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Residual transcendental extensions of valuations, irreducible polynomials, and trace series over $p-$adic fields

by
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Dedicated to the memory of Nicolae Popescu (1937-2010)
on the occasion of his 75th anniversary

Abstract

In this paper we survey some results related to residual transcendental extensions of valuations, irreducible polynomials over a local field, and trace series associated to suitable transcendental elements over $p-$adic fields.

Key Words: Residual transcendental extensions, irreducible polynomials, trace series, local fields.
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1 Introduction

In the present paper we discuss some results concerned with residual transcendental extensions of valuations, irreducible polynomials over a local field, and trace series associated to suitable transcendental elements over $p-$adic fields. In the process we describe two ideas of Professor Nicolae Popescu which connect the above three different topics. Given a field K and a valuation v on K, the r.t. (residual transcendental) extensions of v to $K(X)$ have been considered by Nagata [14] in connection with some problems in field theory. In [14] it was conjectured that if w is an r.t. extension of v to $K(X)$, then the residue field k_w of w is a simple transcendental extension of a finite algebraic extension of k_v, the residue field of v. Nagata's conjecture was proved by Ohm (see [15], [16], [17]) and independently it was proved in [25]. Further questions on r.t. extensions have been considered in the above mentioned papers by Ohm. He also made three natural conjectures concerned with ramification indices and residual degrees. Ohm's conjectures were proved in [6] as consequences of a theorem established in the same paper, which provides one with a very satisfactory description of r.t. extensions of a valuation. Further results on the same subject have been obtained in [7] and [8]. One of the ideas of Professor Popescu mentioned above was that the theory of r.t. extensions of valuations developed in these papers offers us a way to construct irreducible polynomials over a local field from irreducible polynomials of smaller degree by a certain procedure which also involves lifting polynomials defined over the residue field to polynomials defined over the local field. This idea was employed in [27], were a description of irreducible polynomials over a local field was provided. The theory developed in this paper led to further investigations in various directions. One of them was the study of saturated distinguished chains over a local field. Such chains were introduced by Okutsu in [18], [19] and independently in [27], and further studied in [9], [23], [24], [28] and in a series of papers by Ota [20], [21], [22]. The second idea of Professor Popescu mentioned above was to use saturated distinguished chains associated to certain transcendental elements over a $p-$adic field in order to construct power series whose role would be analogous to the role played by minimal polynomials associated to algebraic elements. In this connection, a natural question is to define and study a notion of trace for certain elements in the topological closure $\mathbb{C}_p$ of a fixed algebraic closure $\overline{\mathbb{Q}}_p$ of the field $\mathbb{Q}_p$ of $p-adic$ numbers, which are transcendental over $\mathbb{Q}_p$. This question was addressed in [10], and later studied also in [26]. A natural way to define a notion of trace for certain elements T in $\mathbb{C}_p$ which are transcendental over $\mathbb{Q}_p$ is via an integral over the Galois orbit of T. This naturally leads to various questions concerned with these Galois orbits. For further work in this area the reader is referred to [1], [2], [3], [4], [5], [11], [12], [13], [29].

2 Residual transcendental extensions of valuations

Here and in the next sections we very briefly describe a few of the main concepts and results mentioned in the Introduction. Let K be a field and v a valuation on K. We denote by k_v the residue field, by Γ_v the value group and by O_v the valuation ring of v. If $x \in O_v$ we denote by x^* the image of x in k_v. A valuation w on the field of rational functions $K(X)$ is said to be a residual transcendental (r. t.) extension of v if it is an extension of v and k_w is a transcendental extension of k_v. For any $r \in K(X)$, $r \notin K$, denote $\deg r = [K(X) : K(r)]$. We also let $deg(w/v)$ denote the least positive integer n for which there exists an r in O_w of degree n and such that r^* is transcendental over k_v. Also, one shows that Γ_v has finite index in Γ_w. We denote this index by $e(w/v)$. We also let k denote the algebraic closure of k_v in k_w. One shows that k is a finite extension of k_v. We denote the degree of this extension by $f(w/v)$. Ohm [17] proved the following inequality. For any r.t extension w of v,

$$\deg(w/v) \geq f(w/v)e(w/v). \tag{2.1}$$

Let $\bar{K}$ be a fixed algebraic closure of K and let $\bar{v}$ be a fixed extension of v to $\bar{K}$. If w is an extension of v to $K(X)$, then there exists an extension $\bar{w}$ of w to $\bar{K}(X)$ such that $\bar{w}$ is also an extension of $\bar{v}$. If moreover w is an r.t. extension of v, then $\bar{w}$ is an r.t. extension of $\bar{v}$. One shows that there exist $\alpha \in \bar{K}$ and $\delta \in \Gamma_{\bar{v}}$ such that for any $f(X) \in \bar{K}[X]$, if one writes $f(X)$ in the form

$$f(X) = a_0 + a_1(X - \alpha) + \cdots + a_n(X - \alpha)^n,$$

then one has

$$\bar{w}(f(X)) = \inf_j (\bar{v}(a_j) + j\delta).$$

One calls such a pair (α, δ) a pair of definition for $\bar{w}$. One also says that $\bar{w}$ is the valuation defined by inf, $\bar{v}$, α and δ. By a minimal pair of definition for $\bar{w}$ one means a pair of definition

