{64}{21}e^{\frac{7}{16}t} - \tfrac{16}{3}e^{\frac{3}{8}t} + \tfrac{8}{3}e^{\frac{1}{4}t} - \tfrac{8}{21})e^{\frac{1}{2}t}, \\ &u_4(t) = (\tfrac{1024}{315}e^{\frac{15}{32}t} - \tfrac{128}{21}e^{\frac{7}{16}t} + \tfrac{32}{9}e^{\frac{3}{8}t} - \tfrac{16}{21}e^{\frac{1}{4}t} + \tfrac{16}{315})e^{\frac{1}{2}t}, \\ &u_5(t) = (\tfrac{32768}{9765}e^{\frac{31}{64}t} - \tfrac{2048}{315}e^{\frac{15}{32}t} + \tfrac{256}{63}e^{\frac{7}{16}t} - \tfrac{64}{63}e^{\frac{3}{8}t} + \tfrac{32}{315}e^{\frac{1}{4}t} - \tfrac{32}{9765})e^{\frac{1}{2}t}, \\ &u_6(t) = (\tfrac{2097152}{615195}e^{\frac{63}{128}t} - \tfrac{65536}{9765}e^{\frac{31}{64}t} + \tfrac{4096}{945}e^{\frac{15}{32}t} - \tfrac{512}{441}e^{\frac{7}{16}t} + \tfrac{128}{945}e^{\frac{3}{8}t} - \tfrac{64}{9765}e^{\frac{1}{4}t} + \tfrac{64}{615195})e^{\frac{1}{2}t}. \end{aligned}$$

Here the exact solution is e^t. In [16], it is pointed out that the Taylor methods has better results than the spline methods. And the absolute errors of Adomian decomposition methods and Taylor methods seem like each other. Comparison of the approximate solution $\widehat{u}_5(t)$ and $\widehat{u}_6(t)$ with the computed results by Taylor methods is illustrated in Table 2. Numerical results show that HPM approximates the exact solution with a high degree of accuracy, using only few terms. Moreover, if we consider following perturbation equation

$$u'(t) = u(t) + p(\frac{1}{2}e^{\frac{1}{2}t}u(\frac{1}{2}t) - \frac{1}{2}u(t)).$$

Then we can get $u_0(t) = e^t$, and $u_n(t) = 0,\ \forall n \geq 1$. Namely $\widehat{u}_n(t) = u(t)$.

Table 2. Numerical comparison of $\widehat{u}_n(t)$ of Example 2 with $u(t)$

t	0.2	0.4	0.6	0.8	1.0
$\frac{\lvert u(t)-\widehat{u}_5(t)\rvert}{\lvert u(t)\rvert}$	4.13E-14	2.57E-12	2.84E-11	1.55E-10	5.76E-10
$\frac{\lvert u(t)-\widehat{u}_6(t)\rvert}{\lvert u(t)\rvert}$	0	1.19E-15	1.86E-14	1.41E-13	6.54E-13
Taylor (N=8)	1.18E-12	5.04E-10	1.62E-8	1.81E-7	1.13E-6
Taylor (N=12)	1.80E-16	8.93E-16	1.20E-13	4.21E-12	6.36E-11

Example 3. **[18]** Consider the multi-pantograph delay equation

$$\begin{cases} u'(t) = -\frac{5}{6}u(t) + 4u(\frac{1}{2}t) + 9u(\frac{1}{3}t) + t^2 - 1, & 0 < t \leq 1, \\ u(0) = 1. \end{cases}$$

According to Eqs. (6)-(7), we obtain

$$\begin{aligned}
u_0(t) = &\ \tfrac{6}{5}t^2 - \tfrac{72}{25}t + \tfrac{282}{125} - \tfrac{157}{125}e^{-\frac{5}{6}t}, \\
u_1(t) = &\ -\tfrac{7536}{625}e^{-\frac{5}{12}t} + \tfrac{72}{25}t^2 - \tfrac{3024}{125}t + \tfrac{8028}{125} - \tfrac{12717}{625}e^{-\frac{5}{18}t} - \tfrac{19887}{625}e^{-\frac{5}{6}t}, \\
u_2(t) = &\ \tfrac{3737448}{3125} - \tfrac{241152}{3125}e^{-\frac{5}{24}t} + \tfrac{864}{125}t^2 - \tfrac{101088}{625}t - \tfrac{1610847}{3125}e^{-\frac{5}{18}t} - \tfrac{4272912}{15625}e^{-\frac{5}{36}t} \\
&\ -\tfrac{954576}{3125}e^{-\frac{5}{12}t} - \tfrac{3090231}{12500}e^{-\frac{5}{54}t} + \tfrac{13925343}{62500}e^{-\frac{5}{6}t}, \\
u_3(t) = &\ -\tfrac{3157056}{3125}t + \tfrac{62092656}{3125} - \tfrac{46301184}{109375}e^{-\frac{5}{48}t} - \tfrac{5821995204}{1328125}e^{-\frac{5}{108}t} - \tfrac{541244592}{78125}e^{-\frac{5}{36}t} \\
&\ -\tfrac{2011931136}{859375}e^{-\frac{5}{72}t} - \tfrac{30546432}{15625}e^{-\frac{5}{24}t} + \tfrac{1127952783}{312500}e^{-\frac{5}{18}t} + \tfrac{167104116}{78125}e^{-\frac{5}{12}t} \\
&\ -\tfrac{391435821}{62500}e^{-\frac{5}{54}t} - \tfrac{2252778399}{812500}e^{-\frac{5}{162}t} - \tfrac{2932264952559}{5317812500}e^{-\frac{5}{6}t} + \tfrac{10368}{625}t^2.
\end{aligned}$$

