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Bimonads in a 2-category

by
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Abstract

We associate to a given 2-category $\mathfrak{K}$ a new 2-category $\mathfrak{Bimon}(\mathfrak{K})$, whose 0-cells are the bimonads in $\mathfrak{K}$. We show that this construction defines an endofunctor of the category 2-CAT of all 2-categories, which is represented by a certain 2-category $\mathfrak{Bimon}$.

Key Words: 2-category, bimonad in a 2-category.
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Introduction

Bimonads and Hopf monads have been introduced and investigated in [MW, W]. In this note we are going to show that bimonads make sense in an arbitrary 2-category. Furthermore, for any 2-category $\mathfrak{K}$ we shall prove that the bimonads in $\mathfrak{K}$ define a new 2-category $\mathfrak{Bimon}(\mathfrak{K})$, so that the mapping

$$\mathfrak{K} \to \mathfrak{Bimon}(\mathfrak{K})$$

is an endofunctor of 2-CAT, the (large) category of all 2-categories. In the main result of the paper we shall prove that this 2-functor is representable, in the sense that there are a 2-category $\mathfrak{Bimon}$ and an isomorphism of 2-categories

$$[\mathfrak{Bimon}, \mathfrak{K}] \simeq \mathfrak{Bimon}(\mathfrak{K}),$$

which is natural in $\mathfrak{K}$. Here, $[\mathfrak{K}, \mathfrak{L}]$ denotes the 2-category of all strict 2-functors between the 2-categories $\mathfrak{K}$ and $\mathfrak{K}$. We also prove a similar result for Hopf monads.

1 Bimonads in a 2-category.

Throughout this paper $\mathfrak{K}$ will denote a given 2-category. Recall that a 2-category is by definition a category enriched in the category of all categories. In other words, a 2-category is given by a class of objects (0-cells), morphism between objects (that are called 1-cells) and morphism between morphisms (that are called 2-cells). The 0-cells will be denoted by capital letters X, Y,

X', etc. The 1-cells will also be denoted by capital letters, e.g. $F : X \longrightarrow X'$. We will say that X and Y are the source and the target of F. For 2-cells we will use the notation $\alpha : F \Longrightarrow F'$. The 1-cells F and F' will be called the source and the target of α. To each 0-cell corresponds an 1-cell Id_X, the identity of X. Similarly, for every 1-cell F we can speak about the identity of F, which is a 2-cell denoted by Id_F.

As in an usual category 1-cells can be composed. If there is no danger of confusion, for the composition $F \circ G$ of two 1-cells we will write FG. On the other hand, 2-cells can be composed horizontally and vertically with respect to the operations that will be denoted by $\circ$ and $\bullet$. To simplify the notation we shall write $F\alpha$ and αG instead of $Id_F \circ \alpha$ and $\alpha \circ Id_G$, respectively (of course, whenever these compositions make sense). For details on 2-categories the reader is referred to [LR, S1, S2].

We start by recalling some well-known definitions that we will need later on.

1.1 (Monads in $\mathfrak{K}$.) A monad in $\mathfrak{K}$ consists of a 0-cell X, an 1-cell $T : X \to X$ and a pair of 2-cells $m : TT \Rightarrow T$ and $u : Id_X \Rightarrow T$ (the multiplication and the unit of the monad), such that the following equalities hold

$$m \bullet Tm = m \bullet mT, \tag{1}$$

$$m \bullet Tu = Id_T = m \bullet uT. \tag{2}$$

A morphism between two monads (X, T, m, u) and (X', T', m', u') consists of a pair (F, σ), where $F : X \to X'$ is an 1-cell and $\sigma : T'F \Longrightarrow FT$ is a 2-cell such that the following identities hold:

$$Fm \bullet \sigma T \bullet T'\sigma = \sigma \bullet m'F, \tag{3}$$

$$\sigma \bullet u'F = Fu. \tag{4}$$

1.2 (Comonads in $\mathfrak{K}$.) Comonads in a 2-category are defined by duality. Therefore a comonad consists of a 0-cell X, an 1-cell $S : X \to X$ and a pair of 2-cells $\delta : S \Rightarrow SS$ and $\varepsilon : S \Rightarrow Id_X$ (the counit and the comultiplication of the comonad), such that the following equalities hold:

$$S\delta \bullet \delta = \delta S \bullet \delta, \tag{5}$$

$$\varepsilon S \bullet \delta = Id_S = S\varepsilon \bullet \delta. \tag{6}$$

A morphism of comonads from $(X, S, \delta, \varepsilon)$ to $\left(X', S', \delta', \varepsilon'\right)$ is a pair (G, τ), where $G : X \to X'$ is an 1-cell and $\tau : S'G \Rightarrow GS$ is a 2-cell such that the following identities hold:

$$\tau S \bullet S'\tau \bullet \delta' G = G\delta \bullet \tau, \tag{7}$$

$$G\varepsilon \bullet \tau = \varepsilon' G. \tag{8}$$

1.3 (Bimonads in $\mathfrak{K}$.) To define a bimonad in $\mathfrak{K}$ we need a monad (X, H, m, u) and a comonad $(X, H, \delta, \varepsilon)$ in $\mathfrak{K}$. Note that these structures share the same 0-cell X and the same 1-cell H. Of course, in addition these structures has to be compatible in a certain sense. In order to state the compatibility condition we need the definition of entwining maps. Let (X, T, m, u)

and (X,S,δ,ε) be a monad and a comonad, respectively. A 2-cell $\lambda : TS \Rightarrow ST$ is an entwining structure if the following four conditions hold:

$$Sm \bullet \lambda T \bullet T\lambda = \lambda \bullet mS, \tag{9}$$
$$S\lambda \bullet \lambda S \bullet T\delta = \delta T \bullet \lambda, \tag{10}$$
$$\varepsilon T \bullet \lambda = T\varepsilon, \tag{11}$$
$$\lambda \bullet uS = Su. \tag{12}$$

Now we can say what a bimonad in $\mathfrak{K}$ is. Namely, the sextuple $(X,H,m,u,\delta,\varepsilon)$ is a bimonad if (X,H,m,u) is a monad, (X,H,δ,ε) is a comonad and $\lambda : H^2 \Rightarrow H^2$ is an entwining structure that satisfy the following conditions:

i) The unit u is a morphism of comonads from $(X, Id_X, Id_{Id_X}, Id_{Id_X})$ to (X,H,δ,ε), i.e.

