Bull. Math. Soc. Sci. Math. Roumanie
Tome 56(104) No. 2, 2013, 237–242

A restriction theorem for torsion-free sheaves on some elliptic manifolds

by
Victor Vuletescu

Abstract

We prove that if X is the total space of an elliptic principal bundle $\pi : X \to B$ which is non-kähler, then the restriction of any torsion-free sheaf on X to the general fiber of π is semi-stable.

Key Words: Non-Kähler principal elliptic bundles, holomorphic vector bundles, stable vector bundles.
2010 Mathematics Subject Classification: Primary 32L05, Secondary 32L10.

1 Introduction

In the study of holomorphic vector bundles over a given compact complex manifold X, especially in the study of (semi)stable ones, a very useful tool is the study of their restrictions to general members of a given family of subvarieties of X. However, the restriction of a (semi)stable vector bundle to a submanifold is not always semistable. Still, under some strong hypothesis, such as X is projective and the family of subvarieties is a family of divisors "ample enough", the restriction of a stable vector bundle to the general memeber remains (semi)stable: this is "Flenner's restriction theorem", see [5]. Flenner's theorem has been extended to the more general context of algebraic varieties in arbitrary characteristic (see e.g [6]), but, to the author's knowledge, there is no such extension to the case of non-projective manifolds. The present note tackles this case.

2 Notations and basic facts

The context we are working is the following. We fix a compact complex manifold B and an elliptic curve F. To every principal elliptic bundle

$$\pi : X \to B$$

one can associate (up to the obvious action of $SL(2, \mathbb{Z})$) a couple of elements

$$(c_1'(\pi), c''_1(\pi)) \in H^2(B, \mathbb{Z}) \times H^2(B, \mathbb{Z})$$

called *the Chern classes* of the bundle π (see e.g. [2]).

If at least one of the Chern classes is non vanishing in $H^2(B,\mathbb{R})$, one can prove by a standard argument using the Leray spectral sequence of the fibration that the homology class of any fiber $[F] \in H_2(X,\mathbb{R})$ vanishes; as the fibers are compact complex submanifolds, this shows that X is not of Kähler type.

We also recollect the notion of stability; since we will use this concept for vector bundles on curves, we will only recall the definition in this case. Hence, a vector bundle E on a smooth projective curve will be called *stable* (respectively *semistable*) if for any subbundle $\mathcal{F} \subset E$ with $0 < rank(\mathcal{F}) < rank(E)$ one has

$$\frac{deg(\mathcal{F})}{rank(\mathcal{F})} < \frac{deg(E)}{rank(E)}$$

(resp " $\leq$ " for semistability). A vector bundle which is not semistable is called *unstable.*

Eventually, let us recall a concept which is of relevance only on non-algebraic complex manifolds. If X is a compact complex manifold and $\mathcal{F}$ is a coherent sheaf on X, then $\mathcal{F}$ is called *reducible* if there exist a coherent subsheaf $\mathcal{F}' \subset \mathcal{F}$ with $0 < rank(\mathcal{F}') < rank(\mathcal{F})$; if no such subsheaf exist then $\mathcal{F}$ is called *irreducible.* Notice that on projective manifolds all coherent sheaves are reducible; still, on general compact complex manifolds this is not always the case, as one can see for instance looking at the tangent bundle of a K3 surface X with $Pic(X) = 0$ (the general K3 surface is so).

3 Some Lemmas

In the following we collect some lemmas, which are most likely classical and well-known; but since we don't have any precise reference, we include the proofs here.

Lemma 1. *Let $\pi : X \to B$ be an elliptic principal bundle. If the homology class $[F] \in H_2(X,\mathbb{R})$ vanishes (i.e the Poincaré dual PD_F is zero), then any proper closed analytic subset $Y \subset X, dim(Y) < dim(X)$, does not meet the general fiber.*

Proof: The only non-obvious case is when Y is a hypersurface. But in this case, if Y meets all the fibers, then it meets the general fiber transversely in finitely many points. But then

$$0 < \#(Y \cap F) = \int_X PD_Y \wedge PD_F = 0$$

since $PD_F = 0$ by the assumption that $0 = [F] \in H_2(X,\mathbb{R})$. $\square$

Lemma 2. *For X as in the previous Lemma and for any torsion-free sheaf E on X we have*

$$deg(E_{|F}) = 0$$

for F =general fiber of π.

