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On sums of distinct odd squares arising from a class of totally symmetric plane partitions

by
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Abstract

We prove some results about the coefficients r_n of $\prod_{i\geq 0}(1+q^{3i^2+3i+1})$. These coefficients count the number of a special type of partitions of n, namely totally symmetric plane partitions with self conjugate main diagonal. In particular, we prove the conjecture that $n = 860$ is the largest n such that $r_n = 0$.

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1 Introduction

For $n \geq 0$, let r_n be such that

$$\begin{aligned}\prod_{i\geq 0}(1+q^{3i^2+3i+1}) &= \sum_{n\geq 0} r_n q^n \\ &= 1+q+q^7+q^8+q^{19}+q^{20}+q^{26}+q^{27}+q^{37}+\cdots\end{aligned}$$

The coefficient r_n counts the number of a special type of partitions of n, namely *totally symmetric plane partitions with self conjugate main diagonal* (see [3] for the definition of such partitions and some of their properties). Although it appears that there are a lot of gaps in the above expansion at the beginning, we noticed computationally that $n = 860$ is the largest $n \leq 10000$ such that $r_n = 0$. So, we set ourselves the task to prove that this is indeed so without the computational restriction $n \leq 10000$. This is our first result.

Theorem 1. *If $r_n = 0$, then $n \leq 860$.*

Since r_n counts also the number of partitions of n in distinct parts of the form $3i^2+3i+1$ each, it follows that r_n cannot exceed the total number $q(n)$ of partitions of n into distinct parts, whose asymptotic behaviour is well known:

$$q(n) \sim \frac{1}{4\cdot 3^{1/4} n^{3/4}} \exp\left(\pi\sqrt{n/3}\right) \qquad \text{as} \qquad n\to\infty \tag{1.1}$$

(see [1]). Our next theorem addresses the growth rate of r_n as $n \to \infty$.

Theorem 2. *The asymptotic formula*

$$r_n \sim c_1 \cdot n^{-5/6} \cdot \exp(c_0 n^{1/3}) \tag{1.2}$$

holds as $n \to \infty$, where

$$c_0 = \left(\frac{9\pi}{16}\left(1 - \frac{1}{\sqrt{2}}\right)^2 \zeta(3/2)^2\right)^{1/3}$$

and

$$c_1 = \left(\frac{(1 - 1/\sqrt{2})\zeta(3/2)}{36\pi}\right)^{1/3}.$$

Our proof of Theorem 1 is elementary. The deepest tool we use is the characterization of those positive integers n which admit a representation as a sum of three squares, and this is well known. Theorem 2 is obtained as a corollary to a more general result on partitions into distinct elements from a given set.

2 An auxiliary result

We start with the following lemma.

Lemma 1. *If m is a positive integer with $m \equiv 3 \pmod 8$, $m \equiv 0 \pmod 5$ but $25 \nmid m$, then $m = x_1^2 + x_2^2 + x_3^2$, where $x_1 < x_2 < x_3$ are odd positive integers.*

Proof: It is well known that if m is a positive integer not of the form $4^e(8k+7)$, then m can be written as a sum of three squares. So, we certainly have that $m = x_1^2 + x_2^2 + x_3^2$. Since m is odd, either all three numbers $x_1,\ x_2,\ x_3$ are odd, or one is odd and two are even. If one is odd and two are even, then $x_1^2 + x_2^2 + x_3^2 \equiv 1,\ 5 \pmod 8$, which is not the case for us. Thus, $x_1,\ x_2,\ x_3$ are all odd. If they are all equal, then $m = 3x_1^2$, but this is impossible since $5 \mid m$ but $25 \nmid m$. If only two of them are equal, then we get a representation of m of the form $m = 2x^2 + y^2$ with integers x and y. Since $5 \mid m$ but $25 \nmid m$, it follows that x and y are coprime to 5 and reducing the above relation modulo 5, we get $2x^2 \equiv -y^2 \pmod 5$, or $(yx^{-1})^2 \equiv 3 \pmod 5$, which is not possible because 3 is not a quadratic residue modulo 5. □

