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Recollement of Grothendieck categories. Applications to schemes

by
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Dedicated to the memory of Nicolae Popescu (1937-2010)
on the occasion of his 75th anniversary

Abstract

We prove that under suitable conditions the recollement of two locally finite categories is locally finite and similar statements for locally coirreducible categories and categories that have Gabriel dimension and locally Krull dimension. We present applications to (right) comodule categories over semiperfect coalgebras.

Key Words: Grothendieck category, locally finite category, k-category, Gabriel dimension, Krull dimension.
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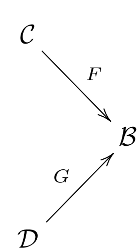
1 Introduction

We consider the following diagram:



(D)

where $\mathcal{C}$ and $\mathcal{D}$ are abelian categories and F and G are exact functors. Given such a diagram, P. Gabriel [4] defined a new category that he called the recollement of $\mathcal{C}$ and $\mathcal{D}$, and denoted by $\mathcal{C}\prod_{\mathcal{B}}\mathcal{D}$. For the definition of $\mathcal{C}\prod_{\mathcal{B}}\mathcal{D}$ see the next section. It is not difficult to see that this category is abelian. If $\mathcal{C}$ and $\mathcal{D}$ are Grothendieck categories and F and G commute with direct sums then $\mathcal{C}\prod_{\mathcal{B}}\mathcal{D}$ is also a Grothendieck category.

In this paper we consider diagrams (D) that satisfy the following conditions:

- $\mathcal{C},\mathcal{D}$ and $\mathcal{B}$ are Grothendieck categories, F and G are exact functors
- the quotient categories $\mathcal{C}/\mathrm{Ker}\ F$ and $\mathcal{D}/\mathrm{Ker}\ G$ are equivalent to $\mathcal{B}$
- F and G are the the canonical functors
- F and G commute with direct sums (equivalently, Ker F and Ker G are localizing subcategories of $\mathcal{C}$ and, respectively, $\mathcal{D}$)

$(*)$

We prove (Theorem 1) that if we assume $(*)$, $\mathcal{C}$ and $\mathcal{D}$ are locally finite and F and G commute with direct products then $\mathcal{C}\prod_{\mathcal{B}}\mathcal{D}$ is locally finite. Moreover, we show that if $\mathcal{C}$ and $\mathcal{D}$ have enough projectives then $\mathcal{C}\prod_{\mathcal{B}}\mathcal{D}$ have enough projectives. We present applications of this result to k-categories, see Theorem 2.

In the last section we show that if $\mathcal{C}$ and $\mathcal{D}$ are locally coirreducible then their recollement is locally coirreducible as well. The same type of result holds if $\mathcal{C}$ and $\mathcal{D}$ have Gabriel dimension and if $\mathcal{C}$ and $\mathcal{D}$ have locally Krull dimension.

2 Preliminaries

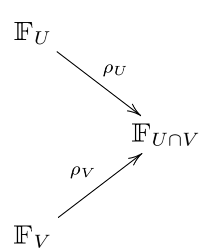
We recall first the definition of $\mathcal{C}\prod_{\mathcal{B}}\mathcal{D}$.

- An object is a triple (C, D, σ) where C is an object of $\mathcal{C}$, D is an object of $\mathcal{D}$ and σ is an isomorphism between $F(C)$ and $F(D)$.
- If (C, D, σ) and (C', D', σ') are two objects, a morphism between them is a couple (u, v) of morphisms $u : \mathcal{C} \to \mathcal{C}'$, $v : \mathcal{D} \to \mathcal{D}'$, such that $\sigma' \circ F(u) = G(v) \circ \sigma$.
- The composition of morphisms is given by $(u, v) \circ (u', v') = (u \circ u', v \circ v')$.

Remarks. By Proposition 1 of [4], page 440, if $\mathcal{C}$ and $\mathcal{D}$ are Grothendieck categories then $\mathcal{C}\prod_{\mathcal{B}}\mathcal{D}$ is in fact a pullback of $\mathcal{C}$ and $\mathcal{D}$ over $\mathcal{B}$.