Proof: Indeed, as E is torsion-free, we see $Sing(E)$ has codimension at least two. Let $L = det(E)^{\vee\vee}$ be the bidual of the determinant of E; it is a reflexive sheaf of rank one on X, so it is a line bundle (cf e.g. [7]). Moreover, the map $det(E) \to L$ is an isomorphism outside $sing(E)$, so if F is any fiber not meeting $Sing(E)$ we have

$$deg(E_{|F}) = deg(det(E)_{|F}) = deg(L_{|F}) = i^*(c_1(L))$$

where $i : F \to X$ is the inclusion of the fiber F. But as $[F] = 0$ in $H_2(X, \mathbb{R})$ we see $i^*(c_1(L)) = 0$. $\square$

Lemma 3. *If F is an elliptic curve and if E is a vector bundle of degree zero on F which is generated by its global sections, then E is trivial.*

We use the following argument from L. Ein (cf [4], Proposition 1.1):
"**Lemma**. If X is a compact complex manifold, and E is a globally generated vector bundle on E such that its dual $E^\vee$ has a section, then E splits as $E = \mathcal{O}_X \oplus F$. "

We do induction of $rank(E)$. For $rank(E) = 1$ the assertion is immediate. If $rank(E) \geq 2$, letting $K = Ker\left(H^0(F,E) \otimes \mathcal{O}_F \to E\right)$ we get an extension:

$$0 \to K \to H^0(F,E) \otimes \mathcal{O}_F \to E \to 0. \tag{1}$$

Now, either the extension splits (and hence E is trivial), or

$$H^1(F, E^\vee \otimes K) \neq 0.$$

As $deg(E) = 0$ we have also $deg(K) = 0$ so we further get by Riemann-Roch on F that

$$H^0(F, E^\vee \otimes K) \neq 0. \tag{2}$$

Twisting the above extension (1) by $E^\vee$ we get

$$0 \to K \otimes E^\vee \to H^0(F,E) \otimes E^\vee \to E \otimes E^\vee \to 0$$

hence, from (2), we get

$$H^0(F, E^\vee) \neq 0$$

Applying Ein's Lemma, we get $E = \mathcal{O}_F \oplus E_1$. But E_1 has degree zero and is generated by its global sections too, so by the induction hypothesis, E_1 is trivial. Consequently, E is trivial too.

Lemma 4. *Let F be an elliptic curve and L a semistable vector bundle on F such that $deg(L) = 0$. Then there is a Zariski-open subset $U \subset Pic_0(F)$ such that $H^0(F, L \otimes I) = 0$ for all $I \in U$.*

Proof: (See also [8]). Again, we do induction on $rank(L)$. For $rank(L) = 1$ the claim is immediate (take $U = Pic_0(F) \setminus \{L^\vee)\}$), so assume $rank(L) > 0$.

In the case $H^0(F,L) = 0$, from the existence of the Poincaré bundle and Grauert's upper continuity theorem we get $H^0(F, L \otimes I) = 0$ for all I in a Zariski neighborhood of $\mathcal{O}_F$.

In the case $h^0(F, L) > 0$ take some $s \in H^0(F, L), s \neq 0$; it defines a map

$$0 \to \mathcal{O}_F \xrightarrow{s} L$$

We infer that this map has torsion-free cokernel; since otherwise, moding out by the torsion of the cokernel, we would get a nontrivial map into L from a nontrivial, effective divisor on F, contradicting the hypothesis that L is semistable. So L sits in an exact sequence

$$0 \to \mathcal{O}_F \to L \to L' \to 0$$

with L' =torsion-free (hence locally free, as F is a curve); in particular, $deg(L') = 0$. It is easy to see that L' is semistable too, so by the induction hypothesis $H^0(F, L' \otimes I) = 0$ for all I is some open subset $U \subset Pic_0(F)$. So

$$H^0(F, L \otimes I) = 0$$

for all $I \in U \setminus \{\mathcal{O}_F\}$. $\square$

Eventually, we recollect a fact which is true more generally

Lemma 5. *Let F be an elliptic curve and*

$$0 \to L \to M \to R \to 0$$

an exact sequence of vector bundles of F with

$$deg(L) = deg(R) = 0.$$

If L and R are semistable, then M is semistable too.

Proof: Using Lemma 4 we get a line bundle $I \in Pic_0(F)$ such that

$$H^0(F, R \otimes I) = H^0(F, L \otimes I) = 0;$$

this implies $H^0(F, M \otimes I) = 0$ as well.

So, replacing M by $M \otimes I$ we can further assume $H^0(F, M) = 0$. Now, if M would be unstable, we would get a destabilizing vector subbundle $D \subset M$ with $deg(D) > 0$. But $deg(D) > 0$ implies $H^0(F, D) \neq 0$; so $H^0(F, M) \neq 0$ as well, contradiction. $\square$

4 The main result

We are now in position to state and prove the main result.