3 Proof of Theorem 1

Let $\mathcal{L} = \{5, 6, 7, 8, 9, 10\}$. For each $n \geq 10000$ let $L \in \mathcal{L}$ be such that $n \equiv L \pmod 6$. We search for a representation of n of the form

$$n = \sum_{j=1}^{L} (3i_j^2 + 3i_j + 1) \qquad \text{where} \qquad i_1 > i_2 > \cdots > i_L. \tag{3.1}$$

Clearly the existence of such a representation for n will imply that $r_n \neq 0$. The above expression (3.1) can be rewritten as

$$\frac{4n-L}{3} = x_1^2 + \cdots + x_L^2, \qquad \text{where} \qquad x_j = 2i_j + 1 \qquad \text{for} \quad j = 1, \ldots, L.$$

To make things simpler, we put $X = (4n - L)/3$. Here is how we choose the odd numbers $x_1 > x_2 > \cdots > x_L$. First, we let

$$x_j \in \left\{ \left\lfloor \sqrt{\frac{X}{L-3}} \right\rfloor - (2j-2), \ \left\lfloor \sqrt{\frac{X}{L-3}} \right\rfloor - (2j-1) \right\} \quad \text{for} \quad j = 1, \ldots, L-5.$$

We choose them in such a way that x_j is odd. They are uniquely determined and mutually distinct for $j = 1, \ldots, L-5$. For x_{L-4} and x_{L-3}, we choose them in the following way. Let $r \in \{0, 1, 2, 3, 4\}$ be such that

$$r \equiv X - \sum_{i=1}^{L-5} x_i^2 \pmod 5.$$

We choose x_{L-4} and x_{L-3} to be odd distinct numbers in the interval

$$\mathcal{I} = \left[\left\lfloor \sqrt{\frac{X}{L-3}} \right\rfloor - (2L-1), \left\lfloor \sqrt{\frac{X}{L-3}} \right\rfloor - (2L-10) \right], \tag{3.2}$$

in such a way that $x_{L-4}^2 + x_{L-3}^2 \equiv r \pmod 5$. To see why this can be done, note that every residue class modulo 5 admits a representation as a sum of two squares $a^2 + b^2 \pmod 5$, where not both a and b are zero modulo 5 (that is, for the class 0 just take the representation $1^2 + 2^2 \pmod 5$). So, represent $r \equiv a^2 + b^2 \pmod 5$, where $b \not\equiv 0 \pmod 5$. Next choose some odd x_{L-4} in the interval $\mathcal{I}$ shown at (3.2) such that $x_{L-4} \equiv a \pmod 5$. There exists one such choice for x_{L-4} since $\mathcal{I}$ contains 10 consecutive integers. So, pick such an x_{L-4}. Next choose x_{L-3} in $\mathcal{I}$ which is odd and such that $x_{L-3} \equiv \pm b \pmod 5$. There are two such possibilities for x_{L-3}, and at least one of them will be different from the already chosen x_{L-4}. Put $m = X - \sum_{i=1}^{L-3} x_i^2$. Observe that $m \equiv 3 \pmod 8$ and $m \equiv 0 \pmod 5$. It might be the case that $m \equiv 0 \pmod{25}$. In this case, we replace x_{L-3} by $x_{L-3} - 10$. Since $x_{L-3} \equiv \pm b \pmod 5 \not\equiv 0 \pmod 5$, then x_{L-3}^2 and $(x_{L-3} - 10)^2$ are congruent modulo 8 and 5 but not modulo 25. Thus our new m has indeed the property that $m \not\equiv 0 \pmod{25}$. Furthermore, since $x_j \leq \lfloor \sqrt{X/(L-3)} \rfloor$ holds for all $j = 1, \ldots, L-3$, it follows that $m \geq 0$. Hence, m is positive (since it is congruent to 3 modulo 8).

Lemma 1 tells us that m is a sum of squares of three odd distinct positive integers, let us call them $x_{L-2} > x_{L-1} > x_L$. It remains to show that $x_{L-3} > x_{L-2}$. For this, let us estimate m from above. Clearly,

$$\begin{aligned}
x_j &> \sqrt{\frac{X}{L-3}} - 2j \qquad \text{for} \qquad j = 1, \ldots, L-5; \\
x_{L-4} &> \sqrt{\frac{X}{L-3}} - (2L-2); \\
x_{L-3} &> \sqrt{\frac{X}{L-3}} - 2L - 10.
\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{i=1}^{L-3} x_i^2 &> X - 4\sqrt{\frac{X}{L-3}}\left(\sum_{j=1}^{L-5} j + (L-1) + (L+5)\right)\\ &= X - \frac{2(L^2-5L+28)}{\sqrt{L-3}}\sqrt{X}\\ &\geq X - 59\sqrt{X},\end{aligned}$$