Note that the canonical functors $(C, D, \sigma) \to C$ and $(C, D, \sigma) \to D$ are exact.

The typical example of a recollement category comes from the theory of schemes. Indeed, if $(X, \mathcal{O}_X)$ is a scheme we denote by $\mathbb{F}_X$ the Grothendieck category of $\mathcal{O}_X$-modules. If U and V are two open subsets of X such that $X = U \cup V$, we consider the diagram



where ρ_U and ρ_V are the restrictions $M \to M_{|U\cap V}$, $N \to N_{|U\cap V}$ for $M \in \mathbb{F}_U$, $N \in \mathbb{F}_V$. By Proposition 2, page 441, in [4] the category $\mathbb{F}_X$ is the recollement of $\mathbb{F}_U$ and $\mathbb{F}_V$ over $\mathbb{F}_{U\cap V}$.

The notion of quotient category of an abelian category through an "épaisse" subcategory was introduced in [5]. If $\mathcal{A}$ is an abelian category and $\mathcal{C}$ is a full subcategory of $\mathcal{A}$ then $\mathcal{C}$ is called épaisse if it is closed under subobjects, quotients, and extensions. For an épaisse subcategory one can define the quotient category $\mathcal{A}/\mathcal{C}$. If $\mathcal{C}$ the canonical functor $T : \mathcal{A} \to \mathcal{A}/\mathcal{C}$ has a right adjoint then $\mathcal{C}$ is called localizing (see [4], page 372). It follows from Proposition 8, page 377, in [4] that if $\mathcal{A}$ is a Grothendieck category the an épaisse subcategory is localizing if and only if it is is closed under (arbitrary) direct sums.

If the localizing subcategory $\mathcal{C}$ is moreover close under direct products then $\mathcal{C}$ is called a TTF class.

By Proposition 3, page 441, in [4] if $(X, \mathcal{O}_X)$ is a scheme and U is an open subset of X such that the injection morphism $j : U \hookrightarrow X$ is quasi-compact (i.e there exists a cover of X by open affine subsets V_i such that $V_i \cap U$ is quasi-compact for each i (see [6]) we can consider the functor T that associates to an $\mathcal{O}_X$-module M the restriction $M|U$ (which is an $(U, \mathcal{O}_X|U)$-module). The functor induced by T by passing to quotient categories is an equivalence between the categories $\mathbb{F}_X/\text{Ker } T$ and $\mathbb{F}_U$.

A Grothendieck category $\mathcal{C}$ is called locally noetherian (locally finite) if $\mathbb{C}$ has a family of noetherian (finite) generators. Obviously a locally finite category is also locally noetherian. $\mathcal{C}$ is called semi-artinian if every non-zero object of $\mathcal{C}$ has a simple subobject. It is not difficult to see that if $\mathcal{C}$ is at the same time locally noetherian and semi-artinian then it is locally finite (because every noetherian object has finite length).

We recall as well that a Grothendieck category has the property $AB4^*$ (see [5]) if the direct product is an exact functor. An example is the category $\mathcal{A} = R\text{-}mod$ (the category of R-modules).

For a Grothendieck category $\mathcal{A}$, we denote by $\text{Sp}(\mathcal{A})$ the set of idencomposable type injectives of $\mathcal{A}$.

3 Recollement of locally finite quotient categories

Lemma 1. *We assume that the diagram (D) satisfies $(*)$. Moreover we assume that $\mathcal{C}$ has the property $AB4^*$. Then* Ker F *is a TTF class if and only if F commutes with direct products.*

Proof: If F commutes with direct products then for any family $(M_i)_{i\in I}$ of objects of Ker F we have that $F(M_i) = 0$ for every $i \in I$. Therefore $F(\prod_{i\in I} M_i) = \prod_{i\in I} F(M_i) = 0$ and hence $\prod_{i\in I} M_i$ is an object of Ker F. It follows that Ker F is a TTF class.

