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A generating operator of inequalities for polynomials

by
M.Ahmadi Baseri, M. Bidkham and Madjid Eshaghi

Abstract

Let $P(z)$ be a polynomial of degree $n \geq 1$. In this paper we consider an operator B, which carries a polynomial $P(z)$ into

$$B[P(z)] := \lambda_0 P(z) + \lambda_1 (\frac{nz}{2}) \frac{P'(z)}{1!} + \lambda_2 (\frac{nz}{2})^2 \frac{P''(z)}{2!},$$

where λ_0, λ_1 and λ_2 are such that all the zeros of

$$u(z) = \lambda_0 + c(n,1)\lambda_1 z + c(n,2)\lambda_2 z^2$$

lie in half plane

$$|z| \leq |z - \frac{n}{2}|,$$

and obtain new generalizations of some well-known results.

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1 Introduction and statement of results

Let P_n be the class of polynomials of degree at most n then,

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|, \tag{1.1}$$

and

$$\max_{|z|=R} |P(z)| \leq R^n \max_{|z|=1} |P(z)|, \quad R > 1. \tag{1.2}$$

Inequality (1.1) is an immediate consequence of Bernstein's theorem on the derivative of a polynomial (see[5]). Inequality (1.2) is a simple deduction from the maximum modulus principle (see[12]). If we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$, then the inequalities (1.1) and (1.2) can be respectively replaced by following [9,1]

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|, \tag{1.3}$$

and

$$\max_{|z|=R}|P(z)| \leq \frac{R^n+1}{2}\max_{|z|=1}|P(z)|, \quad R>1. \tag{1.4}$$

Recently Aziz and Rather [3] have investigated the dependence of

$$\left|P(Rz)-\alpha P(z)+\beta\left\{\left(\frac{R+1}{2}\right)^n-|\alpha|\right\}P(z)\right| for|z|=1 on \max_{|z|=1}|P(z)|$$

for every complex numbers α,β with $|\alpha|\leq 1$, $|\beta|\leq 1$ and $R\geq 1$. In fact, they proved

Theorem A *If P(z) is a polynomial of degree n, then for every complex numbers α,β with $|\alpha|\leq 1$, $|\beta|\leq 1$ and $R\geq 1$,*

$$\begin{aligned}&\left|P(Rz)-\alpha P(z)+\beta\left\{\left(\frac{R+1}{2}\right)^n-|\alpha|\right\}P(z)\right|\\ &\leq\left|R^n-\alpha+\beta\left\{\left(\frac{R+1}{2}\right)^n-|\alpha|\right\}\right||z|^n\max_{|z|=1}|P(z)|,\ for\ |z|\geq 1\end{aligned} \tag{1.5}$$

and

$$\begin{aligned}&\left|P(Rz)-\alpha P(z)+\beta\left\{\left(\frac{R+1}{2}\right)^n-|\alpha|\right\}P(z)\right|\\ &+\left|Q(Rz)-\alpha Q(z)+\beta\left\{\left(\frac{R+1}{2}\right)^n-|\alpha|\right\}Q(z)\right|\\ &\leq\left[\left|R^n-\alpha+\beta\left\{\left(\frac{R+1}{2}\right)^n-|\alpha|\right\}\right||z|^n\right.\\ &\left.+\left|1-\alpha+\beta\left\{\left(\frac{R+1}{2}\right)^n-|\alpha|\right\}\right|\right]\max_{|z|=1}|P(z)|,\end{aligned} \tag{1.6}$$

for $|z|\geq 1$, where $Q(z)=z^n\overline{P(\frac{1}{\bar{z}})}$. The results are sharp and equalities in (1.5) and (1.6) holds for $P(z)=\lambda z^n$, $\lambda\neq 0$. For the class of polynomial having no zeros in $|z|<1$, we have the following result due to Aziz and Rather which is a generalization of inequality (1.4).

