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Morphic objects in categories

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Abstract

An R-module ${}_RM$ is called *morphic* if $M/\mathrm{im}\alpha \cong \ker\alpha$ for every endomorphism α of M, that is, if the dual of the Noether isomorphism theorem holds.

In this paper we consider this notion in categories with kernels and images and recover most of its properties under suitable conditions. Connection with unit-regular and regular objects is made.

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1 Introduction

In [5] (1976) Ehrlich proved that an endomorphism α of a module ${}_RM$ is unit-regular if and only if it is regular and $M/\mathrm{im}(\alpha) \cong \ker\alpha$, that is, the dual of the Noether isomorphism theorem holds for α. After nearly 30 years, the interest for this dual florished in 2003-2004, when Nicholson and Sánchez Campos published a series of papers ending with [8]. Nowadays, the subject continues to be investigated in several papers, including (Abelian) groups (see [1], [9]).

An endomorphism α was called *morphic* if $M/\mathrm{im}(\alpha) \cong \ker\alpha$, a module ${}_RM$ was called *morphic* if all its endomorphisms are morphic and a ring R was called *left morphic* provided that ${}_RR$ is a morphic module.

Since an endomorphism α is morphic if and only if there is an endomorphism β such that $\mathrm{im}\beta = \ker\alpha$ and $\mathrm{im}\alpha = \ker\beta$, we can define morphic (endomorphisms and) objects in a category $\mathcal{C}$, provided $\mathcal{C}$ has kernels and images. In this note, some results are generalized in so called Puppe-exact categories (p-exact for short, see [3] or [2]), and most of them are recovered in abelian categories. Connection with unit-regular and regular objects is made. The main source of inspiration was [8].

To be precise, hereafter a category with zero is called *p-exact* if every morphism factors as a conormal epi followed by a normal mono. As a consequence, kernels and cokernels exist, all monos are normal and all epis are conormal. In such categories, products, pullbacks and pushouts need not exist: a p-exact category is abelian (with a well-determined additive structure) if and only if it has finite products.

Note that such categories were called exact by Mitchell and Herrlich, Strecker (see [7] or [6]).

2 Morphic objects

Definition. Let M be an object in a category $\mathcal{C}$ with kernels and images. An endomorphism α of M is called *morphic* if there is an endomorphism β of M such that the following sequence is exact

$$M \xrightarrow{\alpha} M \xrightarrow{\beta} M \xrightarrow{\alpha} M.$$

The object M is itself *morphic* if every endomorphism α of M is morphic. Hence *an endomorphism α is morphic if and only if there is an endomorphism β such that* $\mathrm{im}\beta = \ker\alpha$ *and* $\mathrm{im}\alpha = \ker\beta$.

Examples. 1) If R is any ring with identity, the morphic objects in the category R-**Mod** (left R-modules) are exactly the morphic modules (see [8]).

2) In **pEns**, the category of pointed sets, one can check a simple characterization: *a pointed set (A, a) is morphic if and only if the set A is finite.*

3) Let $\mathcal{T}$ (see [6], Ex. 39B) be the full subcategory of **pTop** (pointed topological spaces) whose objects are a singleton pointed space $\{*\}$, and a 3 elements pointed space (T, a), with the open sets: $\varnothing$, T, $\{a\}$ and $\{b, c\}$ if $T = \{a, b, c\}$.

Thus, $\mathcal{T}$ has kernels and cokernels, is normal and conormal but is not p-exact. One can check (T, a) has only 5 endomorphisms and that *both objects in $\mathcal{T}$ are morphic.*

4) Since **Grp**, the category of groups, is not normal (and so, nor p-exact), one has to define morphic endomorphisms by requiring the image to be a normal subgroup. This is done in [9].

Remarks. 1) Every *automorphism* $M \to M$ is morphic (take $\beta = 0$). In particular, identity endomorphisms are morphic, and since zero morphisms are trivially morphic, so is the zero object. More, if all nonzero endomorphisms of an object are automorphisms, the object itself is morphic.

2) *An object which is not morphic in a category $\mathcal{C}$ may be morphic in a subcategory of $\mathcal{C}$.* Indeed, the additive group of the real numbers $(\mathbf{R}, +)$, is torsion-free divisible of continuum rank in **Ab**, the category of Abelian groups, i.e., $\mathbf{R} = \bigoplus_{\aleph_1} \mathbf{Q}$. As such, it is not morphic (see [1]). However, consider **HausAb**, the (not full) subcategory of all Hausdorff (topological) Abelian groups together with continuous group homomorphisms. Considered in this subcategory, $(\mathbf{R}, +)$ is morphic.