Conversely, assume that Ker F is a TTF class. Since $\mathcal{B}$ is equivalent with $\mathcal{C}/\text{Ker } F$ via the canonical functor $F : \mathcal{C} \to \mathcal{B}$, let S be the right adjoint functor of F. By [4], chapter III, we have that $F \circ S \simeq \mathbb{1}_{\mathcal{B}}$ and the canonical morphism $\psi : \mathbb{1}_{\mathcal{C}} \to S \circ F$ has the property that, for any $M \in \mathcal{C}$, Ker $\psi(M)$ and Coker $\psi(M)$ belong to Ker T.

Let $(M_i)_{i\in I}$ be a family of objects of Ker F. For $i \in I$ we have the following exact sequence:

$$0 \to \text{Ker } (\psi(M_i)) \to M_i \xrightarrow{\psi(M_i)} (S \circ F)(M_i) \to \text{Coker } (\psi(M_i)) \to 0 \quad (1)$$

Since S is the right adjoint functor of F, S commutes with direct products. Therefore $\prod_{i\in I} S(F(M_i)) \simeq S(\prod_{i\in I} F(M_i))$ and hence $F(\prod_{i\in I} S(F(M_i))) \simeq FS(\prod_{i\in I} F(M_i)) \simeq \prod_{i\in I} F(M_i)$. On the other hand, as $\mathcal{C}$ has $AB4^*$ it follows from the exact sequence (1) that we have the following exact sequence:

$$0 \to \prod_{i\in I} \text{Ker } \psi(M_i) \to \prod_{i\in I} M_i \xrightarrow{\prod \psi(M_i)} \prod_{i\in I} S(F(M_i)) \to \prod_{i\in I} \text{Coker } \psi(M_i) \to 0$$

As we have $\prod_{i\in I} \text{Ker } \psi(M_i) = 0$ and $\prod_{i\in I} \text{Coker } \psi(M_i) = 0$, we deduce that $F(\prod_{i\in I} M_i) \simeq F(\prod_{i\in I}(S \circ F)(M_i)) \simeq \prod_{i\in I} F(M_i)$. That means that F commutes with direct products. $\square$

Lemma 2. *Let $\mathcal{A}$ be a locally noetherian Grothendieck category and $\mathcal{C}$ a TTF-subcategory of $\mathcal{A}$. If $\mathcal{C}$ and $\mathcal{A}/\mathcal{C}$ are locally finite then $\mathcal{A}$ is locally finite.*

Proof: Let $T : \mathcal{A} \to \mathcal{A}/\mathcal{C}$ be the canonical functor and $M \neq 0$ be a noetherian object of $\mathcal{A}$. Let $t : \mathcal{A} \to \mathcal{C}$ be the radical associated to the localizing subcategory $\mathcal{C}$ (hence $t(X) = \sum_{Y \leq X} Y$). We will prove that M contains a simple object. If $t(M) \neq 0$ this assertions is clear since $\mathcal{C}$ is locally finite. Let's assume that $t(M) = 0$ and hence M is $\mathcal{C}$-torsion free. Let S be the right adjoint functor of T. In this case we have $M \leq (S \circ T)(M)$ and moreover $(S \circ T)(M)$ is an essential extension of M. It is clear that it suffices to prove that $(S \circ T)(M)$ contains a simple object. So we may assume that $M = (S \circ T)(M)$ (hence M is $\mathcal{C}$-closed). Since $M \neq 0$ we have that $T(M) \neq 0$ and therefore there exists a simple object $Y \in \mathcal{A}/\mathcal{C}$ such that $Y \leq T(M)$. Therefore $0 \neq S(Y) \leq (S \circ T)(M) = M$.