$$\delta \bullet u = u \circ u, \tag{13}$$
$$\varepsilon \bullet u = Id_{Id_X}. \tag{14}$$

ii) The counit ε is a morphism of monads from (X,H,m,u) to $(X, Id_X, Id_{Id_X}, Id_{Id_X})$, i.e.

$$\varepsilon \bullet m = \varepsilon \circ \varepsilon. \tag{15}$$

Note that the condition expressing that ε is compatible with the unit u is equivalent to (14). Consequently, it was omitted.

iii) The following identity hold:

$$\delta \bullet m = Hm \bullet \lambda H \bullet H\delta. \tag{16}$$

A pair (F,σ) is a morphism of bimonads from $(X,H,m,u,\delta,\varepsilon,\lambda)$ to $\left(X',H',m',u',\delta',\varepsilon',\lambda'\right)$ if it is a morphism of monads and comonads, and the following identity holds:

$$F\lambda \bullet \sigma H \bullet H'\sigma = \sigma H \bullet H'\sigma \bullet \lambda' F. \tag{17}$$

1.4 (The 2-category $\mathfrak{Bimon}(\mathfrak{K})$.) Let $\mathfrak{K}$ be a 2-category. We are going to define a new 2-category, whose 0-cells are the bimonads in $\mathfrak{K}$. By construction, the 1-cells in $\mathfrak{Bimon}(\mathfrak{K})$ are the morphisms between arbitrary bimonads in $\mathfrak{K}$. A 2-cell $\alpha : (F,\sigma) \Longrightarrow (G,\tau)$ in $\mathfrak{Bimon}(\mathfrak{K})$, where

$$(X,H,m,u,\delta,\varepsilon,\lambda) \overset{(F,\sigma)}{\underset{(G,\tau)}{\rightrightarrows}} \left(X',H',m',u',\delta',\varepsilon',\lambda'\right),$$

is a 2-cell $\alpha : F \Longrightarrow G$ in $\mathfrak{K}$ such that

$$\alpha H \bullet \sigma = \tau \bullet H'\alpha. \tag{18}$$

To simplify the notation, we will write (co)monads and bimonads as pairs, whose components are the corresponding 0-cells and 1-cells (we will omit all other underlying structures). For example, if there is no danger of confusion, the monad (X,T,m,u) will be denoted (X,T).

By definition, the composition of two morphisms in $\mathfrak{Bimon}(\mathfrak{K})$

$$(X,H)\xrightarrow{(F,\sigma)}(X',H')\xrightarrow{(G,\tau)}(X'',H'') \tag{19}$$

is given by:

$$(G,\tau)\circ(F,\sigma)=(GF,G\sigma\bullet\tau F)\,. \tag{20}$$

Clearly, for every bimonad (X,H), the pair (Id_X,Id_H) is a morphism of bimonads, that will be called the identity of (X,H). For the 2-cells α and β in $\mathfrak{Bimon}\,(\mathfrak{K})$

$$(X,H)\underset{(F',\sigma')}{\overset{(F,\sigma)}{\Downarrow\alpha}}(X',H')\underset{(G',\tau')}{\overset{(G,\tau)}{\Downarrow\beta}}(X'',H'')\,, \tag{21}$$

we define the horizontal composition $\beta\circ\alpha:GF\Rightarrow G'F'$ as follows:

$$\beta\circ\alpha:=\beta F'\bullet G\alpha=G'\alpha\bullet\beta F. \tag{22}$$

Note that the second equality holds as $\mathfrak{K}$ is a 2-category, so the above definition makes sense. Now, let us consider the 2-cells $\alpha:(F,\sigma)\Longrightarrow(F',\sigma')$ and $\alpha':(F',\sigma')\Longrightarrow(F'',\sigma'')$ in $\mathfrak{Bimon}(\mathfrak{K})$, where (F,σ), (F',σ') and (F'',σ'') are some morphisms of bimonads as in the figure below:

$$(X,H)\;\overset{(F,\sigma)}{\underset{(F'',\sigma'')}{\xrightarrow{(F',\sigma')}}}\;(X',H')\,. \tag{23}$$

We define the vertical composition of α' and α in $\mathfrak{Bimon}\,(\mathfrak{K})$ to be $\alpha'\bullet\alpha$ (the vertical composition of α' and α regarded as 2-cells in $\mathfrak{K}$). The identity 2-cell of (F,σ) is by definition Id_F.

Theorem 1.5 *The above constructions define a* 2*-category that we shall denote by* $\mathfrak{Bimon}\,(\mathfrak{K})$ *.*

Proof: We must check that the above constructions are well-defined. First we prove that, for two morphisms of bimonads as in (19), the composition $(GF,G\sigma\bullet\tau F)$ is also a morphism of bimonads from (X,H) to (X'',H''). Let m,m' and m'' denote the multiplications in $(X,H)\,,(X',H')$ and (X'',H''), respectively. We have

$$\begin{aligned}(G\sigma\bullet\tau F)\bullet m''GF &= G\sigma\bullet Gm'F\bullet\tau H'F\bullet H''\tau F\\ &= GFm\bullet G\sigma H\bullet GH'\sigma\bullet\tau H'F\bullet H''\tau F\\ &= GFm\bullet G\sigma H\bullet\tau FH\bullet H''G\sigma\bullet H''\tau F\\ &= GFm\bullet[(G\sigma\bullet\tau F)\,H]\bullet[H''\,(G\sigma\bullet\tau F)]\,.\end{aligned}$$

Note that for the first and the second equalities we used the relation (3), written for τ and σ, respectively. To deduce the last two equalities we used the interchange law in $\mathfrak{K}$. The compatibility with the units u,u',u'' follows by the computation below:

$$(G\sigma\bullet\tau F)\bullet u''GF=G\sigma\bullet[(\tau\bullet u''G)\,F]=G\sigma\bullet Gu'F=G\,(\sigma\bullet u'F)=GFu.$$

By duality one shows that $(GF, G\sigma \bullet F\tau)$ is a morphism of comonads. In order to prove that this pair is a morphism of bimonads we still have to check that the identity (17) holds. Indeed,

$$\begin{aligned}[(G\sigma \bullet \tau F)\, H] \bullet [H''\,(G\sigma \bullet \tau F)] \bullet \lambda'' GF &\overset{(A)}{=} G\sigma H \bullet GH'\sigma \bullet \tau H'F \bullet H''\tau F \bullet \lambda'' GF \\ &\overset{(B)}{=} G\sigma H \bullet GH'\sigma \bullet G\lambda' F \bullet \tau H'F \bullet H''\tau F \\ &\overset{(C)}{=} GF\lambda \bullet G\sigma H \bullet GH'\sigma \bullet \tau H'F \bullet H''\tau F \\ &\overset{(D)}{=} GF\lambda \bullet G\sigma H \bullet \tau FH \bullet H''G\sigma \bullet H''\tau F \\ &= GF\lambda \bullet [(G\sigma \bullet \tau F)\, H] \bullet [H''\,(G\sigma \bullet \tau F)]\,.\end{aligned}$$

In the above computations for (A) and (D) we used the interchange law, while (B) and (C) are consequences of (17), written for τ and σ, respectively.