Theorem B *If $P(z)$ is a polynomial of degree n which does not vanish in $|z|<1$, then for every complex numbers α,β with $|\alpha|\leq 1$, $|\beta|\leq 1$ and $R\geq 1$,*

$$\begin{aligned}&\left|P(Rz)-\alpha P(z)+\beta\left\{\left(\frac{R+1}{2}\right)^n-|\alpha|\right\}P(z)\right|\\ &\leq\frac{1}{2}\left[\left|R^n-\alpha+\beta\left\{\left(\frac{R+1}{2}\right)^n-|\alpha|\right\}\right||z|^n\right.\\ &\left.+\left|1-\alpha+\beta\left\{\left(\frac{R+1}{2}\right)^n-|\alpha|\right\}\right|\right]\max_{|z|=1}|P(z),\end{aligned} \tag{1.7}$$

for $|z| > 1$. Equality in (1.7) occurs for $P(z) = z^n + 1$.
In this paper, we consider an operator B, which carries $P \in P_n$ in to

$$B[P(z)] := \lambda_0 P(z) + \lambda_1 (\frac{nz}{2}) \frac{P'(z)}{1!} + \lambda_2 (\frac{nz}{2})^2 \frac{P''(z)}{2!}, \tag{1.8}$$

where λ_0, λ_1 and λ_2 are such that all the zeros of

$$u(z) = \lambda_0 + c(n,1)\lambda_1 z + c(n,2)\lambda_2 z^2 \tag{1.9}$$

lie in the half plane

$$|z| \leq |z - \frac{n}{2}|, \tag{1.10}$$

and prove the following generalization of Theorems A and B thus as well of inequalities (1.1) and (1.2).

Theorem 1. *If $P(z)$ is a polynomial of degree n, then for every complex numbers α, β with $|\alpha| \leq 1$, $|\beta| \leq 1$ and $R \geq 1$,*

$$\begin{aligned} &\left| B[P(Rz)] - \alpha B[P(z)] + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} B[P(z)] \right| \\ &\leq \left| (R^n - \alpha) + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} \right| |B[z^n]| \max_{|z|=1} |P(z)|, \ for \ |z| \geq 1. \end{aligned} \tag{1.11}$$

Equality holds in (1.11) for $P(z) = \lambda z^n, \lambda \neq 0$.

Remark 1. For $\lambda_0 = \lambda_2 = 0$ in (1.11) and note that in this case all the zeros of $u(z)$ defined by (1.9) lie in (1.10), we get

$$\begin{aligned} &\left| RP'(Rz) - \alpha P'(z) + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} P'(z) \right| \\ &\leq n \left| (R^n - \alpha) + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} \right| |z|^{n-1} \max_{|z|=1} |P(z)|, \ for \ |z| \geq 1. \end{aligned} \tag{1.12}$$

Equality holds in (1.12) for $P(z) = \lambda z^n, \lambda \neq 0$. If we take $\beta = 0$, $\alpha = 1$ and dividing the both sides of (1.12) by $R - 1$ and then allowing $R \to 1$, we get

$$|zP''(z) + P'(z)| \leq n^2 |z|^{n-1} \max_{|z|=1} |P(z)|, \quad for \ |z| \geq 1. \tag{1.13}$$

Equality holds in (1.13) for $P(z) = \lambda z^n, \lambda \neq 0$. For $\alpha = \beta = 0$ and $R = 1$, inequality (1.12) gives

$$|P'(z)| \leq n|z|^{n-1} \max_{|z|=1} |P(z)|, \quad for \quad |z| \geq 1. \tag{1.14}$$

which in particular gives inequality (1.1).

Remark 2. For $\lambda_1 = \lambda_2 = 0$. Theorem 1 reduces to inequality (1.5). Next as an application of Theorem 1, we prove the following theorem which is a generalization of a results prove by Rahman [11] , Jain [6], Aziz and Rather [3].