We will prove now that $S(Y)$ contains a simple object. If $Z \leq S(Y)$, $Z \neq 0$, then $0 \neq T(Z) \leq (T\circ)(Y) = Y$ and hence, because Y is simple, we have that $T(Z) = Y$. Since T is an exact functor we have that $S(Y)/Z \in \mathcal{C}$. According to our hypothesis, $\mathcal{C}$ is a TTF-class (i.e. it is closed to direct products). This implies that $\prod_{(Z \leq Y, Z\neq 0)} S(Y)/Z \in \mathcal{C}$. As $S(Y)/\bigcap_{(Z\leq S(Y), Z\neq 0)} Z$ is a subobject of $\prod_{(Z \leq Y, Z\neq 0)} S(Y)/Z \in \mathcal{C}$, we have that $S(Y)/\bigcap_{(Z\leq S(Y), Z\neq 0)} Z \in \mathcal{C}$. It follows that $\bigcap_{(Z\leq S(Y), Z\neq 0)} Z \neq 0$ (because otherwise we will get that $S(Y) \in \mathcal{C}$) and clearly $\bigcap_{(Z\leq S(Y), Z\neq 0)} Z$ is a simple object in $\mathcal{A}$.

Therefore we have proved that if M is a non-zero noetherian object of $\mathcal{A}$ then M has a simple subobject. Hence $\mathcal{A}$ is semi-artinian and therefore it is locally finite. $\square$

We recall that a Grothendieck category $\mathcal{A}$ has enough projective objects if for every $M \in \mathcal{A}$ there exists a projective object $P \in \mathcal{A}$ and an epimorhism $P \to M \to 0$

Theorem 1. *We assume that the diagram (D) satisfies $(*)$. We assume also that the functors F and G commute with direct products. If $\mathcal{C}$ and $\mathcal{D}$ are locally finite then $\mathcal{C}\prod_{\mathcal{B}}\mathcal{D}$ is locally finite. Moreover, if $\mathcal{C}$ and $\mathcal{D}$ have enough projective objects, then $\mathcal{C}\prod_{\mathcal{B}}\mathcal{D}$ has enough projective objects.*

Proof: By Lemma 1 in [4], page 442, if T is the functor $(C, D, \sigma) \to D$ then the functor $M \to (M, 0, 0)$ is an isomorphism between the category Ker F and Ker T. Moreover, Ker T is a localizing subcategory of $\mathcal{C} \prod_{\mathcal{B}} \mathcal{D}$ and T defines by passing to the quotient category an equivalence between $\mathcal{C} \prod_{\mathcal{B}} \mathcal{D}/\text{Ker } T$ and $\mathcal{D}$.

Since F is an exact functor, Ker F is a localizing subcategory. According to our hypothesis, the two functors F and G commute with direct products. It follows that $\prod_{i\in I}(C_i, D_i, \sigma_i) = (\prod_{i\in I} C_i, \prod_{i\in I} D_i, \prod_{i\in I} \sigma_i)$ for any family $(C_i, D_i, \sigma_i)_{i\in I}$ of objects in $\mathcal{C} \prod_{\mathcal{B}} \mathcal{D}$. Because F commutes with direct products we have that Ker F is TTF-class. As Ker F and Ker T are isomorphic we have that Ker T is also TTF-class. Since $\mathcal{C}$ and $\mathcal{D}$ are locally finite it follows that Ker T is locally finite. Using Lemma 2 we deduce that $\mathcal{C} \prod_{\mathcal{B}} \mathcal{D}$ is locally finite.

Let's assume now that $\mathcal{C}$ and $\mathcal{D}$ are locally finite and have enough projective objects. According to Theorem 3.1 in [2], a locally finite Grothendieck category satisfies $AB4^*$ if and only if it has enough projective objects. It follows that both $\mathcal{C}$ and $\mathcal{D}$ satisfy $AB4^*$.

Let $(C_i, D_i, \sigma_i)_{i\in I}$ be a family of objects of $\mathcal{C} \prod_{\mathcal{B}} \mathcal{D}$. As above, because F and G commute with direct products, we have that the direct product of this family is $(\prod_{i\in I} C_i, \prod_{i\in I} D_i, \prod_{i\in I} \sigma_i)$. It follows that the direct product is an exact functor in $\mathcal{C} \prod_{\mathcal{B}} \mathcal{D}$ (i.e. the recollement category satisfies $AB4^*$). Applying again Theorem 3.1 in [2] we deduce that $\mathcal{C} \prod_{\mathcal{B}} \mathcal{D}$ has enough projective objects.