3) *Subobjects of morphic objects need not be morphic.* Take $\mathbf{Z}$ and $\mathbf{Q}$ as $\mathbf{Z}$-modules.

4) *Factor objects of morphic objects need not be morphic.* Take $\mathbf{Q}$ and $\mathbf{Q}/\mathbf{Z}$.

5) *Composition of morphic endomorphisms need not be morphic.* Indeed, consider $M = \mathbf{Z}_2 \oplus \mathbf{Z}_4$, as $\mathbf{Z}$-module, $p : M \longrightarrow M$ the projection on $\mathbf{Z}_2$ and $q : M \longrightarrow M$ defined by $q(\widehat{x}, \overline{y}) = (\widehat{y}, \overline{2x})$. It is readily checked that q is well-defined, both p and q are morphic, but the composition $q \circ p$ is not. Actually, both $\ker(q \circ p)$ and $M/\mathrm{im}(q \circ p)$ have order 4 but the first is cyclic and the second, of Klein type. Hence $\ker(q \circ p) \ncong M/\mathrm{im}(q \circ p)$.

3 p-exact categories

First recall that in a p-exact category, for a (short) exact sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

we frequently denote (up to an isomorphism) C by B/A.

This way, for $A \xrightarrow{\alpha} B$, if $A \xrightarrow{q} \mathrm{im}\alpha \xrightarrow{v} B$ is the (co)image mono-epi decomposition, then

$$0 \longrightarrow \ker\alpha \longrightarrow A \xrightarrow{q} \mathrm{im}\alpha \longrightarrow 0$$

is a short exact sequence and so (1-st Noether Theorem) $A/\ker\alpha \cong \mathrm{im}\alpha$.

Lemma 1. *In a p-exact category an endomorphism α is morphic if and only if $M/\mathrm{im}\alpha \cong \ker\alpha$.*

Proof: If α is morphic, let β be associated to α. Since the 1-st Noether Theorem holds for β, i.e., $A/\ker\beta \cong \mathrm{im}\beta$, by replacement we obtain $A/\mathrm{im}\alpha \cong \ker\alpha$. Conversely, suppose $\sigma : M/\mathrm{im}\alpha \longrightarrow \ker\alpha$ is an isomorphism. Using the notations in the following diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & \ker\alpha & \xrightarrow{u} & M & \xrightarrow{q} & \mathrm{im}\alpha & \longrightarrow & 0 \\
 & & & & & \alpha \searrow & \downarrow v & & \\
 & & & & \nwarrow \sigma & & M & & \\
 & & & & & & \downarrow p & & \\
 & & & & & & M/\mathrm{im}\alpha & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

consider $\beta = u \circ \sigma \circ p$. Then $\ker\beta = \ker(u\sigma p) = \ker p = v$ and so $\ker\beta = \mathrm{im}\alpha$. Finally, since the mono-epi decomposition of β is $M \xrightarrow{p} M/\mathrm{im}\alpha \xrightarrow{\sigma} \ker\alpha \xrightarrow{u} M$, we obtain $\mathrm{im}\beta = \ker\alpha$. $\square$

Corollary 2. *A morphic endomorphism in a p-exact category is monic if and only if it is epic.*

Proof: Suppose α is monic. Since $M \xrightarrow{\alpha} M \xrightarrow{\beta} M \xrightarrow{\alpha} M$ is exact, $\mathrm{im}\beta = \ker\alpha = 0$, which makes β the zero morphism. Hence $\mathrm{im}\alpha = \ker\beta = M$ and so α is epic (in a p-exact category, using the mono-epi decomposition and $\mathrm{im}\alpha = \mathrm{coim}\alpha = M$, it follows that $\alpha = 1_M\alpha$ is epic).

Conversely, suppose α is epic. Then again, $\alpha = 1_M\alpha$ is the mono-epi decomposition and so $\ker\beta = M$. Thus β is a zero morphism and similarly $\mathrm{im}\beta = \ker\alpha = 0$ and α mono. $\square$

Example. Let R be a (noncommutative) ring with identity which has a left (right) 0-divisor which is not right (left) 0-divisor. Then R, viewed as a category with only one object, is not morphic.