(α, δ) for which the degree $[K(\alpha) : K]$ is minimal. The main result from [6], which was the basis for some further work in this area, is the following theorem (see Theorem 2.1. and its corollaries in [6]):

Let v be a valuation on K and let w be an r.t. extension of v to $K(X)$. Then there exists a pair of definition (α, δ) for $\bar{w}$ such that:

a) If we denote $[K(\alpha) : K] = n$, then for any polynomial $g(X) \in K[X]$ of degree $\deg g(X) < n$, one has

$$w(g(X)) = \bar{v}(g(\alpha)).$$

b) Denote by v_1 the restriction of $\bar{v}$ to $K(\alpha)$. Let $f(X)$ be the monic minimal polynomial of α over K, let $\gamma = w(f(X))$ and let $e = e(\gamma, K(\alpha))$ be the smallest positive integer for which $e\gamma$ belongs to Γ_{v_1}. There exists $l(X) \in K[X]$ with $\deg l < n$ such that if $r = f^e/l$, then $w(r) = 0$ and r^* is transcendental over k_v.

c) One has the equalities

$$\deg(w/v) = n \cdot e(\gamma, K(\alpha)); \quad e(w/v) = e(v_1/v)e(\gamma, K(\alpha)).$$

d) The residue field k_{v_1} can be canonically identified with the algebraic closure of k_v in k_w, and

$$f(w/v) = f(v_1/v).$$

e) One has

$$k_w = k_{v_1}(r^*).$$

As consequences of the above theorem we mention the following three cases where, as conjectured by Ohm, the inequality (2.1) becomes an equality.

If v is Henselian and char $k_v = 0$, then

$$\deg(w/v) = f(w/v)e(w/v). \tag{2.2}$$

If v is of rank one, and char $k_v = 0$, then

$$\deg(w/v) = f(w/v)e(w/v). \tag{2.3}$$

If v is of rank one and discrete, then

$$\deg(w/v) = f(w/v)e(w/v). \tag{2.4}$$

3 Irreducible polynomials over a local field

A description of irreducible polynomials in one variable over a given local field was obtained in [27]. The connection with the theorem from the previous section is as follows. Suppose K is a field, complete with respect to a rank one and discrete valuation v. In order to produce irreducible polynomials over K, one may take an arbitrary r.t. extension w of v to $K(X)$ and apply the above theorem. One may take a monic irreducible polynomial $G(Y) \neq Y$ defined over the algebraic closure of the residue field k_v in k_w and lift $G(Y)$ to a polynomial $g(X)$ defined over K. Here the general idea is to consider the element $G(r^*)$ in $k_{v_1}(r^*) = k_w$, then lift r^* to

r and also lift each coefficient of G. These coefficients can be lifted to algebraic elements, more specifically to elements of $K(\alpha)$, but can also be lifted to polynomials (or rational functions) in X over K of degrees strictly less than $[K(\alpha) : K]$. At the same time r is a rational function in X over K. One can then combine these individual liftings to a lifting of $G(r^*)$ to a rational function of the form g/l^m for some monic polynomial $g(X)$ in $K(X)$ of degree $em \deg f$, where $e, f(X), l(X)$ are as in the theorem from the previous section, and $m = \deg G$. Then Theorem 2.1 from [27] states the following:

With the above notations, if $G(Y)$ is irreducible, then any lifting $g(X)$ of G is irreducible over K.

A pair (a, b) of elements from $\bar{K}$ is said to be a distinguished pair if the following conditions hold:

(i) $\deg a > \deg b$,

(ii) If $c \in \bar{K}$ and $\deg c < \deg a$, then $\bar{v}(a - c) \leq \bar{v}(a - b)$,

(iii) If $c \in \bar{K}$ and $\deg c < \deg b$, then $\bar{v}(a - c) < \bar{v}(a - b)$.

Here the degrees are over K. Next, given irreducible polynomials f and g over K, one says that (g, f) is a distinguished pair provided there is a root a of g and a root b of f such that (a, b) is a distinguished pair.

Theorems 3.1 and 3.2 in [27] show that there is an intimate connection between the notion of distinguished pairs of polynomials and the lifting procedure described above. Theorem 3.1 states the following:

Notations being as above, if w is an r.t. extension of v to $K(X)$, if $G(Y) \neq Y$ is a monic irreducible polynomial over the algebraic closure of k_v in k_w, and if g is a nontrivial lifting of G as above, then (g, f) is a distinguished pair of polynomials.