where we used the fact that $L \in \mathcal{L}$ (so, $L \leq 10$). Hence, $m \leq 59\sqrt{X}$, which implies that $x_{L-2} < 59^{1/2}X^{1/4}$. Since

$$x_{L-3} > \sqrt{\frac{X}{L-3}} - 2L - 10 \geq \sqrt{\frac{X}{7}} - 30,$$

it follows that it suffices that

$$\sqrt{\frac{X}{7}} - 30 \geq 59^{1/2}X^{1/4},$$

which is fulfilled for $X \geq 314167$. Since $(4n-L)/3 = X$, we get that $n = (3X+L)/4$, so $r_n \neq 0$ for $n \geq 235627$.

It remains to check the values of r_n for $n < 250000$ and this can be done computationally using Mathematica, for example.

4 Proof of Theorem 2

A general asymptotic scheme due to Meinardus ([5], see also [1]) provides a formula for the coefficients of a generating function of the form

$$\prod_{j=0}^{\infty}(1-x^{\Lambda_j})^{-1},$$

which enumerates partitions into elements from a given increasing sequence $\Lambda_0, \Lambda_1, \ldots$ of positive integers under certain technical conditions. A similar result can be obtained for generating functions of the form

$$\prod_{j=0}^{\infty}(1+x^{\Lambda_j}),$$

which enumerates partitions into distincts elements of the sequence. In our case, we have $\Lambda_j = 3j^2+3j+1$ for $j \geq 0$. Hwang [4] studies, in more generality, the bivariate generating function

$$\prod_{j=0}^{\infty}(1+ux^{\Lambda_j}),$$

in which the variable u marks the *length* (number of parts) of the associated partitions. In the case of totally symmetric plane partitions with self-conjugate main diagonal, this corresponds to the number of *layers* in the decomposition described in [3]. Hwang proves a central limit theorem for this quantity, which implies that we also have a central limit theorem for the number of layers in TSPPs with self-conjugate main diagonal. Let us first state Hwang's results explicitly:

Theorem 3 (Hwang [4]). *Suppose that the sequence Λ satisfies the following conditions:*

1. *The associated Dirichlet series $D(s) = \sum_{k\geq 0} \Lambda_k^{-s}$ converges in the half-plane $\Re(s) > \alpha > 0$, and can be analytically continued into the half-plane $\Re(s) \geq -\alpha_0$ for some $\alpha_0 > 0$. Within this half-plane, $D(s)$ is analytic except for a simple pole at $s = \alpha$ with residue A.*

2. *There exists an absolute constant C_1 such that $D(s) \ll |t|^{C_1}$ uniformly for $\Re(s) \geq -\alpha_0$ as $|t| \to \infty$.*

3. *Define $g(\tau) = \sum_{k\geq 0} e^{-\Lambda_k \tau}$, where $\tau = r + iy$ with $r > 0$ and $|y| \leq \pi$. There exists a positive constant C_2 such that $g(r) - \Re(g(\tau)) \geq C_2(\log(1/r))^{2+4/\alpha^2}$ uniformly for $\pi/2 \leq |y| \leq \pi$ as $r \to 0^+$.*

Then the asymptotic formula

$$r_n = [x^n] \prod_{j=0}^{\infty} (1 + x^{\Lambda_j}) \sim \mathcal{A} n^{\lambda} \exp\left(\mathcal{B} n^{\alpha/(\alpha+1)}\right)$$

holds with

$$\begin{aligned}
\mathcal{A} &= 2^{D(0)} \left(2\pi(1+\alpha)\right)^{-1/2} \left(A\Gamma(\alpha+1)\zeta(\alpha+1)(1-2^{-\alpha})\right)^{1/(2\alpha+2)}, \\
\mathcal{B} &= \left(1 + \frac{1}{\alpha}\right) \left(A\Gamma(\alpha+1)\zeta(\alpha+1)(1-2^{-\alpha})\right)^{1/(\alpha+1)}, \\
\lambda &= -\frac{1+\frac{\alpha}{2}}{1+\alpha}.
\end{aligned}$$