Theorem 2. *If $P(z)$ is a polynomial of degree n, then for every complex numbers α, β with $|\alpha| \leq 1$, $|\beta| \leq 1$ and $R \geq 1$,*

$$\begin{aligned}
&\left|B[P(Rz)] - \alpha B[P(z)] + \beta \left\{ \left(\frac{R+1}{2}\right)^n - |\alpha| \right\} B[P(z)]\right| \\
&+ \left|B[Q(Rz)] - \alpha B[Q(z)] + \beta \left\{ \left(\frac{R+1}{2}\right)^n - |\alpha| \right\} B[Q(z)]\right| \\
&\leq \left[\left|R^n - \alpha + \beta \left\{ \left(\frac{R+1}{2}\right)^n - |\alpha| \right\}\right| |B[z^n]| \right. \\
&+ \left. \left|1 - \alpha + \beta \left\{ \left(\frac{R+1}{2}\right)^n - |\alpha| \right\}\right| |\lambda_0| \right] \max_{|z|=1} |P(z)|, \qquad (1.15)
\end{aligned}$$

for $|z| \geq 1$, where $Q(z) =^n \overline{P(\frac{1}{\bar{z}})}$. If we take $\lambda_0 = \lambda_2 = \beta = 0$ and $\alpha = 1$ in (1.15), we obtain the following result.

Corollary 1. *If $P(z)$ is a polynomial of degree n, then for every complex number α with $|\alpha| \leq 1$ and $R \geq 1$,*

$$\begin{aligned}
&|RP'(Rz) - P'(z)| + |RQ'(Rz) - Q'(z)| \\
&\qquad \leq n(R^n - 1)|z|^{n-1} \max_{|z|=1} |P(z)|, \quad for \quad |z| \geq 1. \qquad (1.16)
\end{aligned}$$

*Equality holds in (1.16) for $P(z) = \lambda z^n, \lambda \neq 0$.*If $P(z)$ is a polynomial of degree n, then for every complex number α with $|\alpha| \leq 1$ and $R \geq 1$,

$$\begin{aligned}
&|RP'(Rz) - P'(z)| + |RQ'(Rz) - Q'(z)| \\
&\qquad \leq n(R^n - 1)|z|^{n-1} \max_{|z|=1} |P(z)|, \quad for \quad |z| \geq 1. \qquad (1.16)
\end{aligned}$$

Equality holds in (1.16) for $P(z) = \lambda z^n, \lambda \neq 0$. Theorem 2 includes a result due to Rahman [11] as a special case for $\lambda_1 = \lambda_2 = \alpha = \beta = 0$, where as inequality (1.15) reduces to a result due to Jain [6,Theorem 1] for $\lambda_1 = \lambda_2 = \alpha = 0$. For $\lambda_1 = \lambda_2 = 0$, inequality (1.15) reduces to inequality (1.6).
Lastly, for class of polynomial having no zeros in $|z| < 1$, we prove the following generalization of Theorem B.

Theorem 3. *If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, then for every complex numbers α, β with $|\alpha| \leq 1$, $|\beta| \leq 1$ and $R \geq 1$,*

$$\begin{aligned}&\left|B[P(Rz)] - \alpha B[P(z)] + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}B[P(z)]\right| \\ &\leq \frac{1}{2}\left[\left|R^n - \alpha + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}\right||B[z^n]|\right. \\ &\left.+\left|1 - \alpha + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}\right||\lambda_0|\right]\max_{|z|=1}|P(z)|, \quad for \quad |z| \geq 1. \qquad (1.17)\end{aligned}$$

Equality holds in (1.17) for $P(z) = z^n + 1$. If we take $\alpha = \beta = 0$ in Theorem 3, we get the following result.

Corollary 2. *If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, then for $R \geq 1$,*

$$|B[P(Rz)]| \leq \frac{1}{2}\{R^n|B[z^n]| + |\lambda_0|\}\max_{|z|=1}|P(z)|, \quad for \quad |z| \geq 1. \qquad (1.18)$$

The result is sharp and equality holds for $P(z) = z^n + 1$. For $R = 1$, inequality (1.18) reduces to a results due to Shah and Liman [13].

