Filippov lemma for a certain differential inclusion of fourth-order

by

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Abstract

We consider a boundary value problem for a fourth-order nonconvex differential inclusion and we establish some Filippov type existence results for this problem.

Key Words: Boundary value problem, differential inclusion, contractive set-valued map.

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1 Introduction

This paper is concerned with the following boundary value problem for fourth-order differential inclusions

\[
\begin{aligned}
L_4 x(t) + a(t) x(t) &\in F(t, x(t)) \quad \text{a.e. } ([0, T]), \\
L_i x(0) &= L_i x(T), \quad i = 0, 1, 2, 3,
\end{aligned}
\]

where \( L_0 x(t) = a_0(t) x(t), \) \( L_i x(t) = a_i(t) (L_{i-1} x(t))', \) \( i = 1, 2, 3, \)
\( L_4 x(t) = (a_3(t) (a_2(t) (a_1(t) (a_0(t) x(t)))'))', \)
\( a(\cdot), a_i(\cdot) : [0, T] \to \mathbb{R} \) are continuous mappings,
\( a_0(t) \equiv 1, a(t) \geq 0, a_i(t) > 0, i = 1, 2, t \in [0, T], \)
a_1(t) \equiv a_3(t) \) and \( F(\cdot, \cdot) : [0, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R}) \) is a set-valued map.

The present paper is motivated by relatively recent papers of Švec ([15]) and Arara, Benchohra, Ntouyas and Ouahab ([3]), where several existence results for problem (1.1)-(1.2) are provided. In [15], by means of the Ky Fan fixed point theorem it is obtained an existence result for problem (1.1)-(1.2), when \( F(\cdot, \cdot) \) is upper semicontinuous and has convex compact values. In [3] the situation when \( F(\cdot, \cdot) \) has nonconvex values is investigated and two existence results are obtained using the Covitz and Nadler multivalued contraction principle and the Bressan and Colombo selection theorem for lower semicontinuous set-valued maps with decomposable values.

The aim of our paper is to present two additional existence results for problem (1.1)-(1.2) which are Filippov type existence results for this problem. The first one is obtained by the application of the set-valued contraction principle in the space of derivatives of solutions instead of the space of solutions as in [3]. In addition, as usual at a Filippov existence type theorem,
our result provides an estimate between the starting "quasi" solution and the solution of the differential inclusion. The idea of applying the set-valued contraction principle in the space of derivatives of the solutions belongs to Tallos ([11,16]) and it was already used for other classes of differential inclusions.

In our second approach we show that Filippov's ideas ([10]) can be suitably adapted in order to obtain the existence of solutions for problem (1.1)-(1.2). Recall that for a differential inclusion defined by a lipschitzian set-valued map with nonconvex values, Filippov’s theorem ([10]) consists in proving the existence of a solution starting from a given "quasi" solution.

For the motivation of study of problem (1.1)-(1.2) we refer to [3,15] and references therein. Several qualitative properties of the solutions of forth-order differential equations and inclusions may be found in [1,2,5,8,9,12,14] etc..

The paper is organized as follows: in Section 2 we recall some preliminary results that we need in the sequel and in Section 3 we prove our main results.

2 Preliminaries
In this short section we sum up some basic facts that we are going to use later.

Let \((X,d)\) be a metric space and consider a set valued map \(T\) on \(X\) with nonempty values in \(X\). \(T\) is said to be a \(\lambda\)-contraction if there exists \(0 < \lambda < 1\) such that:

\[
d_H(T(x), T(y)) \leq \lambda d(x,y) \quad \forall x, y \in X,
\]

where \(d_H(\cdot,\cdot)\) denotes the Pompeiu-Hausdorff distance. Recall that the Pompeiu-Hausdorff distance of the closed subsets \(A, B \subset X\) is defined by

\[
d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\},
\]

where \(d(x, B) = \inf_{y \in B} d(x, y)\).

The set-valued contraction principle ([7]) states that if \(X\) is complete, and \(T : X \to \mathcal{P}(X)\) is a set valued contraction with nonempty closed values, then \(T(.)\) has a fixed point, i.e. a point \(z \in X\) such that \(z \in T(z)\).

We denote by \(Fix(T)\) the set of all fixed points of the set-valued map \(T\). Obviously, \(Fix(T)\) is closed.