Furthermore, if ω_n denotes the length of a random partition of n into distinct elements of the sequence Λ, then the distribution of ω_n is asymptotically Gaussian, with mean

$$\mu_n \sim (\kappa\alpha)^{1/(\alpha+1)} \cdot \frac{(1-2^{1-\alpha})\zeta(\alpha)}{\alpha(1-2^{-\alpha})\zeta(\alpha+1)} \cdot n^{\alpha/(1+\alpha)}$$

and variance

$$\sigma_n^2 \sim (\kappa\alpha)^{1/(\alpha+1)} \left(\frac{(1-2^{2-\alpha})\zeta(\alpha-1)}{\alpha(1-2^{-\alpha})\zeta(\alpha+1)} - \frac{(1-2^{1-\alpha})^2\zeta(\alpha)^2}{(\alpha+1)(1-2^{-\alpha})^2\zeta(\alpha+1)^2}\right) n^{\alpha/(1+\alpha)}$$

with $\kappa = A\Gamma(\alpha)(1-2^{-\alpha})\zeta(\alpha+1)$.

In our specific case, the Dirichlet series $D(s)$ can be rewritten as follows:

$$\begin{aligned} D(s) &= \sum_{\ell=0}^{\infty}(3\ell^2+3\ell+1)^{-s} = 3^{-s}\sum_{\ell=0}^{\infty}\left(\left(\ell+\frac{1}{2}\right)^2+\frac{1}{12}\right)^{-s} \\ &= 3^{-s}\sum_{\ell=0}^{\infty}\left(\ell+\frac{1}{2}\right)^{-2s}\sum_{k=0}^{\infty}\binom{-s}{k}\left(12\left(\ell+\frac{1}{2}\right)^2\right)^{-k} \\ &= 3^{-s}\sum_{k=0}^{\infty}\binom{s+k-1}{k}\left(-\frac{1}{12}\right)^k\sum_{\ell=0}^{\infty}\left(\ell+\frac{1}{2}\right)^{-2s-2k} \\ &= 3^{-s}\sum_{k=0}^{\infty}\binom{s+k-1}{k}\left(-\frac{1}{12}\right)^k\zeta\left(2s+2k,\frac{1}{2}\right). \end{aligned}$$

The sum converges for arbitrary s and thus provides an analytic continuation with poles at $\alpha = \frac{1}{2}$ (with residue $A = \frac{1}{2\sqrt{3}}$) and all negative half-integers. This shows that the first condition of Hwang's theorem is satisfied. The second condition immediately follows from growth properties of the Hurwitz zeta function. Note also that the binomial coefficient $\binom{s+k-1}{k}$ vanishes at $s=0$ for all $k>0$, so that we obtain

$$D(0) = \zeta\left(0,\frac{1}{2}\right) = 0.$$

Finally, the third condition can be checked by well-known exponential sum techniques: we have $g(\tau) = \sum_{j\geq 0} e^{-(3j^2+3j+1)\tau}$ and thus

$$\begin{aligned} g(r) - \Re(g(\tau)) &= \sum_{j\geq 0} e^{-(3j^2+3j+1)r}\left(1-\cos((3j^2+3j+1)t)\right) \\ &\gg \sum_{0\leq j<r^{-1/2}}\left(1-\cos((3j^2+3j+1)t)\right). \end{aligned}$$

Hence we study the sum

$$\sum_{0\leq j<x}\cos((3j^2+3j+1)t) = \Re\left(\sum_{0\leq j<x} e((3j^2+3j+1)\alpha)\right),$$

where $\alpha = \frac{t}{2\pi}$ satisfies $1/4 \leq |\alpha| \leq 1/2$, and $e(x) = \exp(2\pi i x)$ as usual. We distinguish two cases: if $\alpha = \frac{a}{q} + \beta$ for coprime integers a, q with $1 \le q \le x^{1/4}$ and $|\beta| \le x^{-3/2}$, then we have

$$\begin{aligned}
\Re\left(\sum_{0\le j<x} e((3j^2+3j+1)\alpha)\right) &\le x - \sqrt{x} + \Re\left(\sum_{0\le j<\sqrt{x}} e((3j^2+3j+1)\alpha)\right) \\
&= x - \sqrt{x} + \Re\left(\sum_{0\le j<\sqrt{x}} e((3j^2+3j+1)a/q) + O(\beta j^2)\right) \\
&= x - \sqrt{x} + \Re\left(\frac{\sqrt{x}}{q}\sum_{0\le j<q} e((3j^2+3j+1)a/q)\right) + O(q+\beta x^{3/2}) \\
&= x - \sqrt{x} + \frac{\sqrt{x}}{q}\Re\left(\sum_{0\le j<q} e((3j^2+3j+1)a/q)\right) + O(x^{1/4}).
\end{aligned}$$