Proposition 3. *The following are equivalent for an object M in a p-exact category:*
(1) M is morphic.
(2) If $M/K \cong N$ where K and N are subobjects of M, then $M/N \cong K$.

Proof: For a subobject (K, u) of a morphic object M, denote (up to isomorphism) by $M/K = (\mathrm{co}\ker(u), p)$, that is, we have a short exact sequence

$$0 \longrightarrow K \xrightarrow{u} M \xrightarrow{p} M/K \longrightarrow 0.$$

Suppose (N, v) is another subobject of M together with the corresponding short exact sequence

$$0 \longrightarrow N \xrightarrow{v} M \xrightarrow{q} M/N \longrightarrow 0$$

and $M/K \overset{\phi}{\cong} N$; we have to prove $M/N \cong K$.

Consider $\alpha : M \xrightarrow{p} M/K \xrightarrow{\phi} N \xrightarrow{v} M$. If β is associated to α, then $\mathrm{im}\beta = \ker\alpha = \ker(v \circ \phi \circ p) = \ker p = K$ and $\ker\beta = \mathrm{im}\alpha = N$, because α is already decomposed mono-epi and so N is its image. By 1-st Noether Theorem for β, $M/\ker\beta \cong \mathrm{im}\beta$ and so $M/\mathrm{im}\alpha \cong \ker\alpha$.

Conversely, let α be any endomorphism of M. Again, by 1-st Noether Theorem, $M/\ker\alpha \cong \mathrm{im}\alpha$ and so, by hypothesis, $M/\mathrm{im}\alpha \cong \ker\alpha$. Finally, we use Lemma 1. □

We just mention

Theorem 4. *The following conditions are equivalent for a morphic object M in a p-exact category:*
(1) Every subobject of M is isomorphic to an image of M.
(2) Every image of M is isomorphic to a subobject of M.
In this case, if N and N' are subobjects of M then $M/N \cong M/N'$ if and only if $N \cong N'$.

We just note that for all results in this section, no additive structure (i.e., an abelian group structure on the Hom-sets), nor finite products were necessary.

4 Abelian categories

Having in mind our first example and Mitchell Embedding Theorem, a natural question is whether proofs of results concerning *morphic objects in abelian categories* may be reduced to the corresponding proofs, already known, for R-modules.

Indeed, this can be done as follows.

Proposition 5. *Let $\mathcal{C}$ be a small subcategory of an abelian category $\mathcal{A}$. Then there exists a small abelian full subcategory $\mathcal{D}$ of $\mathcal{A}$ such that $\mathcal{C}$ is a subcategory of $\mathcal{D}$.*

Theorem 6. *(Mitchell) For any small abelian category there is a full and faithful exact functor (embedding) to a suitable category of modules.*

Theorem 7. *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a full and faithful exact functor between two abelian categories, and $A \in Ob(\mathcal{A})$. Then A is morphic if and only if $F(A)$ is morphic.*

Proof: Suppose A is morphic (in $\mathcal{A}$) and let $\overline{\alpha}$ be an endomorphism of $F(A)$. The functor F being full, there exists an endomorphism α of A such that $\overline{\alpha} = F(\alpha)$. Since A is morphic, there is an endomorphism $\beta \in end_{\mathcal{A}}(A)$ such that the following sequence is exact

$$A \xrightarrow{\alpha} A \xrightarrow{\beta} A \xrightarrow{\alpha} A.$$

Hence

$$F(A) \xrightarrow{\overline{\alpha}} F(A) \xrightarrow{F(\beta)} F(A) \xrightarrow{\overline{\alpha}} F(A)$$

is (together with F) also exact and $F(A)$ is morphic.

Conversely, if $F(A)$ is morphic and α is an endomorphism of A, there is an endomorphism $\overline{\beta}$ of $F(A)$ such that

$$F(A) \xrightarrow{F(\alpha)} F(A) \xrightarrow{\overline{\beta}} F(A) \xrightarrow{F(\alpha)} F(A)$$

is exact. Again, F being full, $\overline{\beta} = F(\beta)$, for a suitable $\beta \in end_{\mathcal{A}}(A)$ and so

$$F(A) \xrightarrow{F(\alpha)} F(A) \xrightarrow{F(\beta)} F(A) \xrightarrow{F(\alpha)} F(A)$$

is exact. Now, since F is exact, it preserves zero, and so, being faithful, it reflects exact sequences (see [7], Theorem 7.1). Finally

$$A \xrightarrow{\alpha} A \xrightarrow{\beta} A \xrightarrow{\alpha} A.$$

is exact and so A is morphic. ∎

Therefore, using, in abelian categories, the (traditional) term of finite *direct sum*, instead of finite coproduct (or product), we can recover the following results