**Proposition 2.1** ([13]) Let \(X\) be a complete metric space and suppose that \(T_1, T_2\) are \(\lambda\)-contractions with closed values in \(X\). Then

\[
d_H(Fix(T_1), Fix(T_2)) \leq \frac{1}{1 - \lambda} \sup_{z \in X} d(T_1(z), T_2(z)).
\]

Let \(I = [0, T]\), we denote by \(C(I, \mathbb{R})\) the Banach space of all continuous functions from \(I\) to \(\mathbb{R}\) with the norm \(||x(.)||_C = \sup_{t \in I} |x(t)|\) and \(L^1(I, \mathbb{R})\) is the Banach space of integrable functions \(u(.) : I \to \mathbb{R}\) endowed with the norm \(||u(.)||_1 = \int_0^T |u(t)| dt\). If \(J \subset \mathbb{R}\) is an interval, by \(AC^i(J, \mathbb{R}), i = 0, 1, 2, 3\) we denote the space of \(i\)-times differentiable functions \(x(.) : J \to \mathbb{R}\) whose \(i^{th}\) derivative \(x^{(i)}(.)\) is absolutely continuous.
A function \( x(.) \in AC^3(I, \mathbb{R}) \) is called a solution of problem (1.1)-(1.2) if there exists a function \( f(.) \in L^1(I, \mathbb{R}) \) with \( f(t) \in F(t, x(t)) \) a.e. \( I \) such that \( L_i x(0) = L_i x(T), \ i = 0, 1, 2, 3. \)

**Lemma 2.2** ([15]) The boundary value problem

\[
L_4 x(t) + a(t)x(t) = 0, \quad (2.1)
\]

\[
L_i x(0) = L_i x(T), \quad i = 0, 1, 2, 3, \quad (2.2)
\]

has only the trivial solution \( x(t) \equiv 0 \) on \( I \).

Therefore, if \( f(.) : [0, T] \rightarrow \mathbb{R} \) is an integrable function, there exists the Green function \( G(., .) \) for problem

\[
L_4 x(t) + a(t)x(t) = f(t), \quad (2.3)
\]

\[
L_i x(0) = L_i x(T), \quad i = 0, 1, 2, 3, \quad (2.4)
\]

and the solution of problem (2.3)-(2.4) is given by

\[
x(t) = \int_0^T G(t, s)f(s)ds. \quad (2.5)
\]

According to [15] the Green function \( G(., .) \) is bounded. Let \( G_0 := \sup_{t,s \in I} |G(t, s)|. \)

### 3 The main results

We study first problem (1.1)-(1.2) with fixed point techniques. In order to do this we introduce the following hypothesis.

**Hypothesis 3.1** (i) \( F(., .) : I \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R}) \) has nonempty closed values and for every \( x \in \mathbb{R}, F(., x) \) is measurable.

(ii) There exists \( L(.) \in L^1(I, \mathbb{R}_+) \) such that for almost all \( t \in I, F(., .) \) is \( L(t) \)-Lipschitz in the sense that

\[
d_H(F(t, x), F(t, y)) \leq L(t)|x - y| \quad \forall \ x, y \in \mathbb{R}.
\]

(iii) \( d(0, F(t, 0)) \leq L(t) \quad \text{a.e.} \ (I) \)

Denote \( L_0 := \int_0^1 L(s)ds. \)

**Theorem 3.2** Assume that Hypothesis 3.1 is satisfied and \( G_0L_0 < 1 \). Let \( y(.) \in AC^3(I, \mathbb{R}) \) be such that \( L_i y(0) = L_i y(T), \ i = 0, 1, 2, 3 \) and there exists \( q(.) \in L^1(I, \mathbb{R}_+) \) with \( d(L_4 y(t) + a(t)y(t), F(t, y(t))) \leq q(t) \), a.e. \( I \).

Then for every \( \varepsilon > 0 \) there exists \( x(.) \) a solution of problem (1.1)-(1.2) satisfying for all \( t \in I \)

\[
|x(t) - y(t)| \leq \frac{G_0}{1 - G_0L_0} \int_0^T q(t)dt + \varepsilon.
\]

**Proof.** For \( u(.) \in L^1(I, \mathbb{R}) \) define the following set-valued maps

\[
M_0(t) = F(t, \int_0^T G(t, s)u(s)ds), \quad t \in I.
\]
\[ T(u) = \{ \phi(.) \in L^1(I, \mathbb{R}); \ \phi(t) \in M_u(t) \ \text{a.e. (I)} \}. \]

It follows from Lemma 2.2 that \( x(.) \) is a solution of problem (1.1) if and only if \( L_4 x(.) + a(.) x(.) \) is a fixed point of \( T(.) \).