Theorem 3.2 shows that any distinguished pair of polynomials arises in this way. More precisely:

Let (g, f) be a distinguished pair of polynomials over a local field K and let a be a root of g. Let b be a root of f for which $\bar{v}(a-b)$ is largest. Let $\delta = \bar{v}(a-b)$, and let w be the r.t. extension of v to $K(x)$ having (a, δ) as a minimal pair of definition. Then g is a lifting with respect to w of a certain monic irreducible polynomial $G(Y) \neq Y$ with coefficients in the algebraic closure of k_v in k_w.

For a more detailed presentation and for proofs of these and other results on this topic, the reader is referred to [27].

4 Trace series over $p-$adic fields

Let p be a prime number, and let $\mathbb{C}_p$ be the topological closure of a fixed algebraic closure $\overline{\mathbb{Q}}_p$ of the field $\mathbb{Q}_p$ of $p - adic$ numbers. Let T be an element of $\mathbb{C}_p$ which is transcendental over $\mathbb{Q}_p$. The element T can be approximated in various ways by elements which are algebraic over $\mathbb{Q}_p$. One particularly good way to do this is to approximate T by a sequence of elements in $\overline{\mathbb{Q}}_p$ which converges to T and has the property that any two consecutive elements of the sequence form a distinguished pair. Say $(\alpha_n)_{n\in\mathbb{N}}$ is such a distinguished sequence for T. Next, there is a well defined notion of trace for algebraic elements. To be precise, for any α in $\overline{\mathbb{Q}}_p$ and any finite field extension L of $\mathbb{Q}_p$ containing α, we may consider the ratio

$$Tr(\alpha) := \frac{Tr_{L/\mathbb{Q}_p}(\alpha)}{[L:\mathbb{Q}_p]}\,.$$

This expression depends on α, but it is independent of the choice of the finite field extension L of $\mathbb{Q}_p$ containing α. Here $Tr(\alpha)$ is not the sum, but the arithmetic mean of the conjugates of α over $\mathbb{Q}_p$. Returning to the sequence $(\alpha_n)_{n\in\mathbb{N}}$, with $Tr(\alpha_n)$ well defined for each n, one may then try and define a notion of trace of T by letting

$$Tr(T) := \lim_{n\to\infty} Tr(\alpha_n),$$

provided the limit exists. For some classes of transcendental elements T over $\mathbb{Q}_p$ it is proved that the above limit exists, and is independent of the choice of the sequence $(\alpha_n)_{n\in\mathbb{N}}$., see [10]. An alternative way to define a notion of trace for some classes of elements of $\mathbb{C}_p$ which are transcendental over $\mathbb{Q}_p$ is as follows. Denote by G the group of all continuous automorphisms of $\mathbb{C}_p$ over $\mathbb{Q}_p$. The group G is canonically isomorphic to the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. For any $T \in \mathbb{C}_p$, denote by $O(T)$ the set of all conjugates of T with respect to G. The group G acts continuously on $\mathbb{C}_p$, and the natural map $G \to O(T)$ which sends any $\sigma \in G$ to $\sigma(T)$ is continuous and defines a homeomorphism between $G/H(T)$ and $O(T)$, where $H(T)$ is the stabilizer of T. In such a way, $O(T)$ is endowed with a Haar measure. This Haar measure has real values, which are rational numbers, and one can view them as being $p-$adic numbers. So one can view this measure as being a $p-$adic Haar measure, denoted by π_T. One then defines the trace of T as the integral with respect to π_T of the canonical embedding $O(T) \to \mathbb{C}_p$,

$$Tr(T) := \int_{O(T)} y \, d\pi_T(y),$$

provided this integral exists. The problem of existence of this integral was investigated in [26]. The trace $Tr(T)$ does not always exist. However, when $Tr(T)$ exists, it has similar properties to the usual trace of algebraic elements. If $Tr(T^n)$ exists for all $n \geq 1$, then one defines the so called trace series

$$F(T,Z) = 1 + \sum_{n\geq 1} Tr(T^n)Z^n.$$

This series provides an example of an equivariant rigid analytic function. For more details and further work on this topic the reader is referred to the papers cited at the end of the Introduction.

References

[1] S. Achimescu, V. Alexandru, N. Popescu, M. Vâjâitu, A. Zaharescu, The behavior of rigid analytic functions around orbits of elements of $\mathbb{C}_p$, *Rend. Semin. Mat. Univ. Padova* **127** (2012), 193–211.

[2] V. Alexandru, N. Popescu, M. Vâjâitu, A. Zaharescu, The p-adic measure on the orbit of an element of C_p, *Rend. Semin. Mat. Univ. Padova* **118** (2007), 197–216.