We now take α and β to be 2-cells in $\mathfrak{Bimon}(\mathfrak{K})$ as in (21). We claim that $\beta \circ \alpha$ is a 2-cell between $(GF, G\sigma \bullet \tau F)$ and $(G'F', G'\sigma' \bullet \tau' F')$, that is $G'\alpha \bullet \beta F = \beta F' \bullet G\alpha$ satisfies the condition (18). Indeed, we have:

$$\begin{aligned}(\beta \circ \alpha)\, H \bullet (G\sigma \bullet \tau H) &\overset{(A)}{=} G'\alpha H \bullet \beta FH \bullet G\sigma \bullet \tau F \\ &\overset{(B)}{=} G'\alpha H \bullet G'\sigma \bullet \beta H'F \bullet \tau F \\ &\overset{(C)}{=} G'\sigma' \bullet G'H'\alpha \bullet \tau' F \bullet H''\beta F \\ &\overset{(D)}{=} G'\sigma' \bullet \tau' F' \bullet H''G'\alpha \bullet H''\beta F \\ &= (G'\sigma' \bullet \tau' F') \bullet H''\,(\beta \circ \alpha)\,.\end{aligned}$$

In the above computation we used the definition of the horizontal composition in $\mathfrak{Bimon}\,(\mathfrak{K})$ to deduce (A) and (D). In (B) and (D) we also used the interchange law. Finally, we got (C) by applying the relation (18) twice.

Let $\alpha : (F, \sigma) \Rightarrow (F', \sigma')$ and $\alpha' : (F', \sigma') \Rightarrow (F'', \sigma'')$ be 2-cells in $\mathfrak{Bimon}\,(\mathfrak{K})$, where (F, σ), (F', σ') and (F'', σ'') are morphisms of bimonads with the same source and the same target. Our aim now is to check that $\alpha' \bullet \alpha$ is a 2-cell in $\mathfrak{Bimon}\,(\mathfrak{K})$, i.e.

$$((\alpha' \bullet \alpha)\, H) \bullet \sigma = \sigma'' \bullet (H''\,(\beta \bullet \alpha)).$$

Since α' and α are 2-cells in $\mathfrak{Bimon}\,(\mathfrak{K})$, we get:

$$((\alpha' \bullet \alpha)\, H) \bullet \sigma = \alpha' H \bullet \alpha H \bullet \sigma = \alpha' H \bullet \sigma' \bullet H'\alpha = \sigma'' \bullet H'\alpha' \bullet H'\alpha = \sigma'' \bullet (H'\,(\alpha' \bullet \alpha))\,.$$

Obviously, (Id_X, Id_H) satisfies the axioms of the identity 1-cell in a 2-category, for any bimonad (X, H). Moreover, it is easy to see that and Id_F is the identity 2-cell of (F, σ), for every bimonad morphism (F, σ).

In order to show that the composition of 1-cells in $\mathfrak{Bimon}\,(\mathfrak{K})$ is associative we take three composable morphisms (F, σ), (G, τ) and (H, δ). By applying the relation (20) four times we get

$$\begin{aligned}(H, \delta) \circ [(G, \tau) \circ (F, \sigma)] =& (H, \delta) \circ (GF, G\sigma \bullet \tau F) = (HGF, HG\sigma \bullet H\tau F \bullet \delta GF) \\ =& (HG, H\tau \bullet \delta G) \circ (F, \sigma) = [(H, \delta) \circ (G, \tau)] \circ (F, \sigma).\end{aligned}$$

The horizontal composition in $\mathfrak{Bimon}(\mathfrak{K})$ is also associative. To check that we take the 2-cells α and β as in (21). If $\gamma : (H, \delta) \Longrightarrow (H', \delta')$ is another 2-cell such that $\gamma \circ (\beta \circ a)$ exists, then

$$(\gamma \circ \beta) \circ \alpha = (\gamma G' \bullet H\beta) F' \bullet HG\alpha = \gamma G' F' \bullet H\beta F' \bullet HG\alpha = \gamma G' F' \bullet H (\beta F' \bullet G\alpha) = \gamma \circ (\beta \circ \alpha).$$

The vertical composition in $\mathfrak{Bimon}(\mathfrak{K})$ is associative, as it coincides with that one in $\mathfrak{K}$. It remains to show that interchange law holds in $\mathfrak{Bimon}(\mathfrak{K})$. We take the 2-cells α and β as in (21). We assume that α' and β' are other 2-cells in $\mathfrak{Bimon}(\mathfrak{K})$ such that $\beta' \circ \alpha'$, $\alpha' \bullet \alpha$ and $\beta' \bullet \beta$ make sence. Therefore, the source of α' and β' are F' and G', respectively. Let F'' and G'' be their targets. Since $\mathfrak{K}$ is a 2-category, the interchange law holds, so we have

$$G'\alpha' \bullet \beta F' = \beta \circ \alpha' = \beta F'' \bullet G\alpha'.$$

Now we can prove that the interchange law holds in $\mathfrak{Bimon}(\mathfrak{K})$ too. Indeed, by the definition of the horizontal composition in $\mathfrak{Bimon}(\mathfrak{K})$, and the fact the vertical composition in this 2-category coincides to that one $\mathfrak{K}$, we get

$$\begin{aligned}(\beta' \bullet \beta) \circ (\alpha' \bullet \alpha) &= \left[(\beta' \bullet \beta) F''\right] \bullet [G(\alpha' \bullet \alpha)] = \beta' F'' \bullet \beta F'' \bullet G\alpha' \bullet G\alpha \\ &= \beta' F'' \bullet G'\alpha' \bullet \beta F' \bullet G\alpha = (\beta' \circ \alpha') \bullet (\beta \circ \alpha).\end{aligned}$$

In conclusion we have just proved that $\mathfrak{Bimon}(\mathfrak{K})$ is a 2-category. □

1.6 (The 2-functor $\mathfrak{Bimon}(\mathfrak{F})$.) Our goal now is to show that the construction of $\mathfrak{Bimon}(\mathfrak{K})$ is functorial in $\mathfrak{K}$. More precisely, if 2-CAT denotes the (large) category of 2-categories with strict 2-functors as morphisms, then the mapping

$$\mathfrak{K} \to \mathfrak{Bimon}(\mathfrak{K})$$

defines an endofunctor of 2-CAT. We have already defined $\mathfrak{Bimon}(-)$ on the objects of 2-CAT. It remains to construct $\mathfrak{Bimon}(\mathfrak{F})$, for every strict 2-functor $\mathfrak{F} : \mathfrak{K} \to \mathfrak{L}$.