Remark 3. Theorem 3 includes some well-known inequalities as special case. For example inequality (1.17) reduces to a result due to Aziz and Rather [4] for $\lambda_1 = \lambda_2 = \beta = 0$. For $\lambda_1 = \lambda_2 = \alpha = 0$ inequality (1.17) reduces to result due to Jain [7] where as for $\lambda_1 = \lambda_2 = 0$ inequality (1.18) reduces to

$$|P(Rz)| \leq \frac{1}{2}\{R^n + 1\}\max_{|z|=1}|P(z)| \quad R \geq 1.$$

If we take $\lambda_0 = \lambda_2 = \alpha = \beta = 0$, inequality (1.17) reduces to inequality (1.3).

2 Lemmas

For the proofs of the theorems, we need the following lemmas. The first lemma was proved by Aziz [2].

Lemma 1 *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k (k \leq 1)$, then for every $R > 1$,*

$$|P(Rz)| \geq \left(\frac{R+k}{1+k}\right)^n |P(z)|, \quad for \quad |z| = 1. \qquad (2.1)$$

The following lemma follows from corollary 18.3 of [10].

Lemma 2 *If all the zeros of a polynomial $P(z)$ of degree n lie in a circle $|z| \leq 1$, then all the zeros of the polynomial $B[P(z)]$ also lie in the circle $|z| \leq 1$.*

Lemma 3. *If $P(z)$ is a polynomial of degree n such that $P(z) \neq 0$, in $|z| < 1$, then*

$$|B[P(z)]| \leq |B[Q(z)]|, \quad for \quad |z| \geq 1, \tag{2.2}$$

where $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$.

Lemma 4 *If $P(z)$ is a polynomial of degree n, then for $|z| \geq 1$,*

$$|B[P(z)]| + |B[Q(z)]| \leq \{|B[z^n]| + |\lambda_0|\} \max_{|z|=1} |P(z)|, \tag{2.3}$$

where $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$.
The above two lemmas are due to Shah and Liman [13].

Lemma 5 *If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, then for every complex numbers α, β with $|\alpha| \leq 1, |\beta| \leq 1$ and $R \geq 1$,*

$$\begin{aligned} &\left| B[P(Rz)] - \alpha B[P(z)] + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} B[P(z)] \right| \\ &\leq \left| B[Q(Rz)] - \alpha B[Q(z)] + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} B[Q(z)] \right|, \end{aligned} \tag{2.4}$$

for $|z| \geq 1$, where $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$.

Proof. If $P(z) \neq 0$ in $|z| < 1$, then by Lemma 3 we have $|B[P(z)]| \leq |B[Q(z)]|$ for $|z| \geq 1$ and hence for $R = 1$, we have nothing to prove . For $R > 1$, since $|P(z)| = |Q(z)|$ for $|z| = 1$, it follows by Rouche's theorem that for every complex number λ with $|\lambda| > 1$, the polynomial $T(z) = P(z) - \lambda Q(z)$ does not vanish in $|z| > 1$, with at least one zero in $|z| < 1$. Let $T(z) = (z - re^{i\delta})F(z)$ where $r < 1$ and $F(z)$ is a polynomial of degree $n - 1$ having no zeros in $|z| > 1$. Applying Lemma 1 with $k = 1$, for every $R > 1$, $0 \leq \theta \leq 2\pi$

$$\begin{aligned} |T(Re^{i\theta})| &\geq |Re^{i\theta} - re^{i\delta}| \left(\frac{R+1}{2} \right)^{n-1} |F(e^{i\theta})| \\ &= \left(\frac{R+1}{2} \right)^{n-1} \left| \frac{Re^{i\theta} - re^{i\delta}}{e^{i\theta} - re^{i\delta}} \right| |(e^{i\theta} - re^{i\delta})F(e^{i\theta})| \\ &\geq \left(\frac{R+1}{2} \right)^{n-1} \left(\frac{R+r}{1+r} \right) |T(e^{i\theta})|, \end{aligned}$$