We shall prove first that \( T(u) \) is nonempty and closed for every \( u \in L^1(I, \mathbb{R}) \). The fact that the set valued map \( M_u(.) \) is measurable is well known. For example the map \( t \to \int_0^T G(t,s)u(s)ds \) can be approximated by step functions and we can apply Theorem III. 40 in [6]. Since the values of \( F \) are closed with the measurable selection theorem (Theorem III.6 in [6]) we infer that \( M_u(.) \) admits a measurable selection \( \phi \). One has
\[
|\phi(t)| \leq d(0, F(t,0)) + d_H(F(t,0), F(t, \int_0^T G(t,s)u(s)ds)) \leq L(t)(1 + G_0 \int_0^T |u(s)|ds),
\]
which shows that \( \phi \in L^1(I, \mathbb{R}) \) and \( T(u) \) is nonempty.

On the other hand, the set \( T(u) \) is also closed. Indeed, if \( \phi_n \in T(u) \) and \( ||\phi_n - \phi||_1 \to 0 \) then we can pass to a subsequence \( \phi_{n_k} \) such that \( \phi_{n_k}(t) \to \phi(t) \) for a.e. \( t \in I \), and we find that \( \phi \in T(u) \).

We show next that \( T(.) \) is a contraction on \( L^1(I, \mathbb{R}) \).

Let \( u, v \in L^1(I, \mathbb{R}) \) be given and \( \phi \in T(u) \). Consider the following set-valued map:
\[ H(t) = M_v(t) \cap \{ x \in \mathbb{R}; \ |\phi(t) - x| \leq L(t) \int_0^T G(t,s)(u(s) - v(s))ds \}. \]

From Proposition III.4 in [6], \( H(.) \) is measurable and from Hypothesis 3.1 ii) \( H(.) \) has nonempty closed values. Therefore, there exists \( \psi(.) \) a measurable selection of \( H(.) \). It follows that \( \psi \in T(v) \) and according with the definition of the norm we have
\[
||\phi - \psi||_1 = \int_0^T |\phi(t) - \psi(t)|dt \leq \int_0^T L(t)(\int_0^T |G(t,s)|u(s) - v(s)|ds)dt
= \int_0^T (\int_0^T L(t)|G(t,s)|u(s) - v(s)|ds)dt \leq G_0 L_0 ||u - v||_1.
\]

We deduce that
\[
d(\phi, T(v)) \leq G_0 L_0 ||u - v||_1.
\]
Replacing \( u \) by \( v \) we obtain
\[
d_H(T(u), T(v)) \leq G_0 L_0 ||u - v||_1,
\]
thus \( T(.) \) is a contraction on \( L^1(I, \mathbb{R}) \).

We consider next the following set-valued maps
\[ F_1(t, x) = F(t, x) + q(t)[-1, 1], \quad (t, x) \in I \times \mathbb{R}, \]
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\[ M_1^1(t) = F_1(t, \int_0^T G(t, s)u(s)ds), \]

\[ T_1(u) = \{ \psi(.) \in L^1(I, \mathbb{R}); \quad \psi(t) \in M_1^1(t) \quad \text{a.e. (I)}, \quad u(.) \in L^1(I, \mathbb{R}) \}. \]

Obviously, \( F_1(.,.) \) satisfies Hypothesis 3.1.

Repeating the previous step of the proof we obtain that \( T_1 \) is also a \( G_0L_0 \)-contraction on \( L^1(I, \mathbb{R}) \) with closed nonempty values.

We prove next the following estimate

\[ d_H(T(u), T_1(u)) \leq \int_0^T q(t)dt. \]  \( (3.1) \)

Let \( \phi \in T(u) \) and define

\[ H_1(t) = M_1^1(t) \cap \{ z \in \mathbb{R}; \quad |\phi(t) - z| \leq q(t) \}. \]

With the same arguments used for the set valued map \( H(.,.) \), we deduce that \( H_1(.,.) \) is measurable with nonempty closed values. Hence let \( \psi(.) \) be a measurable selection of \( H_1(.,.) \). It follows that \( \psi \in T_1(u) \) and one has

\[ ||\phi - \psi||_1 = \int_0^T |\phi(t) - \psi(t)|dt \leq \int_0^T q(t). \]

As above we obtain (3.1).