Recall that a 2-functor $\mathfrak{F}$ as above is given by a map $\mathfrak{F}_0 : \mathfrak{K}_0 \to \mathfrak{L}_0$ and a family of functors $(\mathfrak{F}_{X,Y})_{X,Y \in \mathfrak{K}_0}$ where, for all 0-cells X and Y,

$$\mathfrak{F}_{X,Y} : \mathfrak{K}(X, Y) \to \mathfrak{L}(\mathfrak{F}_0(X), \mathfrak{F}_0(Y)).$$

The family $(\mathfrak{F}_{X,Y})_{X,Y \in \mathfrak{K}_0}$ is assumed to be compatible with the composition of 1-cells and with the identity 1-cells. Therefore, to each 1-cell $f : X \to Y$ corresponds a unique 1-cell $F_{X,Y}(f)$, whose source and target are $\mathfrak{F}_0(X)$ and $\mathfrak{F}_0(Y)$, respectively. It will be denoted by $\mathfrak{F}_1(f)$. Analogously, if $\alpha : f \Longrightarrow g$ is a 2-cell such that f and g have the same source and the same target, then we denote the 2-cell $F_{X,Y}(\alpha) : \mathfrak{F}_1(f) \Longrightarrow \mathfrak{F}_1(g)$ by $\mathfrak{F}_2(\alpha)$.

We are going to associate to $\mathfrak{F}$ a 2-functor

$$\mathfrak{Bimon}(\mathfrak{F}) : \mathfrak{Bimon}(\mathfrak{K}) \to \mathfrak{Bimon}(\mathfrak{L}).$$

We first define $\overline{\mathfrak{F}} := \mathfrak{Bimon}(\mathfrak{F})$ on 0-cells. Let $(X, H, m, u, \delta, \varepsilon, \lambda)$ be a bimonad. We set

$$\overline{\mathfrak{F}}_0(X, H, m, u, \delta, \varepsilon, \lambda) := (\mathfrak{F}_0(X), \mathfrak{F}_1(H), \mathfrak{F}_2(m), \mathfrak{F}_2(u), \mathfrak{F}_2(\delta), \mathfrak{F}_2(\varepsilon), \mathfrak{F}_2(\lambda)).$$

It is not difficult to see that this is a bimonad in $\mathfrak{L}$. For a morphism of bimonads in $\mathfrak{K}$, or equivalently an 1-cell $(F,\sigma):(X,H)\to(X',H')$ in $\mathfrak{Bimon}(\mathfrak{K})$, we define:

$$\overline{\mathfrak{F}}_1(F,\sigma) := (\mathfrak{F}_1(F), \mathfrak{F}_2(\sigma)).$$

Hence $\mathfrak{F}_1(F):\mathfrak{F}_0(X)\to\mathfrak{F}_0(X')$ and $\mathfrak{F}_2(\sigma):\mathfrak{F}_1(H')\circ\mathfrak{F}_1(F)\Longrightarrow\mathfrak{F}_1(F)\circ\mathfrak{F}_1(H)$. One shows easily that $\overline{\mathfrak{F}}(F,\sigma)$ is a morphism of bimonads in $\mathfrak{L}$ from $\overline{\mathfrak{F}}(X,H)$ to $\overline{\mathfrak{F}}(X',H')$.

For a 2-cell $\alpha:(F,\sigma)\Longrightarrow(G,\tau)$ in $\mathfrak{Bimon}(\mathfrak{K})$, we put:

$$\overline{\mathfrak{F}}_2(\alpha) = \mathfrak{F}_2(\alpha).$$

Clearly, $\mathfrak{F}_2(\alpha):\mathfrak{F}_1(F)\to\mathfrak{F}_1(G)$ is a 2-cell in $\mathfrak{L}$. In fact by an easy computation one checks that $\mathfrak{F}_2(\alpha)$ is a 2-cell in $\mathfrak{Bimon}(\mathfrak{L})$ from $\overline{\mathfrak{F}}(X,H)$ to $\overline{\mathfrak{F}}(X',H')$.

Theorem 1.7 *The above data define a 2-functor*

$$\mathfrak{Bimon}(\mathfrak{F}):\mathfrak{Bimon}(\mathfrak{K})\to\mathfrak{Bimon}(\mathfrak{L}).$$

Proof: We have to show that

$$\overline{\mathfrak{F}}_{(X,H),(X',H')}:\mathfrak{Bimon}((X,H),(X',H'))\to\mathfrak{Bimon}((\mathfrak{F}_0(X),\mathfrak{F}_1(H)),(\mathfrak{F}_0(X'),\mathfrak{F}_1(H')))$$

is a functor, and that the family of these functors is compatible with the composition of 1-cells. In other words $\overline{\mathfrak{F}}$ is compatible with the compositions of 1-cells, and with horizontal and vertical compositions of 2-cells. Let (F,σ) and (G,τ) be 1-cells as in (19). Their composition is defined by the formula (20), so $\overline{\mathfrak{F}}$ maps $(G,\tau)\circ(F,\sigma)$ to the morphism $(\mathfrak{F}_1(GF),\mathfrak{F}_2(G\sigma\bullet\tau F))$. Since $\mathfrak{F}$ is a 2-functor from $\mathfrak{K}$ to $\mathfrak{L}$, we get

$$\begin{aligned}(\mathfrak{F}_1(GF),\mathfrak{F}_2(G\sigma\bullet\tau F)) &= (\mathfrak{F}_1(G)\circ\mathfrak{F}_1(F),\mathfrak{F}_1(G)\mathfrak{F}_2(\sigma)\bullet\mathfrak{F}_2(\tau)\mathfrak{F}_1(F))\\ &= (\mathfrak{F}_1(G),\mathfrak{F}_2(\tau))\circ(\mathfrak{F}_1(F),\mathfrak{F}_2(\sigma)).\end{aligned}$$