Comparison of the approximate solution $\widehat{u}_3(t)$ with the exact solution $u(t) = 1 + \frac{67}{6}t + \frac{1675}{72}t^2 + \frac{12157}{1296}t^3$ is illustrated in Table 3. This results is different from the variational iteration method's, since we choose the different initial values. Of course, we can get the same results by choosing the proper initial values.

Table 3: Numerical comparison of $\widehat{u}_n(t)$ of Example 3 with $u(t)$

t	0	0.2	0.4	0.6	0.8	1.0
$\widehat{u}_3(t)$	1.00000	4.23202	9.68320	17.5864	28.0635	41.1495
$u(t)$	1.00000	4.23893	9.78923	18.1012	29.6250	44.8110
$\frac{\lvert u(t)-\widehat{u}_3(t)\rvert}{\lvert u(t)\rvert}$	4.09E-12	1.63E-3	1.08E-2	2.84E-2	5.27E-2	8.17E-2

Example 4. **[16]** Consider the following pantograph delay equation

$$\begin{cases} u'(t) = -u(t) - u(0.8t), \\ u(0) = 1. \end{cases}$$

According to Eqs. (6)-(7), we obtain

$$\begin{aligned}
u_0(t) =&\ e^{-t},\\
u_1(t) =&\ (-5e^{\frac{1}{5}t}+5)e^{-t},\\
u_2(t) =&\ (\tfrac{125}{9}e^{\frac{9}{25}t}-25e^{\frac{1}{5}t}+\tfrac{100}{9})e^{-t},\\
u_3(t) =&\ (\tfrac{-15625}{549}e^{\frac{61}{125}t}+\tfrac{625}{9}e^{\frac{9}{25}t}-\tfrac{500}{9}e^{\frac{1}{5}t}+\tfrac{8000}{549})e^{-t},\\
u_4(t) =&\ (\tfrac{9765625}{202581}e^{\frac{369}{625}t}-\tfrac{78125}{549}e^{\frac{61}{125}t}+\tfrac{12500}{81}e^{\frac{9}{25}t}-\tfrac{40000}{549}e^{\frac{1}{5}t}+\tfrac{2560000}{202581})e^{-t},\\
u_5(t) =&\ (\tfrac{-30517578125}{425622681}e^{\frac{2101}{3125}t}+\tfrac{48828125}{202581}e^{\frac{369}{625}t}-\tfrac{1562500}{4941}e^{\frac{61}{125}t}+\tfrac{1000000}{4941}e^{\frac{9}{25}t}\\
&-\tfrac{12800000}{202581}e^{\frac{1}{5}t}+\tfrac{3276800000}{425622681})e^{-t},\\
u_6(t) =&\ (\tfrac{476837158203125}{4907003889249}e^{\frac{11529}{15625}t}-\tfrac{152587890625}{425622681}e^{\frac{2101}{3125}t}+\tfrac{976562500}{1823229}e^{\frac{369}{625}t}\\
&-\tfrac{125000000}{301401}e^{\frac{61}{125}t}+\tfrac{320000000}{1823229}e^{\frac{9}{25}t}-\tfrac{16384000000}{425622681}e^{\frac{1}{5}t}+\tfrac{16777216000000}{4907003889249})e^{-t}.
\end{aligned}$$

Comparison of the approximate solution $\widehat{u}_5(t)$ and $\widehat{u}_6(t)$ with the exact solution by Taylor method [16] is illustrated in Table 4. Clearly, the HPM has better results than the Taylor method. On the other hand, we consider the HAM for solving this problem. By using the iterative method in Section2, we obtain the following equations:

$$\begin{cases} u_0'(t) = -u_0(t),\\ u_0(0) = 1.\end{cases}$$
$$\begin{cases} u_1'(t) = -u_1(t) + hu_0(0.8t),\\ u_1(0) = 0.\end{cases}$$
$$\begin{cases} u_n'(t) = -u_n(t) + (1+h)(u_{n-1}'(t) + u_{n-1}(t)) + hu_{n-1}(0.8t),\\ u_n(0) = 0. \qquad n \geq 2\end{cases}$$