Theorem 8. *Every direct summand of a morphic object is again morphic.*

However, the class of morphic modules is not closed under taking direct sums. In fact, the $\mathbf{Z}$-modules $\mathbf{Z}_2$ and $\mathbf{Z}_4$ are both morphic, but $\mathbf{Z}_2 \oplus \mathbf{Z}_4$ is not morphic.

Proposition 9. *If M and N are morphic objects with $\mathrm{Hom}_{\mathcal{C}}(M,N) = 0 = \mathrm{Hom}_{\mathcal{C}}(N,M)$, then $M \oplus N$ is morphic.*

Proposition 10. *Let M and N be objects with $\mathrm{Hom}_{\mathcal{C}}(M,N) = 0$. If there is an epimorphism $\pi : N \longrightarrow M$ then $M \oplus N$ is not morphic.*

Rephrasing a previous remark, if $\mathrm{Morf}(\mathcal{C})$ denotes the full subcategory of all the morphic objects in $\mathcal{C}$, this subcategory has no (finite) (co)products, and this opens another subject: to determine the properties of this subcategory. Unfortunately, this is a bad category. Here are some examples (already **Ab** will do).

No kernels: consider $\mathbf{Z}_4 \oplus \mathbf{Z}_4 \xrightarrow{p_1} \mathbf{Z}_4 \xrightarrow{2} 2\mathbf{Z}_4$ with p_1 the projection on the left summand and 2 the multiplication, where both $\mathbf{Z}_4 \oplus \mathbf{Z}_4$ and $2\mathbf{Z}_4 \cong \mathbf{Z}_2$ are morphic. However $\ker(2 \circ p_1) = 2\mathbf{Z}_4 \oplus \mathbf{Z}_4$ is not morphic. Hence there are no pullbacks, nor equalizers.

No images: now take $2\mathbf{Z}_4 \xrightarrow{i} \mathbf{Z}_4 \xrightarrow{i_1} \mathbf{Z}_4 \oplus \mathbf{Z}_4$ with i the inclusion and i_1 the injection in the left summand. Again $\mathrm{im}(i_1 \circ i) = 2\mathbf{Z}_4 \oplus \mathbf{Z}_4$ is not morphic.

No cokernels: again $\mathrm{co}\ker(i_1 \circ i) = (\mathbf{Z}_4 \oplus \mathbf{Z}_4)/(2\mathbf{Z}_4 \oplus \mathbf{0}) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_4$ is not morphic.

5 Unit regular, regular and morphic

An endomorphism $\alpha : M \to M$ is called *unit regular* (in the sense of G. Ehrlich) if $\alpha = \alpha\sigma\alpha$ for some automorphism σ of M, or equivalently, if $\alpha = \pi\sigma$, where $\pi^2 = \pi$ and σ is an automorphism.

As in [7], we denote by αM the image of α, and, for a morphism $\alpha : A \longrightarrow B$ and a monomorphism $A' \longrightarrow A$, we denote the image of the composition $A' \longrightarrow A \longrightarrow B$ by $\alpha(A')$.

Lemma 11. *Let $\alpha \in \mathrm{end}M$ be a morphic endomorphism in a category with kernels and images. If $\sigma : M \to M$ is an automorphism, then $\alpha\sigma$ and $\sigma\alpha$ are both morphic. In particular, every unit regular endomorphism is morphic.*

Proof: By definition, choose $\beta \in \mathrm{end}M$ such that $\beta M = \ker\alpha$ and $\ker\beta = \alpha M$. Then $\sigma\alpha M = \alpha M = \ker\beta = \ker(\beta s^{-1})$, and $\ker(\sigma\alpha) = \sigma^{-1}(\ker\alpha) = (\beta\sigma^{-1})M$, so $\sigma\alpha$ is morphic. Similarly, $\alpha\sigma M = \sigma(\ker\beta) = \ker(\sigma^{-1}\beta)$ and $\ker(\alpha\sigma) = \ker\alpha = \beta M = (\sigma^{-1}\beta)M$, so $\alpha\sigma$ is morphic. The last claim follows from the equivalent definition above. □

Regular objects M in an abelian category $\mathcal{C}$ are naturally defined by asking $\mathrm{end}_{\mathcal{C}}M$ to be a regular ring (for a more general notion of *relative regular* object see [4]). In a similar way, we can define *unit-regular objects* by asking $\mathrm{end}_{\mathcal{C}}M$ to be a unit-regular ring.