This means that $\overline{\mathfrak{F}}$ is compatible with the composition of 1-cells. In order to prove the compatibility of $\overline{\mathfrak{F}}$ with the horizontal composition of 2-cells we take α and β to be 2-cells as in (21). In view of (22) and taking into account that $\mathfrak{F}$ is a 2-functor, we have

$$\overline{\mathfrak{F}}_2(\beta\circ\alpha)=\mathfrak{F}_2(\beta F'\bullet G\alpha)=\mathfrak{F}_2(\beta F')\bullet\mathfrak{F}_2(G\alpha)=\mathfrak{F}_2(\beta)\mathfrak{F}_1(F')\bullet\mathfrak{F}_1(G)\mathfrak{F}_2(\alpha)=\overline{\mathfrak{F}}_2(\beta)\circ\overline{\mathfrak{F}}_2(\alpha).$$

Suppose that α and α' are 2-cells that can be composed vertically. Then, since $\mathfrak{F}$ is a 2-functor,

$$\overline{\mathfrak{F}}_2(\alpha\bullet\alpha')=\mathfrak{F}_2(\alpha\bullet\alpha')=\mathfrak{F}_2(\alpha)\bullet\mathfrak{F}_2(\alpha').$$

By construction $\overline{\mathfrak{F}}$ maps identity cells to identity cells, so $\overline{\mathfrak{F}}$ is a strict 2-functor indeed. To conclude the proof of the theorem we have to show that

$$\mathfrak{Bimon}(\mathfrak{F}\circ\mathfrak{G})=\mathfrak{Bimon}(\mathfrak{F})\circ\mathfrak{Bimon}(\mathfrak{F})\quad\text{and}\quad\mathfrak{Bimon}(Id_{\mathfrak{K}})=Id_{\mathfrak{Bimon}(\mathfrak{K})}.$$

Both relations are immediate consequences of the definitions. □

2 The main result.

In this section we prove our main result, stating that the 2-functor $\mathfrak{Bimon}(-)$ is representable. We will also establish a similar result for Hopf monads in a 2-category.

2.1 (The 2-category $\mathfrak{Bimon}$.) An useful method to produce new examples of 2-categories is explained in [S2]. We will follow the terminology from loc. cit. To every 2-category $\mathfrak{K}$ one associates in a canonical way a computad $\mathbb{U}\mathfrak{K}$ (the underlying computad of $\mathfrak{K}$, cf. [S2, p. 538]). We obtain a functor $\mathbb{U}$ from 2-CAT to the category of computads. This functor has a left adjoint $\mathbb{F}$, which maps a computad $\mathbf{\Gamma}$ the free 2-category $\mathbb{F}\mathbf{\Gamma}$ of Γ, see [S2, p. 538]. Therefore, for every computad $\mathbf{\Gamma}$ and every 2-category $\mathfrak{K}$ there is an one-to-one correspondence between the morphisms of computads $\mathbf{\Gamma} \to \mathbb{U}\mathfrak{K}$ and the strict 2-functors $\mathbb{F}\mathbf{\Gamma} \to \mathfrak{K}$.

Furthermore, an arbitrary 2-category $\mathfrak{K}$ can by quotient out modulo a congruence relation R. One obtains a new 2-category $\mathfrak{K}/R$ such that the 0-cells and 1-cells in $\mathfrak{K}$ and $\mathfrak{K}/R$ are identical, but the 2-cells in the latter 2-category are the equivalence classes of those in the former one. By construction, for every 2-category $\mathfrak{L}$, there is an one-to-one correspondence between the 2-functors $\mathfrak{K}/R \to \mathfrak{L}$ and the functors $\mathfrak{K} \to \mathfrak{L}$ that maps equivalent 2-cells in $\mathfrak{K}$ to the same 2-cell in $\mathfrak{L}$.

We apply this strategy to construct the 2-category $\mathfrak{Bimon}$. For, we start with the computad $\mathbf{\Gamma}$ that has an unique 0-cell X_0 and an unique 1-cell $H_0 : X_0 \to X_0$. The 2-cells of $\mathbf{\Gamma}$ are:

$$m_0 : H_0^2 \Longrightarrow H_0, \quad u_0 : Id_{X_0} \Longrightarrow H_0, \quad \delta_0 : H_0 \Longrightarrow H_0^2, \quad \varepsilon_0 : H_0 \Longrightarrow Id_{X_0}, \quad \lambda_0 : H_0^2 \Longrightarrow H_0^2.$$

On the free 2-category $\mathbb{F}\mathbf{\Gamma}$ we impose the relations:

$$\begin{aligned}
&m_0 \bullet (H_0 m_0) \equiv m_0 \bullet (m_0 H_0)\,, && m_0 \bullet (H_0 u_0) \equiv \mathbf{1}_{H_0} \equiv m_0 \bullet (u_0 H_0)\,,\\
&(H_0 \delta_0) \bullet \delta_0 \equiv (\delta_0 H_0) \bullet \delta_0, && (H_0 \varepsilon_0) \bullet \delta_0 \equiv \mathbf{1}_{H_0} \equiv (\varepsilon_0 H_0) \bullet \delta_0,\\
&\lambda_0 \bullet (m_0 H_0) \equiv (H_0 m_0) \bullet (\lambda_0 H_0) \bullet (H_0 \lambda_0)\,, && \lambda_0 \bullet (u_0 H_0) \equiv H_0 u_0,\\
&(\delta_0 H_0) \bullet \lambda_0 \equiv (H_0 \lambda_0) \bullet (\lambda_0 H_0) \bullet (H_0 \delta_0)\,, && \lambda_0 \bullet (H_0 \varepsilon_0) \equiv \varepsilon_0 H_0,\\
&\delta_0 \bullet m_0 \equiv (H_0 m_0) \bullet (\lambda_0 H_0) \bullet (H_0 \delta_0)\,.
\end{aligned}$$

We define R to be the congruence generated by the above nine relations, and we set $\mathfrak{Bimon} := \mathbb{F}\mathbf{\Gamma}/\boldsymbol{R}$. The equivalence classes of the 2-cells m_0, u_0, etc. will be denoted by $\widehat{m}_0$, $\widehat{u}_0$, etc.