We apply Proposition 2.1 and we infer that

\[ d_H(Fix(T), Fix(T_1)) \leq \frac{1}{1 - G_0L_0} \int_0^T q(t)dt. \]

Since \( v(.) = L_4y(.) + a(.)y(.) \in Fix(T_1) \) it follows that for any \( \varepsilon > 0 \) there exists \( u(.) \in Fix(T) \) such that

\[ ||v - u||_1 \leq \frac{1}{1 - G_0L_0} \int_0^T q(t)dt + \frac{\varepsilon}{G_0}. \]

We define \( x(t) = \int_0^T G(t, s)u(s)ds, \quad t \in I \) and we have

\[ |x(t) - y(t)| \leq \int_0^T |G(t, s)| |u(s) - v(s)|ds \leq \frac{G_0}{1 - G_0L_0} \int_0^T q(t)dt + \varepsilon \]

which completes the proof.

The assumption in Theorem 3.2 is satisfied, in particular, for \( y(.) = 0 \) and therefore, via Hypothesis 3.1 (iii), with \( q(.) = L(.) \). We obtain the following consequence of Theorem 3.2.

**Corollary 3.3** Assume that Hypothesis 3.1 is satisfied and \( G_0L_0 < 1 \). Then for every \( \varepsilon > 0 \) there exists \( x(.) \) a solution of problem (1.1)-(1.2) satisfying for all \( t \in I \)

\[ |x(t)| \leq \frac{G_0L_0}{1 - G_0L_0} + \varepsilon. \]  \( (3.2) \)
Remark 3.4 Corollary 3.3 is an improvement of Theorem 3.3 in [3]. According to Theorem 3.3 in [3] if Hypothesis 3.1 is satisfied, $G_0L_0 < 1$ and $F(\cdot, \cdot)$ has compact values then problem (1.1)-(1.2) has at least a solution. Obviously, in Corollary 3.3 we do not assume that the values of $F(\cdot)$ are compact. Moreover, in (3.2) we obtained a priori bounds for the solution.

We present next the main result of this paper. Its proof uses Filippov’s construction of successive approximations ([10]).

Theorem 3.5 Assume that Hypothesis 3.1 (i), (ii) is satisfied and $G_0L_0 < 1$. Let $y(\cdot) \in AC^3(I, \mathbb{R})$ be such that $L_iy(0) = L_iy(T)$, $i = 0, 1, 2, 3$ and there exists $q(\cdot) \in L^1(I, \mathbb{R}_+)$ with $d(L_qy(t) + a(t)y(t), F(t, y(t))) \leq q(t)$, a.e. $(I)$. Then there exists $x(\cdot)$ a solution of problem (1.1)-(1.2) satisfying for all $t \in I$

$$|x(t) - y(t)| \leq \frac{G_0}{1 - G_0L_0} \int_0^T q(t)dt. \quad (3.3)$$

Proof. The set-valued map $t \to F(t, y(t))$ is measurable with closed values and

$$F(t, y(t)) \cap \{L_qy(t) + a(t)y(t) + q(t)[\cdot - 1, 1]\} \neq \emptyset \quad a.e. \ (I).$$

It follows (e.g., Theorem 1.14.1 in [4]) that there exists a measurable selection $f_1(t) \in F(t, y(t))$ a.e. $(I)$ such that

$$|f_1(t) - L_qy(t) - a(t)y(t)| \leq q(t) \quad a.e. \ (I) \quad (3.4)$$

Define $x_1(t) = \int_0^T G(t, s)f_1(s)ds$ and one has

$$|x_1(t) - y(t)| \leq G_0 \int_0^T q(t)dt.$$

We claim that it is enough to construct the sequences $x_n(\cdot) \in C(I, \mathbb{R})$, $f_n(\cdot) \in L^1(I, \mathbb{R})$, $n \geq 1$ with the following properties

$$x_n(t) = \int_0^T G(t, s)f_n(s)ds, \quad t \in I, \quad (3.5)$$

$$f_n(t) \in F(t, x_{n-1}(t)) \quad a.e. \ (I), \quad n \geq 1, \quad (3.6)$$

$$|f_{n+1}(t) - f_n(t)| \leq L(t)|x_n(t) - x_{n-1}(t)| \quad a.e. \ (I), \quad n \geq 1. \quad (3.7)$$