or

$$\left(\frac{r+1}{R+r} \right) |T(Re^{i\theta})| \geq \left(\frac{R+1}{2} \right)^{n-1} |T(e^{i\theta})|, \ R > 1 \quad and \quad 0 \leq \theta \leq 2\pi, \tag{2.5}$$

since $R > 1 > r$, hence $T(Re^{i\theta}) \neq 0$ and $(\frac{2}{R+1}) > (\frac{r+1}{R+r})$, from inequality (2.5), we have

$$|T(Rz)| > \left(\frac{R+1}{2}\right)^n |T(z)|, \quad |z| = 1\ ,\ R > 1. \tag{2.6}$$

Hence for every complex number α with $|\alpha| \leq 1$, we have

$$\begin{aligned}|T(Rz) - \alpha T(z)| &\geq |T(Rz)| - |\alpha||T(z)| \\ &> \left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\} |T(z)|, \quad for\ |z| = 1\ and\ R > 1. \end{aligned} \tag{2.7}$$

Since $T(Re^{i\theta}) \neq 0$ and $(\frac{R+1}{2})^n > 1$, hance from inequality (2.6), we have

$$|T(Re^{i\theta})| > (|T(e^{i\theta})|, \quad for \quad R > 1\ and \quad 0 \leq \theta \leq 2\pi,$$

equivalently

$$|T(Rz)| > (|T(z)|, \quad for \quad |z| = 1\ and \quad R > 1.$$

Since all the zeros of $T(Rz)$ lie in $|z| < 1$, it follows (by Rouche's theorem for $|\alpha| \leq 1$) that the polynomial $T(Rz) - \alpha T(z)$ does not vanish in $|z| \geq 1$. Hence from inequality (2.7)(by Rouche's theorem for $|\beta| \leq 1$), we have the polynomial

$$S(z) = T(Rz) - \alpha T(z) + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\} T(z),$$

has all its zeros in $|z| < 1$. Therefore, by Lemma 2, all the zeros of $B[S(z)]$ lie in $|z| < 1$. Replacing $T(z)$ by $P(z) - \lambda Q(z)$ and since B is liner, it follows that the polynomial

$$\begin{aligned}&B[P(Rz)] - \alpha B[P(z)] + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\} B[P(z)] \\ &\qquad - \lambda\left\{B[Q(Rz)] - \alpha B[Q(z)] + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\} B[Q(z)]\right\}\end{aligned}$$

having no zeros in $|z| \geq 1$. This implies

$$\begin{aligned}&\left|B[P(Rz)] - \alpha B[P(z)] + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\} B[P(z)]\right| \\ &\leq \left|B[Q(Rz)] - \alpha B[Q(z)] + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\} B[Q(z)]\right|,\end{aligned} \tag{2.8}$$

for $|z| \geq 1$. If this is not true, then there is a point $z = z_0$ with $|z_0| \geq 1$, such that

$$\begin{aligned}&\left|B[P(Rz_0)] - \alpha B[P(z_0)] + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\} B[P(z_0)]\right| \\ &> \left|B[Q(Rz_0)] - \alpha B[Q(z_0)] + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\} B[Q(z_0)]\right|.\end{aligned}$$

Since all the zeros of $Q(z)$ lie in $|z| \leq 1$, hence (As in case of $T(z)$) all the zeros of

$$B[Q(Rz)] - \alpha B[Q(z)] + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} B[Q(z)] \quad lie\ in |z| < 1,$$

for every complex numbers α, β with $|\alpha| \leq 1$, $|\beta| \leq 1$ and $R > 1$. Therefore

$$B[Q(Rz_0)] - \alpha B[Q(z_0)] + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} B[Q(z_0)] \neq 0 \quad with \quad |z_0| \geq 1,$$

we take

$$\lambda = \frac{B[P(Rz_0)] - \alpha B[P(z_0)] + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} B[P(z_0)]}{B[Q(Rz_0)] - \alpha B[Q(z_0)] + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} B[Q(z_0)]}$$

so that $|\lambda| > 1$ and for this value λ, $B[S(z_0)] = 0$ for $|z_0| \geq 1$, which contradicts the fact that all the zeros of $B[S(z)]$ lie in $|z| < 1$. This proves the desired result.