Extending an early result for modules of Azumaya (1960), in [4], for abelian categories it is proved

Proposition 12. *Let α be an endomorphism of M in an abelian category. Then α is regular if and only if both* $\mathrm{im}\alpha$ *and* $\ker\alpha$ *are direct summands of M.*

In order to relate unit-regular, regular and morphic objects in abelian categories directly (not using Mitchell's Embedding Theorem), we need the following

Lemma 13. *In an abelian category, let $\alpha \in \mathrm{end}M$ and $M = N \oplus \ker\alpha$. Then $\alpha M = \alpha N$.*

Proof: For $(K, u) = \ker\alpha$ and $(I, i) = \text{im}\alpha$, consider the crossed exact sequences

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & N & & & & \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & K & \stackrel{u}{\longrightarrow} & M & \longrightarrow & I & \longrightarrow & 0 \\ & & & & \downarrow p & & & & \\ & & & & K & & & & \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array}$$

where $p \circ u = 1_K$. Since $N \longrightarrow M \longrightarrow I$ is epimorphism if and only if $K \longrightarrow M \longrightarrow K = 1_K$ is epimorphism (see Corollary 16.8, [7]), $N \longrightarrow M \longrightarrow I \stackrel{i}{\longrightarrow} M$ is a mono-epi decomposition for $N \longrightarrow M \stackrel{\alpha}{\longrightarrow} M$ and so $\alpha M = \alpha N$. □

Remark. Embedding this diagram into a 9 Lemma type diagram (0 pull-back for the left upper square), shows that, more, $\overline{\alpha} : N \longrightarrow M \longrightarrow I$ is an isomorphism.

Using this, we can give a direct proof of the following characterization which generalizes a result of Ehrlich ([5]).

Theorem 14. *In any abelian category, an endomorphism α is unit regular if and only if it is both regular and morphic.*

Proof: If α is unit regular then it is morphic by Lemma 11. Conversely, if M is regular, by the previous Proposition, $M = \text{im}\alpha \oplus K = N \oplus \ker\alpha$, and so (since the short exact sequence

$$0 \longrightarrow \text{im}\alpha \longrightarrow M \longrightarrow K \longrightarrow 0$$

splits) $K \cong M/\text{im}\alpha$ and this is $\cong \ker\alpha$ because α is morphic, say via $\gamma : K \to \ker\alpha$.

Using the previous Lemma, from $M = \text{im}\alpha \oplus K = N \oplus \ker\alpha$ we deduce $\text{im}\alpha = \alpha N$ and so we have $M = \alpha N \oplus K$.

Further, using also the previous Remark, define $\sigma : M \to M$, that is, $\sigma : \alpha N \oplus K \longrightarrow N \oplus \ker\alpha$ by the following matrix $\sigma = \begin{bmatrix} \overline{\alpha}^{-1} & 0 \\ 0 & \gamma \end{bmatrix}$ which is clearly a unit in end$M$.

Finally, $\alpha\sigma\alpha = \alpha$ holds because of the following matrix computation:
$\begin{bmatrix} \overline{\alpha} & 0 \\ 0 & \gamma^{-1} \end{bmatrix} \begin{bmatrix} \overline{\alpha}^{-1} & 0 \\ 0 & \gamma \end{bmatrix} \begin{bmatrix} \overline{\alpha} & 0 \\ 0 & \gamma^{-1} \end{bmatrix} = \begin{bmatrix} \overline{\alpha} & 0 \\ 0 & \gamma^{-1} \end{bmatrix}$. □

An object M is *kernel-direct* if for every $\alpha \in \text{end}_{\mathcal{C}}(M)$, $\ker\alpha$ is a direct summand of M and *image-direct* if for every α, $\text{im}\alpha$ is a direct summand of M. Since for a morphic object, kernel-direct implies image-direct and conversely, we can also prove the following extension

Theorem 15. *The following are equivalent for an object M in an abelian category:*
(1) end$M$ *is unit regular.*
(2) M is morphic and kernel-direct.
(3) M is morphic and image-direct.

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