$\square$

Example: A scheme $(X, \mathcal{O}_X)$ is called locally artinian if X can be covered by open affine subsets Spec (A_i) where A_i is an artinian ring. $(X, \mathcal{O}_X)$ is called artinian if it is locally artinian and X is quasi-compact. Since every artinian ring is noetherian, it follows that $(X, \mathcal{O}_X)$ is noetherian.

We claim that if $(X, \mathcal{O}_X)$ is an artinian scheme then the category $\mathbb{F}_X$ is equivalent to an artinian affine scheme (Spec $A, \tilde{A}$) where A is an artinian ring.

Indeed, by Theorem 1 of [4], page 443, $\mathbb{F}_X$ is locally noetherian. In fact by Theorem 1 $\mathbb{F}_X$ is locally finite with $\text{Sp}(\mathbb{F}_X)$ being finite. Therefore $\mathbb{F}_X$ has finitely many simple objects. We have that $X = \bigcup X_i$ where $X_i = \text{Spec } A_i$ and A_i are artinian rings. Hence X is finite and its topology is the discrete one. Also, for $x \in X$ we have that $(\mathcal{O}_X)_x$ is a local artinian ring. We put then $A = \prod_{x\in X}(\mathcal{O}_X)_x$. Clearly Spec $A = X$.

4 Applications to k-categories

Let k be a field (or, more generally, a commutative ring) and $\mathcal{A}$ be an abelian category. $\mathcal{A}$ is called a k-abelian category if $\text{Hom}(M, N)$ is a k-vector space for any objects $M, N, \in \mathcal{A}$ and the composition is k-bilinear. It is clear that if in the diagram (D) the three categories $\mathcal{C}$, $\mathcal{D}$ and

$\mathcal{B}$ are k-abelian categories and the functors F and G are k-linear the the recollement category $\mathcal{C}\prod_{\mathcal{B}}\mathcal{D}$ is k-linear.

If C is a k-coalgebra, we denote by $\mathcal{M}^C$ the category of right C-comodule. In [9], M. Takeuchi proved the following theorem:

Let $\mathcal{A}$ be a k-abelian category. Then $\mathcal{A}$ is k-linear equivalent to $\mathcal{M}^C$ for some k-coalgebra C, if and only if $\mathcal{A}$ is a Grothendieck locally finite category and $\mathrm{Hom}(M, N)$ is finite dimensional over k for any two objects of finite length of $\mathcal{A}$.

In [3], page 105, it is proved that if C is a coalgebra, then every localizing subcategory of $\mathcal{M}^C$ is a TTF-class. In the same book, page 131, it is shown that if the coalgebra C is right semiperfect, i.e. the category $\mathcal{M}^C$ has enough projectives, then $\mathcal{M}^C$ has the property $AB4^*$ of Grothendieck.

Using Theorem 1 and Takeuchi's we obtain the following result:

Theorem 2. *We consider the diagram D satisfying $(*)$. We assume that $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{B}$ are k-linear categories that satisfy the conditions of Takeuchi's theorem mentioned above and that F and G are k-linear. Moreover, we assume that $\mathcal{C}$ and $\mathcal{D}$ have enough projective objects (this means that $\mathcal{C}$, and $\mathcal{D}$ are equivalent to comodule categories over semiperfect coalgebras). Then $\mathcal{C}\prod_{\mathcal{B}}\mathcal{D}$ is k-linear and $\mathcal{C}\prod_{\mathcal{B}}\mathcal{D}$ is equivalent to a comodule category $\mathcal{M}^T$ for some right semiperfect coalgebra T.*

Proof: By Theorem 1 we that $\mathcal{C}\prod_{\mathcal{B}}\mathcal{D}$ is a k-linear locally finite category. We want to prove that if X and Y are two objects of finite length of $\mathcal{C}\prod_{\mathcal{B}}\mathcal{D}$ then $\mathrm{Hom}_{\mathcal{C}\prod_{\mathcal{B}}\mathcal{D}}(X, Y)$ has finite dimension over k. This fact follows from the Lemma 3 bellow. From Takeuchi's theorem we get that $\mathcal{C}\prod_{\mathcal{B}}\mathcal{D}$ is equivalent to a right comodule category $\mathcal{M}^T$, for some coalgebra T. The fact that T is a right semiperfect coalgebra follows from the last part of Theorem 1. □

We introduce the following ad-hoc definition: we say that a category $\mathcal{A}$ has the property (P) if, for any two objects X and Y of finite length of $\mathcal{A}$, $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is finite dimensional.