If this construction is realized then from (3.4)-(3.7) we have for almost all $t \in I$

$$|x_{n+1}(t) - x_n(t)| \leq \int_0^T |G(t, t_1)| |f_{n+1}(t_1) - f_n(t_1)|dt_1 \leq$$

$$G_0 \int_0^T L(t_1)|x_n(t_1) - x_{n-1}(t_1)|dt_1 \leq G_0 \int_0^T L(t_1) \int_0^T |G(t_1, t_2)|dt_1.$$
Filippov lemma

\[ |f_n(t_2) - f_{n-1}(t_2)| dt_2 \leq G_0^2 \int_0^T L(t_1) \int_0^T L(t_2) |x_{n-1}(t_2) - x_{n-2}(t_2)| dt_2 dt_1 \]
\[ \leq (G_0)^n \int_0^T L(t_1) \int_0^T L(t_2) \cdots \int_0^T L(t_n) |x_1(t_n) - y(t_n)| dt_n \cdots dt_1 \leq \]
\[ \leq (G_0L_0)^n G_0 \int_0^T q(t) dt. \]

Therefore \( \{x_n(.)\} \) is a Cauchy sequence in the Banach space \( C(I, \mathbb{R}) \), hence converging uniformly to some \( x(.) \in C(I, \mathbb{R}) \). Therefore, by (3.7), for almost all \( t \in I \), the sequence \( \{f_n(t)\} \)

is Cauchy in \( \mathbb{R} \). Let \( f(.) \) be the pointwise limit of \( f_n(.) \).

Moreover, one has

\[ |x_n(t) - y(t)| \leq |x_1(t) - y(t)| + \sum_{i=1}^{n-1} |x_{i+1}(t) - x_i(t)| \]
\[ \leq G_0 \int_0^T q(t) dt + \sum_{i=1}^{n-1} (G_0 \int_0^T q(t) dt)(G_0L_0)^i = \frac{G_0 \int_0^T q(t) dt}{1 - G_0 L_0}. \quad (3.8) \]

On the other hand, from (3.4), (3.7) and (3.8) we obtain for almost all \( t \in I \)

\[ |f_n(t) - L_4 y(t) - a(t)y(t)| \leq \sum_{i=1}^{n-1} |f_{i+1}(t) - f_i(t)| + |f_1(t) - L_4 y(t)| \]
\[ -a(t)y(t)| \leq L(t) \frac{G_0 \int_0^T q(t) dt}{1 - G_0 L_0} + q(t). \]

Hence the sequence \( f_n(.) \) is integrably bounded and therefore \( f(.) \in L^1(I, \mathbb{R}) \).

Using Lebesgue’s dominated convergence theorem and taking the limit in (3.5), (3.6) we deduce that \( x(.) \) is a solution of (1.1)-(1.2). Finally, passing to the limit in (3.8) we obtained the desired estimate on \( x(.) \).

It remains to construct the sequences \( x_n(.) \), \( f_n(.) \) with the properties in (3.5)-(3.7). The

construction will be done by induction.

Since the first step is already realized, assume that for some \( N \geq 1 \) we already constructed

\( x_n(.) \in C(I, \mathbb{R}) \) and \( f_n(.) \in L^1(I, \mathbb{R}) \), \( n = 1, 2, \ldots N \) satisfying (3.5), (3.7) for \( n = 1, 2, \ldots N \) and (3.6) for \( n = 1, 2, \ldots N - 1 \). The set-valued map \( t \rightarrow F(t, x_N(t)) \) is measurable. Moreover, the map \( t \rightarrow L(t)|x_N(t) - x_{N-1}(t)| \) is measurable. By the lipschitzianity of \( F(t, .) \) we have that for almost all \( t \in I \)

\[ F(t, x_N(t)) \cap \{f_N(t) + L(t)|x_N(t) - x_{N-1}(t)|[-1,1]\} \neq \emptyset. \]

Theorem 1.14.1 in [4] yields that there exist a measurable selection \( f_{N+1}(.) \) of \( F(., x_N(.)) \)

such that

\[ |f_{N+1}(t) - f_N(t)| \leq L(t)|x_N(t) - x_{N-1}(t)| \quad a.e. (I). \]

We define \( x_{N+1}(.) \) as in (3.5) with \( n = N + 1 \). Thus \( f_{N+1}(.) \) satisfies (3.6) and (3.7) and the proof is complete.

Remark 3.6 Obviously, Theorem 3.5 extends Theorem 3.2. We do not suppose that \( d(0, F(t, 0)) \leq L(t) \quad a.e. (I) \) and the estimate in (3.3) is better than the one in Theorem 3.2.
Even if Theorem 3.5 improves Theorem 3.2, we chosen to present both results; on one hand because the methods used in their proofs are different and on the other hand to show that there exists situations when the fixed point approaches are less powerful.

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References


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