2.2 (The 2-category $[\mathfrak{K}, \mathfrak{L}]$.) It is well known that, for every 2-categories $\mathfrak{K}$ and $\mathfrak{L}$, there is a 2-category $[\mathfrak{K}, \mathfrak{L}]$ whose 0-cells are the strict 2-functors $\mathfrak{F} : \mathfrak{K} \to \mathfrak{L}$. The 1-cells of $[\mathfrak{K}, \mathfrak{L}]$ are called transformations. By definition, a transformation $\mathfrak{T} : \mathfrak{F} \longrightarrow \mathfrak{G}$ is a pair $\mathfrak{T} = (\{T_X\}_{X \in \mathfrak{K}_0}, \{\tau_F\}_{F \in \mathfrak{K}_1})$ such that, for any X in $\mathfrak{K}_0$ and any $F : X \to Y$,

$$T_X : \mathfrak{F}_0(X) \to \mathfrak{G}_0(X) \quad \text{and} \quad \tau_F : \mathfrak{G}_1(F) \circ T_X \Longrightarrow T_Y \circ \mathfrak{F}_1(F)$$

are an 1-cell and a 2-cells in $\mathfrak{L}$, respectively. The 2-cells τ_F are natural in F, i.e. for $\alpha : F \to F'$ we have the following three relations:

$$T_Y \mathfrak{F}_2(\alpha) \bullet \tau_F = \tau_{F'} \bullet \mathfrak{G}_2(\alpha)\, T_X, \tag{24}$$

$$\tau_{F'F} = \tau_{F'} \mathfrak{F}_1(F) \bullet \mathfrak{G}_1(F')\, \tau_F, \tag{25}$$

$$\tau_{Id_X} = Id_{T_X}. \tag{26}$$

Let us assume that $\mathfrak{T} = (\{T_X\}_X, \{\tau_F\}_F)$ and $\mathfrak{T}' = \left(\{T'_X\}_X, \{\tau'_F\}_F\right)$, where

$$\mathfrak{T} : \mathfrak{F} \Longrightarrow \mathfrak{F}' \quad \text{and} \quad \mathfrak{T}' : \mathfrak{F}' \Longrightarrow \mathfrak{F}''. \tag{27}$$

Then, $\mathfrak{T}' \circ \mathfrak{T}$ is the transformation whose first component is the family of 1-cells $(T'_X T_X)_{X \in \mathfrak{K}_0}$. The second component of $\mathfrak{T}' \circ \mathfrak{T}$ is the family $\{\gamma_F\}_{F \in \mathfrak{K}_1}$, where γ_F is the 2-cell

$$\gamma_F = (T'_Y \tau_F) \bullet (\tau'_F T_X). \tag{28}$$

The 2-cells in $[\mathfrak{K}, \mathfrak{L}]$ are called modifications. If $\mathfrak{T} = (\{T_X\}_X, \{\tau_F\}_F)$ and $\mathfrak{S} = (\{S_X\}_X, \{\sigma_F\}_F)$ are two transformations between the 2-functors $\mathfrak{F}$ and $\mathfrak{G}$ then a modification $\Gamma : \mathfrak{T} \Rightarrow \mathfrak{S}$ is a family of 2-cells $\{\Gamma_X\}_{X \in \mathfrak{K}_0}$, where $\Gamma_X : T_X \Rightarrow S_X$. The 2-cells Γ_X are assumed to satisfy the following identity:

$$\Gamma_Y \mathfrak{F}_1(F) \bullet \tau_F = \sigma_F \bullet \mathfrak{G}_1(F) \Gamma_X \tag{29}$$

Horizontal composition of two modifications is defined using the horizontal pointwise composition in $\mathfrak{K}$. componentwise. Therefore, if we take two modifications

$$\xymatrix{ \mathfrak{F} \ar@/^1.5pc/[r]^{\mathfrak{T}} \ar@/_1.5pc/[r]_{\mathfrak{S}} & {\Downarrow \Gamma} \; \mathfrak{F}' \ar@/^1.5pc/[r]^{\mathfrak{T}'} \ar@/_1.5pc/[r]_{\mathfrak{S}'} & {\Downarrow \Gamma'} \; \mathfrak{F}'' }, \tag{30}$$

then $\Gamma' \circ \Gamma = \{\Gamma'_X \circ \Gamma_X\}_{X \in \mathfrak{K}_0}$. Similarly, for the transformations $\mathfrak{T}$, $\mathfrak{T}'$ and $\mathfrak{T}''$, with the same source and the same target, and the modifications

$$\Gamma : \mathfrak{T} \to \mathfrak{T}' \text{ and } \Sigma : \mathfrak{T}' \to \mathfrak{T}'' \tag{31}$$

one defines the vertical composition by:

$$\Gamma \bullet \Sigma = \{\Gamma_X \bullet \Sigma_X\}_{X \in \mathfrak{K}_0}. \tag{32}$$

The identity 1-cell of a 2-functor $\mathfrak{F}$ is the pair $\left(\left\{Id_{\mathfrak{F}(X)}\right\}_{X \in \mathfrak{K}_0}, \left\{Id_{\mathfrak{F}(F)}\right\}_{F \in \mathfrak{K}_1}\right)$. The identity of a transformation $\mathfrak{T} = (\{T_X\}_X, \{\tau_F\}_F)$ is the family $\{Id_{T_X}\}_{X \in \mathfrak{K}_0}$.