Then we have

$$\begin{aligned}
&u_0(t) = e^{-t},\\
&u_1(t) = 5h(e^{0.2t}-1)e^{-t},\\
&u_2(t) = [(5h-20h^2)e^{\frac{t}{5}} + \tfrac{125h^2}{9}e^{\frac{9t}{25}} + \tfrac{55h^2}{9} - 5h]e^{-t}, \ \cdots .
\end{aligned}$$

So we get the approximation solution $\widehat{u}_2(t)$ of HAM. Now we choose different values of the convergence-control parameter h to compute the second-order approximate solution, numerical results are shown in Figure 1. Obviously, we can see that the optimal values of h is less than -1. If the exact solution is known, we can get the optimal values. It is our further study for the general cases.

Table 4. Numerical comparison of $\widehat{u}_n(t)$ of Example 4 with $u(t)$

t	0	0.2	0.4	0.6	0.8	1.0
Taylor (N=8)	1.00000	0.66469	0.433561	0.276483	0.171494	0.102744
Taylor (N=11)	1.00000	0.66469	0.433561	0.276482	0.171484	0.102670
$\widehat{u}_5(t)$	1.00000	0.664691	0.433561	0.276481	0.171476	0.102643
$\widehat{u}_6(t)$	1.00000	0.664691	0.433561	0.276482	0.171484	0.102671

Example 5. **[17]** Consider the pantograph equation of third order

$$\begin{cases} u'''(t) = -u(t) - u(t-0.3) + e^{-t+0.3}, \quad 0 \leq t \leq 1\\ u(0) = 1,\ u'(0) = -1,\ u''(0) = 1.\end{cases}$$

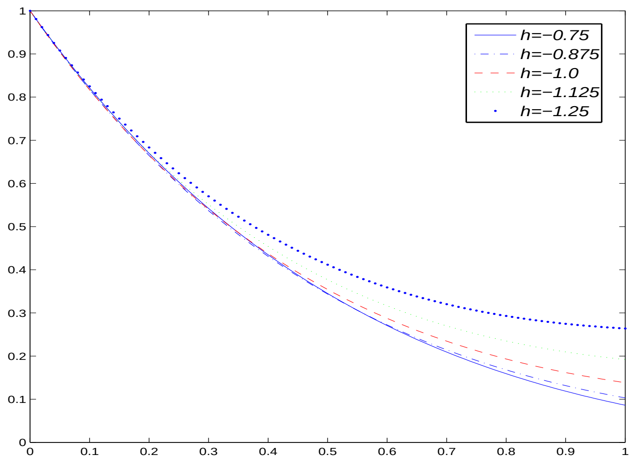

Figure 1: Numerical solutions at different values of convergence-control parameter h.

According to Eqs. (6)-(7), then we get $u_0(t) = e^{-t},\ u_i(t) = 0, i = 1, 2\cdots$. Hence, we obtain $\widehat{u}_n(t) = u_0(t) = e^{-t}$, which is the exact solution. Compared with the Adomian decomposition method [2] and the Taylor method [16], this method is the best way to solve above problem.

Example 6. [2] Consider the pantograph equation of second order

$$\begin{cases} u''(t) = \frac{3}{4}u(t) + u(\frac{1}{2}t) - t^2 + 2, & 0 < t \leq 1, \\ u(0) = 0,\ u'(0) = 0. \end{cases}$$

According to Eqs. (6)-(7), we get the solutions

$$\begin{aligned} u_0(t) &= t^2 - \tfrac{1}{12}t^4, \\ u_1(t) &= \tfrac{1}{12}t^4 - \tfrac{2}{885}t^6, \\ u_2(t) &= \tfrac{2}{885}t^6 - \tfrac{1}{32144}t^8, \\ u_3(t) &= \tfrac{1}{32144}t^8 - \tfrac{1}{3846150}t^{10}, \\ u_4(t) &= \tfrac{1}{3846150}t^{10} - \tfrac{1}{676923060}t^{12}, \\ u_5(t) &= \tfrac{1}{676923060}t^{12} - \tfrac{1}{163963963800}t^{14}, \quad \ldots \end{aligned}$$

Here, the exact solution is $u(t) = t^2$. If we take more terms of the convergence series, we also obtain the same result. Namely, $\lim_{n\to\infty} \widehat{u}_n(t) = t^2$.

4 Conclusions

In this paper, we successfully apply HPM to the multi-pantograph delay equations without any assumptions and restrictions on the parameters. And we obtain the high approximate solutions or the exact solutions within a few iterations. Some numerical experiments have been provided to illustrate that the present method are effective in accuracy and convergence speed. In a word, the HPM is a promising method for many nonlinear problems.

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