3 Proofs of the theorems

Proof of Theorem 1. For $R = 1$, it is already proved by Rahman [11].We assume $R > 1$. On Lemma 1, if we take $k = 1$ and $P(z) \neq 0$ in $|z| \geq 1$,then one can easily obtain

$$|P_1(Rz)| > \left(\frac{R+1}{2} \right)^n |P_1(z)|, \quad |z| = 1 \ and \ R > 1. \tag{3.1}$$

Since $P_1(Re^{i\theta}) \neq 0$, $0 \leq \theta < 2\pi$ and $(\frac{R+1}{2})^n > 1$ from above inequality we have $|P_1(Re^{i\theta})| > |P_1(e^{i\theta})|$, $R > 1$.
Equivalently,

$$|P_1(Rz)| > |P_1(z)|, \quad for \ |z| = 1 \ and \ R > 1.$$

For every complex number α with $|\alpha| \leq 1$ and using inequality (3.1), we have

$$\begin{aligned} |P_1(Rz) - \alpha P_1(z)| &\geq |P_1(Rz)| - |\alpha||P_1(z)| \\ &> \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} |P_1(z)|, \quad for \ |z| = 1 \ and \ R > 1.(3.2) \end{aligned}$$

Since all the zeros of $P_1(Rz)$ lie in $|z| < 1$, it follows (by Rouche's theorem for $|\alpha| \leq 1$) that the polynomial $P_1(Rz) - \alpha P_1(z)$ has all its zeros in $|z| < 1$. Hence from inequality (3.2)(by Rouche's theorem for $|\beta| \leq 1$), we have the polynomial

$$F(z) = P_1(Rz) - \alpha P_1(z) + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} P_1(z),$$

has all its zeros in $|z| < 1$. So, by Lemma 2, all the zeros of $B[F(z)]$ lie in $|z| < 1$. Replacing $P_1(z)$ by $P(z) - \lambda M z^n$ and since B is liner, it follows that the polynomial

$$B[P(Rz)] - \alpha B[P(z)] + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}B[P(z)] \\ - \lambda M B[z^n]\left\{R^n - \alpha + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}\right\}$$

having no zeros in $|z| \geq 1$. This implies

$$\left|B[P(Rz)] - \alpha B[P(z)] + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}B[P(z)]\right| \\ \leq \left|R^n - \alpha + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}\right||B[z^n]|M,$$

for $|z| \geq 1$. If this is not true, then there is a point $z = z_0$ with $|z_0| \geq 1$, such that

$$\left|B[P(Rz_0)] - \alpha B[P(z_0)] + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}B[P(z_0)]\right| \\ > \left|R^n - \alpha + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}\right||B[z_0^n]|M.$$

Since all the zeros of Mz^n lie in $|z| < 1$, hence (As in case of $F(z)$) all the zeros of

$$\left\{R^n - \alpha + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}\right\}MB[z^n]$$

lie in $|z| < 1$. We take

$$\lambda = \frac{B[P(Rz_0)] - \alpha B[P(z_0)] + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}B[P(z_0)]}{\left\{R^n - \alpha + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}\right\}B[z_0^n]M}$$

so that $|\lambda| > 1$ and for this value λ, $B[F(z_0)] = 0$ for $|z_0| \geq 1$, which contradicts the fact that all the zeros of $B[F(z)]$ lie in $|z| < 1$. This proves the Theorem 1.