2.3 (The 2-functor $\Theta(\mathfrak{K}) : [\mathfrak{Bimon}, \mathfrak{K}] \to \mathfrak{Bimon}(\mathfrak{K})$.) On 0-cells (i.e. 2-functors) the 2-functor $\Theta(\mathfrak{K})$ is defined by

$$\Theta(\mathfrak{K})_0(\mathfrak{F}) = \left(\mathfrak{F}_0(X_0), \mathfrak{F}_1(H_0), \mathfrak{F}_2(\widehat{m}_0), \mathfrak{F}_2(\widehat{u}_0), \mathfrak{F}_2(\widehat{\delta}_0), \mathfrak{F}_2(\widehat{\varepsilon}_0), \mathfrak{F}_2(\widehat{\lambda}_0)\right).$$

Since $(X_0, H_0, \widehat{m}_0, \widehat{u}_0, \widehat{\delta}_0, \widehat{\varepsilon}_0, \widehat{\lambda}_0)$ is a bimonad in $\mathfrak{Bimon}$ and $\mathfrak{F}$ is a strict 2-functor it follows that $\Theta(\mathfrak{K})_0(\mathfrak{F})$ is a bimonad on $\mathfrak{K}$.Let $\mathfrak{T}$ be a transformation between $\mathfrak{F}$ and $\mathfrak{G}$. Since $\mathfrak{Bimon}$ has a unique 0-cell X_0, the first component of $\mathfrak{T}$ is a family with one element, namely the 1-cell T_{X_0}. The second component of $\mathfrak{T}$ is a family indexed by the 1-cells of $\mathfrak{Bimon}$, which are $T_0^n = T_0 \ldots T_0$ (n-factors). Hence $\mathfrak{T} = (T_{X_0}, \{\tau_{T_0^n}\}_{n \in \mathbb{N}})$. We define

$$\Theta(\mathfrak{K})_1(\mathfrak{T}) = (T_{X_0}, \tau_{T_0}).$$

Let us show that $\Theta(\mathfrak{K})_1(\mathfrak{T})$ is a morphism of bimonads between $\Theta(\mathfrak{K})_0(\mathfrak{F})$ and $\Theta(\mathfrak{K})_0(\mathfrak{G})$. Since the family $\left\{\tau_{T_0^n}\right\}_n$ is natural, in view of (24), we get:

$$T_{X_0}\mathfrak{F}_2(\widehat{m}_0)\bullet\tau_{T_0^2}=\tau_{T_0}\bullet\mathfrak{G}_2(\widehat{m}_0)T_{X_0}.$$

On the other hand, by (25),

$$\tau_{T_0^2}=\left(\tau_{T_0}\mathfrak{F}_1\left(T_0\right)\right)\bullet\left(\mathfrak{G}_1\left(T_0\right)\tau_{T_0}\right).$$

Since the multiplications of $\left(\mathfrak{F}_0\left(X_0\right),\mathfrak{F}_1\left(H_0\right)\right)$ and $\left(\mathfrak{G}_0\left(X_0\right),\mathfrak{G}_1\left(H_0\right)\right)$ are $\mathfrak{F}_2(\widehat{m}_0)$ and $\mathfrak{G}_2(\widehat{m}_0)$, it follows that $\Theta(\mathfrak{K})_1(\mathfrak{T})$ satisfies the condition (3) for $\alpha=\widehat{u}_0$, one proves that:

$$\tau_{T_0}\bullet\mathfrak{G}_2\left(\widehat{u}_0\right)T_{X_0}=T_{X_0}\mathfrak{F}_2\left(\widehat{u}_0\right)\bullet\tau_{Id_{X_0}}$$

Since $\tau_{Id_{X_0}}=Id_{T_{X_0}}$, cf. (26), and $\mathfrak{F}_2\left(\widehat{u}_0\right)$ and $\mathfrak{S}_2(\widehat{\delta}_0)$ are the units of $\Theta(\mathfrak{K})_0(\mathfrak{F})$ and $\Theta(\mathfrak{K})_0(\mathfrak{G})$ we deduce that $\Theta(\mathfrak{K})_1(\mathfrak{T})$ is compatible with the units of these bimonads. In conclusion, $\Theta(\mathfrak{K})_1(\mathfrak{T})$ is a morphism of monads. One proves $\Theta(\mathfrak{K})_1(\mathfrak{T})$ is a morphism of comonads that satisfies the relation (17). Thus $\Theta(\mathfrak{K})_1(\mathfrak{T})$ is a morphism of bimonads. We take now $\Gamma:\mathfrak{T}\to\mathfrak{S}$ to be a modification, where $\mathfrak{T}$ and $\mathfrak{S}$ have the same target $\mathfrak{F}$ and source $\mathfrak{G}$. In $\mathfrak{Bimon}$ there is only one 0-cell X_0. Hence Γ is a family with one element $\Gamma_{X_0}:T_{X_0}\to S_{X_0}$, where T_{X_0} and S_{X_0} are the 2-cells in $\mathfrak{K}$ that define $\mathfrak{T}$ and $\mathfrak{S}$, respectively. We now set:

$$\Theta(\mathfrak{K})_2(\Gamma)=\Gamma_{X_0}.$$

Clearly the condition (29) written for the modification Γ and the condition (18) written for $\alpha=\Gamma_{X_0}$ are equivalent. Therefore, $\Theta(\mathfrak{K})_2(\Gamma)$ is a 2-cell in $\mathfrak{Bimon}$. We claim that $\Theta(\mathfrak{K})_1$ is compatible with the composition of transformations. Indeed, let $\mathfrak{T}$ and $\mathfrak{T}'$ be two transformations such that $\mathfrak{T}'\circ\mathfrak{T}$ exists. If $\mathfrak{T}=\left(T_{X_0},\left\{\tau_{T_0^n}\right\}_{n\in\mathbb{N}}\right)$ and $\mathfrak{T}'=\left(T'_{X_0},\left\{\tau'_{T_0^n}\right\}_{n\in\mathbb{N}}\right)$, then $\mathfrak{T}'\circ\mathfrak{T}$ is the pair whose first component is $T'_{X_0}T_{X_0}$ and second component is the family $\left\{T'_{X_0}\tau_{T_0^n}\bullet\tau'_{T_0^n}T\right\}_n$. We deduce that

$$\begin{aligned}\Theta(\mathfrak{K})_1(\mathfrak{T}'\circ\mathfrak{T})&=\left(T'_{X_0}T_{X_0},(T'_{X_0}\tau_{T_0})\bullet(\tau'_{T_0}T_{X_0})\right)=(T'_{X_0},\tau'_{T_0})\circ(T_{X_0},\tau_{T_0})\\&=\Theta(\mathfrak{K})_1(\mathfrak{T}')\circ\Theta(\mathfrak{K})_1(\mathfrak{T}).\end{aligned}$$

Let us prove that $\Theta(\mathfrak{K})_2$ is compatible with the horizontal and vertical composition. If Γ and Σ are two modifications defined by Γ_{X_0} and Σ_{X_0}, respectively, then:

$$\Theta(\mathfrak{K})_2(\Gamma\circ\Sigma)=\Gamma_{X_0}\circ\Sigma_{X_0}=\Theta(\mathfrak{K})_2(\Gamma)\circ\Theta(\mathfrak{K})_2(\Sigma).$$

The compatibility with the vertical composition is proven similarly. Clearly, $\Theta(\mathfrak{K})$ maps identity cell to an identity cell, so we have just proven that $\Theta(\mathfrak{K})$ is a strict 2-functor.