The Gauss sum

$$\sum_{0\le j<q} e((3j^2+3j+1)a/q)$$

is well known to be $O(\sqrt{q})$ (see [2]). Moreover, it can only be equal to q if q divides all the numbers $3j^2+3j+1$, $j \ge 0$; this is clearly only the case for $q = 1$, but this is excluded by the condition $1/4 \le |\alpha| \le 1/2$. Therefore, there must be an absolute constant $\eta > 0$ such that

$$\Re\left(\sum_{0\le j<q} e((3j^2+3j+1)a/q)\right) \le (1-\eta)q$$

for all $q > 1$. But this finally yields

$$\Re\left(\sum_{0\le j<x} e((3j^2+3j+1)\alpha)\right) \le x - \eta\sqrt{x} + O(x^{1/4}).$$

If, on the other hand, α does not have a rational approximation as above, then by Dirichlet's approximation theorem, there must be coprime integers a, q such that $1 \le q \le x^{3/2}$

$$\left|\alpha - \frac{a}{q}\right| \le \frac{1}{qx^{3/2}} \le \min\left(\frac{1}{q^2}, \frac{1}{x^{3/2}}\right),$$

and it follows that $q \ge x^{1/4}$ (otherwise, one could write $\alpha = \frac{a}{q} + \beta$ with $|\beta| \le \frac{1}{x^{3/2}}$). Hence by Weyl's inequality [6], one has

$$\sum_{0\le j<x} e((3j^2+3j+1)\alpha) \ll x^{1+\epsilon}\left(x^{-1}+q^{-1}+x^{-2}q\right)^{1/2} \ll x^{7/8+\epsilon}$$

for any $\epsilon > 0$. But then it also follows that

$$\Re\left(\sum_{0\le j<x} e((3j^2+3j+1)\alpha)\right) \ll x^{7/8+\epsilon}.$$

In either case, we have shown that

$$x - \Re\left(\sum_{0\le j<x} e((3j^2+3j+1)\alpha)\right) \ge \eta\sqrt{x} + O(x^{1/4})$$

for sufficiently large x and thus finally

$$g(r) - \Re(g(\tau)) \gg r^{-1/4},$$

which completes the proof of the third condition.

Plugging everything into Theorem 3 yields the desired result, as well as asymptotic normality of the number of layers in TSPPs with self-conjugate main diagonal, with mean and variance given by

$$\mu_n \sim A_1 n^{1/3}$$

and

$$\sigma_n^2 \sim A_2 n^{1/3}.$$

Numerically, the two constants are $A_1 = 0.533049$ and $A_2 = 0.194486$ respectively.

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References

[1] Andrews, George E., *The theory of partitions*, Cambridge Mathematical Library, Reprint of the 1976 original, Cambridge University Press, Cambridge, 1998.

[2] Berndt, Bruce C. and Evans, Ronald J. and Williams, Kenneth S., *Gauss and Jacobi sums*, Canadian Mathematical Society Series of Monographs and Advanced Texts, A Wiley-Interscience Publication, John Wiley & Sons Inc., New York,1998.

[3] Blecher, Aubrey, A generating function for $TSPP$ with self conjugate main diagonal, *Utilitas Math.*, to appear.

[4] Hwang, Hsien-Kuei, Limit theorems for the number of summands in integer partitions, J. Combin. Theory Ser. A, **96** (2001), 89–126.

[5] Meinardus, Günter, Asymptotische Aussagen über Partitionen, *Math. Z.*, **59** (1954), 388–398.

[6] Vaughan, R. C., *The Hardy-Littlewood method*, Cambridge Tracts in Mathematics **125**, Cambridge University Press, Cambridge, 1997.

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