[3] V. Alexandru, N. Popescu, M. Vâjâitu, A. Zaharescu, On the Iwasawa algebra associated to a normal element of $\mathbb{C}_p$, *Proc. Indian Acad. Sci. Math. Sci.* **120** (2010), no. 1, 45–55.

[4] V. Alexandru, N. Popescu, M. Vâjâitu, A. Zaharescu, Representation results for equivariant rigid analytic functions, *Algebr. Represent. Theory* **15** (2012), no. 1, 137–145.

[5] V. Alexandru, N. Popescu, M. Vâjâitu, A. Zaharescu, On the zeros of Krasner analytic functions, to appear in *Algebr. Represent. Theory.*

[6] V. Alexandru, N. Popescu, A. Zaharescu, A theorem of characterization of residual transcendental extensions of a valuation, *J. Math. Kyoto Univ.* **28** (1988), no. 4, 579–592.

[7] V. Alexandru, N. Popescu, A. Zaharescu, Minimal pairs of definition of a residual transcendental extension of a valuation, *J. Math. Kyoto Univ.* **30** (1990), no. 2, 207–225.

[8] V. Alexandru, N. Popescu, A. Zaharescu, All valuations on K(X), *J. Math. Kyoto Univ.* **30** (1990), no. 2, 281–296.

[9] V. Alexandru, N. Popescu, A. Zaharescu, On the closed subfields of C_p, *J. Number Theory* **68** (1998), no. 2, 131–150.

[10] V. Alexandru, N. Popescu, A. Zaharescu, Trace on $\mathbb{C}_p$, *J. Number Theory* **88** (2001), no. 1, 13–48.

[11] V. Alexandru, N. Popescu, A. Zaharescu, A representation theorem for a class of rigid analytic functions, *J. Théor. Nombres Bordeaux* **15** (2003), no. 3, 639–650.

[12] V. Alexandru, N. Popescu, A. Zaharescu, Analytic normal basis theorem, *Cent. Eur. J. Math.* **6** (2008), no. 3, 351–356.

[13] V. Alexandru, A. Zaharescu, A transcendence criterion over $p-$adic fields, *Nihonkai Math. J.* **14** (2003), no. 2, 83–93.

[14] M. Nagata, A theorem on valuation rings and its applications, *Nagoya Math. J.* **29** (1967), 85–91.

[15] J. Ohm, Simple transcendental extensions of valued fields, *J. Math. Kyoto Univ.* **22** (1982), 201–221.

[16] J . Ohm, The ruled residue theorem for simple transcendental extensions of valued field, *Proc. Amer. Math. Soc.* **89** (1983), 16–18.

[17] J. Ohm, Simple transcendental extensions of valued fields, II: A fundamental inequality, *J. Math. Kyoto Univ.* **25** (1985), 583–596.

[18] K. Okutsu, Construction of integral basis I-IV, *Proc. Japan Acad. A* **58** (1982), 47–49, 87–89, 117–119, 167–169.

[19] K. Okutsu, Integral basis of the field $Q(\sqrt[n]{\alpha})$, *Proc. Japan Acad. A* **58** (1982), 219–222.

[20] K. Ota, On saturated distinguished chains over a local field, *J. Number Theory* **79** (1999), 217–248.

[21] K. Ota, On saturated distinguished chains over a local field II, *J. Number Theory* **111** (2005), 86–143.

[22] K. Ota, On saturated distinguished chains over a local field III, The case of p-extensions with two proper higher ramification groups, *Internat. J. Math.* **17** (2006), no. 5, 493–604.

[23] A. Popescu, N. Popescu, M. Vâjâitu, A. Zaharescu, Chains of metric invariants over a local field, *Acta Arith.* **103** (2002), no. 1, 27–40.

[24] A. Popescu, N. Popescu, A. Zaharescu, Metric invariants over Henselian valued fields, *J. Algebra* **266** (2003), no. 1, 14–26.

[25] N. Popescu, On a problem of Nagata in valuation theory, *Rev. Roumaine Math. Pures Appl.* **31** (1986), 639–641.

[26] N. Popescu, M. Vâjâitu, A. Zaharescu, On the existence of trace for elements of $\mathbf{C}_p$, *Algebr. Represent. Theory* **9** (2006), no. 1, 47–66.

[27] N. Popescu, A. Zaharescu, On the structure of the irreducible polynomials over local fields, *J. Number Theory* **52** (1995), no. 1, 98–118.

[28] N. Popescu, A. Zaharescu, On the main invariant of an element over a local field, *Portugal. Math.* **54** (1997), no. 1, 73–83.

[29] M. Vâjâitu, A. Zaharescu, Trace functions and Galois invariant $p-$adic measures, *Publ. Mat.* **50** (2006), no. 1, 43–55.

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