Proof of Theorem 2. The result is trivial if $(R = 1)$ (Lemma 4), so we suppose that $R > 1$. If $M = \max_{|z|=1}|P(z)|$, then $|P(z)| \leq M$ for $|z| = 1$. Now for every complex number λ with $|\lambda| > 1$, we have the polynomial $W(z) = P(z) + \lambda M$ has no zeros in $|z| < 1$ and on applying Lemma 5, we get for $|z| \geq 1$ and $R > 1$,

$$\begin{aligned}&\Bigg|B[P(Rz)] - \alpha B[P(z)] + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}B[P(z)]\\ &+ \lambda\left[1 - \alpha + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}\right]\lambda_0 M\Bigg|\\ &\leq \Bigg|B[Q(Rz)] - \alpha B[Q(z)] + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}B[Q(z)]\\ &+\bar{\lambda}\left[R^n - \alpha + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}\right]B[z^n]M\Bigg| \qquad (3.3)\end{aligned}$$

where $|\alpha| \leq 1$, $|\beta| \leq 1$ and $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$. Choosing the argument of λ, which is possible by (1.11) such that

$$\begin{aligned}&\Bigg|B[Q(Rz)] - \alpha B[Q(z)] + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}B[Q(z)]\\ &+ \bar{\lambda}\left[R^n - \alpha + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}\right]MB[z^n]\Bigg|\\ &= |\lambda|\left|R^n - \alpha + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}\right|M|B[z^n]|\\ &- \left|B[Q(Rz)] - \alpha B[Q(z)] + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}B[Q(z)]\right|,\end{aligned}$$

we get from (3.3)

$$\begin{aligned}&\left|B[P(Rz)] - \alpha B[P(z)] + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}B[P(z)]\right|\\ &- |\lambda||\lambda_0|\left|1 - \alpha + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}\right|M\\ &\leq |\lambda|\left|R^n - \alpha + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}\right|M|B[z^n]|\\ &- \left|B[Q(Rz)] - \alpha B[Q(z)] + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}B[Q(z)]\right|, \qquad (3.4)\end{aligned}$$

for $|z| \geq 1$, $|\alpha| \leq 1$, $|\beta| \leq 1$, and $R > 1$, making $|\lambda| \to 1$ in (3.4), we get (1.15). This completes the proof of Theorem 2.

Proof of Theorem 3. By hypothesis $P(z)$ does not vanish in $|z| < 1$, therefore by Lemma 5 we have

$$\begin{aligned}&\left|B[P(Rz)] - \alpha B[P(z)] + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}B[P(z)]\right|\\ &\leq \left|B[Q(Rz)] - \alpha B[Q(z)] + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}B[Q(z)]\right|,\end{aligned}$$

for $|z| \geq 1$ where $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$.
Equavalently

$$\begin{aligned}
&2\left|B[P(Rz)] - \alpha B[P(z)] + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}B[P(z)]\right| \\
&\leq \left|B[P(Rz)] - \alpha B[P(z)] + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}B[P(z)]\right| \\
&+ \left|B[Q(Rz)] - \alpha B[Q(z)] + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}B[Q(z)]\right|,
\end{aligned} \tag{3.5}$$

on applying Theorem 2, we get

$$\begin{aligned}
&2\left|B[P(Rz)] - \alpha B[P(z)] + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}B[P(z)]\right| \\
&\leq \left|B[P(Rz)] - \alpha B[P(z)] + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}B[P(z)]\right| \\
&+ \left|B[Q(Rz)] - \alpha B[Q(z)] + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}B[Q(z)]\right| \\
&\leq \left[\left|R^n - \alpha + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}\right| |B[z^n]| \right. \\
&\left. + \left|1 - \alpha + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}\right| |\lambda_0|\right] \max_{|z|=1} |P(z)|,
\end{aligned}$$

which is inequality (1.17) and this completes proof of Theorem 3.

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Department of Mathematics, Semnan University, P. O. Box 35195-363, Semnan, Iran
E-mails: `m_ahmadi_baseri@yahoo.com`
`mbidkham@semnan.ac.ir`
`madjid.eshaghi@gmail.com`