Lemma 3. *Let $\mathcal{A}$ be a k-locally finite Grothendieck category and let $\mathcal{C}$ be a localizing subcategory. Assume that $\mathcal{C}$ and $\mathcal{A}/\mathcal{C}$ have the property (P). Then $\mathcal{A}$ has the property (P).*

Proof: Let X and Y be two objects of finite length of $\mathcal{A}$. Using induction it suffices to prove the finite dimensionality of $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ if X and Y are simple objects. If X and Y are both objects of $\mathcal{C}$, as $\mathcal{C}$ has the property (P) we get that $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is finite dimensional. If $X \in \mathcal{C}$ and $Y \notin \mathcal{C}$ then $\mathrm{Hom}_{\mathcal{A}}(X, Y) = 0$. Similarly, if $X \notin \mathcal{C}$ and $Y \in \mathcal{C}$ then $\mathrm{Hom}_{\mathcal{A}}(X, Y) = 0$. It remains to consider the case $X \notin \mathcal{C}$ and $Y \notin \mathcal{C}$. Since X and Y are simple, they are $\mathcal{C}$-torsion free.

Let $T : \mathcal{A} \to \mathcal{A}/\mathcal{C}$ be the canonical functor. We have the canonical morphism of k-vector spaces: $\mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(T(X), T(Y))$, $f \to T(f)$ where $f \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$. Since $T(X)$

and $T(Y)$ are simple objects of $\mathcal{A}/\mathcal{C}$ we have that $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(T(X), T(Y))$ has finite dimension over k. Also if $T(f) = 0$ it follows that $\mathrm{Im}(f) \in \mathcal{C}$. If $f \neq 0$ we would have that $Y = \mathrm{Im}(f)$ and hence $Y \in \mathcal{C}$ which contradicts our assumption. Therefore the morphism $f \to T(f)$ is injective and hence $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is finite dimensional over k. □

5 Krull and Gabriel dimensions of recollement categories

Let $\mathcal{A}$ be a Grothendieck category. $\mathcal{A}$ is called locally coirreducible (we will write l.c.) if every non-zero object contains a non-zero coirreducible object (see [8]). $\mathcal{A}$ has Gabriel dimension if for any localizing subcategory $\mathcal{C} \subsetneq \mathcal{A}$ the quotient category contains a simple non-zero object (see [4]). It is known that if $\mathcal{A}$ has Gabriel dimension then $\mathcal{A}$ is a l.c. category (see [8]). $\mathcal{A}$ is called a locally stable category if every localizing subcategory $\mathcal{C}$ has the form $\mathcal{C} = \cap_{I \in X} \mathcal{C}_I$ where $X \subseteq \mathrm{Sp}(\mathcal{A})$ and $\mathcal{C}_I = \{M \in \mathcal{A} : \mathrm{Hom}(M, I) = 0\}$ (see [1]). Finally, $\mathcal{A}$ has locally Krull dimension if there exists a family of generators $(U_i)_{i \in I}$ of $\mathcal{A}$ such that U_i has Krull dimension for any $i \in I$. We set $\text{K-dim}\,\mathcal{A} = \sup_{i \in I} \text{K-}\dim U_i$. If $\mathcal{A}$ is locally finite then this definition does not depend on the choice of the family of finite length generators. It is clear that if $\mathcal{A}$ has locally Krull dimension then $\mathcal{A}$ has Gabriel dimension. Also if $\mathcal{A}$ is a l.c. category then $\mathcal{A}$ is locally stable.