Theorem 1. *Let $\pi : X \to B$ be an elliptic principal bundle with at least one of the Chern classes non-vanishing in $H^2(B, \mathbb{R})$ (in particular, X is nonKähler). Then the restriction of any torsion-free sheaf E on X to the general fiber of π is semi-stable.*

Before proving it, let us make a small comment. As one can see, the theorem gives the semi-stability of the restriction of E to the general fiber of π with *no apriori assumptions like (semi)stability for* E. This is not completely surprising; in the non-projective context, more exactly on non-projective surfaces, the "Bogomolov inequality" $\Delta(E) \geq 0$, holds similarly for *all torsion-free sheaves* E (cf [1], or [3] for a simpler proof), in contrast to the projective case, when it holds mainly for stable vector bundles.

Proof of the theorem. We do induction on the rank $r = rk(E)$. For $r = 1$ there is nothing to prove, so we assume $r \geq 2$.

Case 1: E **is reducible.** That is, E sits in an exact sequence

$$0 \to L \to E \to R \to 0$$

By the Lemma 2, we see that for a general fiber F of π, $L_{|F}, R_{|F}$ are locally free of degree zero. More, by the induction hypothesis, both $L_{|F}, R_{|F}$ are also semistable, so $E_{|F}$ is semistable too, by Lemma 5.

Case 2: E **is irreducible**. We distinguish again two subcases:

Subcase 2.1: $\pi_*(E) = 0$**.** In this case, $H^0(F, E_{|F}) = 0$ for the general fiber. But as also $deg(E_{|F}) = 0$ for the general fiber F, we see at once that $E_{|F}$ is semistable. Indeed, if this is not the case, then a destabilizing subsheaf $D \subset E$ would have $deg(D) > 0$; but then $h^0(F, D) > 0$ so $h^0(F, E_{|F}) > 0$ too, contradiction.

Subcase 2.2: $\pi_*(E) \neq 0$**.** Let $\alpha : \pi^*\pi_*(E) \to E$ be the canonical morphism and let $\mathcal{F} = Im(\alpha)$. As E is irreducible and as α is non-trivial, we see we have

$$rank(\mathcal{F}) = rank(E).$$

Let $Y = Supp(E/\mathcal{F})$; by Lemma 1, Y cannot meet all the fibers of π so for the general fiber F we have $\mathcal{F}_{|F} = E_{|F}$; more, by Lemma 2 we can assume $deg(E_{|F}) = 0$.

So, for the general fiber F we have a surjection

$$\pi^*\pi_*(E)_{|F} \to E_{|F}.$$

But

$$\pi^*\pi_*(E)_{|F}$$

is trivial, so $E_{|F}$ is spanned by its global sections. As it is also of degree zero, it folllows by Lemma 3 that $E_{|F}$ is trivial, in particular semi-stable.

Acknowledgements. The author would like to thank V. Brînzănescu for asking the question of the semistability of restrictions in the non-Kähler context and for a careful reading of some firsts drafts of the paper. The ellaboration of paper was funded by the grant "Vector Bundle Techniques in the Geometry of Complex Varieties", PN-II-ID-PCE-2011-3-0288, Contract 132/05.10.2011.

References

[1] C. Bănică; J. Le Potier, *Sur l'existence des fibrés vectoriels holomorphes sur les surfaces non- algébriques. (On the existence of holomorphic vector bundles on non- algebraic surfaces).* J. Reine Angew. Math. 378, 1-31 (1987).

[2] V. Brînzănescu, *Neron-Severi group for nonalgebraic elliptic surfaces. I: Elliptic bundle case.* Manuscr. Math. 79, No.2, 187-195 (1993).

[3] V. Brînzănescu, *A simple proof of a Bogomolov type inequality in the case of nonalgebraic surfaces.* Rev. Roum. Math. Pures Appl. 38, No.7-8, 631-633 (1993).

[4] L. Ein, *An analogue of Max Noether's theorem.* Duke Math. J. 52, 689-706 (1985).

[5] H. Flenner, *Restrictions of semistable bundles on projective varieties,* Comment. Math. Helv. 59 (1984), 635650.

[6] A. Langer, *A note on restriction theorems for semistable sheaves*, Math.Res.Lett.17(2010), no.05,823832

[7] Ch. Okonek; M. Schneider; H. Spindler, *Vector bundles on complex projective spaces.* Progress in Mathematics. 3. Boston - Basel - Stuttgart: Birkhuser. VII, 389 p.

[8] M. Raynaud, *Sections des fibrés vectoriels sur une courbe*, Bulletin de la S.M.F., tome 110 (1982), p.103-125

Received: 25.02.2012,
Accepted: 10.11.2012.

University of Bucharest, Faculty of Mathematics and Computer Sciences, Romania and
"Simion Stoilow" Institute of Mathematics of the Romanian Academy, Romania
E-mail: `vuli@fmi.unibuc.ro`