When the category *R-mod* is a l.c. category (respectively it has Gabriel dimension, respectively it has locally Krull dimension, respectively it is locally stable) then the ring R called a left l.c. module (respectively it has locally Krull dimension, respectively it is locally stable).

Theorem 3. *We assume that the diagram* (D) *satisfies conditions* $(*)$ *and that the Grothendieck categories* $\mathcal{C}$ *and* $\mathcal{D}$ *are l.c. (respectively they have Gabriel dimension, respectively they have locally Krull dimension, respectively they are locally stable). Then the recollement category* $\mathcal{C} \prod\limits_{\mathcal{B}} \mathcal{D}$ *is l.c. (respectively it has Gabriel dimension, respectively it has locally Krull dimension, respectively it is locally stable).*

Proof: As in the proof of Theorem 1, we let T be the functor $(C, D, \sigma) \to D$ and we apply Lemma 1 in [4], page 442. We get that Ker F and Ker T are isomorphic and that Ker T is a localizing subcategory of $\mathcal{C} \prod\limits_{\mathcal{B}} \mathcal{D}$.

Assume that $\mathcal{C}$ and $\mathcal{D}$ are l.c. category. It follows from [8] that Ker F is a l.c. category and therefore Ker T is an l.c. category. Also from [8] we have that if we have a Grothendieck category $\mathcal{A}$, $\mathcal{C}$ is a localizing subcategory and both $\mathcal{C}$ and $\mathcal{A}/\mathcal{C}$ are l.c. categories then $\mathcal{A}$ is a l.c. category. We deduce that $\mathcal{C} \prod\limits_{\mathcal{B}} \mathcal{D}$ is a l.c. category. Using similar results from [4], chapter IV and [1] we get that if $\mathcal{C}$ and $\mathcal{D}$ have Gabriel dimension (respectively they are locally stable) then $\mathcal{C} \prod\limits_{\mathcal{B}} \mathcal{D}$ has Gabriel dimension (respectively it is locally stable).

The proof that $\mathcal{C} \prod\limits_{\mathcal{B}} \mathcal{D}$ has Krull dimension provided that both $\mathcal{C}$ and $\mathcal{D}$ have Krull dimension is analogous to the proof of Lemma 2, page 442, in [4]. □

Remark. In [4], page 382, P. Gabriel defined what he called the Krull dimension of an abelian category Nowdays this is called the Gabriel dimension. As a convention, for an abelian category $\mathcal{A}$, we denote by $\mathcal{A}_0$ the category in which the only objects are the null objects of $\mathcal{A}$ (instead of $\mathcal{A}_{-1}$ as Gabriel does) and we carry on Gabriel's construction to define the Gabriel dimension. Then, under the assumptions of Theorem 3, if Gdim $\mathcal{C} = \alpha$ and Gdim $\mathcal{D} = \beta$ (α and β are ordinals) then $\sup\{\alpha, \beta\} \leq \text{Gdim } \mathcal{C} \prod_{\mathcal{B}} \mathcal{D} \leq \inf\{\alpha + \beta, \beta + \alpha\}$. This follows from proposition 1 in [4], page 382 by interchanging $\mathcal{C}$ and $\mathcal{D}$ in the proof of Theorem 3.

A similar inequality can be obtained for the Krull dimension.

Corollary 1. *Let $(X, \mathcal{O}_X)$ be a scheme. Assume that $X = \bigcup_{i=1}^{n} U_i$, where U_i are open quasi-compact subsets of X, and $(U_i, \mathcal{O}_X|U_i) \simeq (\text{Spec } A_i, \mathcal{O}_i)$ where A_i are commutative rings. If each A_i, $1 \leq i \leq n$ is an l.c. ring (respectively it has Gabriel dimension, respectively it has locally Krull dimension, respectively it is locally stable) then the category $\mathbb{F}_X$ is a l.c. category (respectively it has Gabriel dimension, respectively it has locally Krull dimension, respectively it is locally stable).*

The proof of this corollary is immediate using induction on n.

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