2.4 (The 2-functor $\Lambda(\mathfrak{K}):\mathfrak{Bimon}(\mathfrak{K})\to[\mathfrak{Bimon},\mathfrak{K}]$.) Our goal now is to construct an inverse of $\Theta(\mathfrak{K})$. On objects $\Lambda(\mathfrak{K})$ is defined as follows. If $(X,m,u,\delta,\varepsilon,\lambda)$ is a bimonad in $\mathfrak{K}$, then

$$X_0\mapsto X,\quad H_0\mapsto H,\quad m_0\mapsto m,\quad u_0\mapsto u,\quad \delta_0\mapsto\delta,\quad \varepsilon_0\mapsto\varepsilon\quad\text{and}\quad\lambda_0\mapsto\lambda$$

define a morphism of computads from $\mathbf{\Gamma}_0$ to $\mathbb{U}\mathfrak{K}$. By the universal property of the free 2-category, there is a unique 2-functor $\mathfrak{F}' : \mathbb{F}\mathbf{\Gamma}_0 \to \mathfrak{K}$ that lifts the above morphism of computads. Since (X, H) is a bimonad, $\mathfrak{F}'$ factors through a 2-functor $\mathfrak{F}$ from $\mathfrak{Bimon} := \mathbb{F}\mathbf{\Gamma}_0/R$ to $\mathfrak{K}$, cf. the definition of $\mathfrak{Bimon}$. Hence $\mathfrak{F}$ is uniquely defined by the relations:

$$\mathfrak{F}_0(X_0) = X, \;\; \mathfrak{F}_1(H_0) = H, \;\; \mathfrak{F}_2(m_0) = m, \;\; \mathfrak{F}_2(\widehat{u}_0) = u, \;\; \mathfrak{F}_2(\widehat{\delta}_0) = \delta, \;\; \mathfrak{F}_2(\widehat{\varepsilon}_0) = \varepsilon, \;\; \mathfrak{F}_2(\widehat{\lambda}_0) = \lambda.$$

We set $\Lambda(\mathfrak{K})_0\,(X, H) = \mathfrak{F}$. For a morphism of bimonads $(F, \sigma) : (X, H) \to (X', H')$ we define:

$$\Lambda(\mathfrak{K})_1\,(F, \sigma) : \Lambda(\mathfrak{K})_0\,(X, H) \to \Lambda(\mathfrak{K})_0\,(X', H')$$

as follows. Let $\mathfrak{F} := \Lambda(\mathfrak{K})_0\,(X, H)$ and $\mathfrak{F}' := \Lambda(\mathfrak{K})_0\,(X', H')$. We need a transformation from $\mathfrak{F}$ to $\mathfrak{F}'$. Since X_0 is the unique 0-cell of $\mathfrak{Bimon}$ and the 1-cell of this 2-category are H_0^n, a transformation from $\mathfrak{F}$ to $\mathfrak{F}'$ is a pair $\left(T, \{\tau_{T_0^n}\}_{n\in\mathbb{N}}\right)$. We take $T := F$, and we set $\tau_{Id_{X_0}} = Id_T$ and $\tau_{T_0} = \sigma$. Then, for $n \geq 2$, we define inductively $\tau_{T_0^n}$ by using the relation (25). It is routine to check that $\left(T, \{\tau_{T_0^n}\}_{n\in\mathbb{N}}\right)$ is a transformation, indeed. Thus, we define:

$$\Lambda(\mathfrak{K})_1\,(F, \sigma) := \left(F, \{\tau_{T_0^n}\}_{n\in\mathbb{N}}\right)$$

It is remains to construct $\Lambda(\mathfrak{K})_2$. Let $\alpha : (F, \sigma) \Rightarrow (F', \sigma')$ be a 2-cell in $\mathfrak{Bimon}\,(\mathfrak{K})$. We are looking for a transformation from $\Lambda(\mathfrak{K})_1\,(F, \sigma)$ to $\Lambda(\mathfrak{K})_1\,(F', \sigma')$, which has to be a 2-cell in $\mathfrak{K}$ with source F and target F'. Obviously, we take

$$\Lambda(\mathfrak{K})_2\,(\alpha) = \alpha.$$

By definitions it is clear that $\Lambda(\mathfrak{K})_2\,(\alpha)$ is a 2-cell in $[\mathfrak{Bimon}, \mathfrak{K}]$. If (F, σ) and (F', σ') are two morphism of bimonads that can be composed, then

$$\Lambda(\mathfrak{K})_1\,((F', \sigma') \circ (F, \sigma)) = \Lambda(\mathfrak{K})_1\,(F'F, F'\sigma \bullet \sigma'F) = \left(F'F, \{\tau_{T_0^n}\}_n\right)$$

where $\tau_{T_0} = F'\sigma \bullet \sigma'F$. On the other hand

$$\Lambda(\mathfrak{K})_1\,(F, \sigma) = \left(F, \left\{\gamma_{T_0^n}\right\}_n\right) \qquad \text{and} \qquad \Lambda(\mathfrak{K})_1\,(F', \sigma') = \left(F', \left\{\gamma'_{T_0^n}\right\}_n\right),$$

where $\gamma'_{T_0} = \sigma'$ and $\gamma_{T_0} = \sigma$. Note that all $\tau_{T_0^n}, \gamma_{T_0^n}$ and $\gamma'_{T_0^n}$ are uniquely determined by τ_{T_0}, γ_{T_0} and γ'_{T_0}, cf. (25). As

$$\left(F', \left\{\gamma'_{T_0^n}\right\}_n\right) \circ \left(F, \left\{\gamma_{T_0^n}\right\}_n\right) = \left(F'F, \left\{(F'\gamma_{T_0^n}) \bullet (\gamma'_{T_0^n}F)\right\}_n\right),$$

in view of the foregoing remarks, if follows that $\Lambda(\mathfrak{K})_1$ is compatible with the compositions of morphism of bimonads. Since $\Lambda(\mathfrak{K})_2\,(\alpha) = \alpha$, it is clear that $\Lambda(\mathfrak{K})_2$ is compatible with the horizontal and vertical compositions. The compatibility with the identity cells is also obvious, so $\Lambda(\mathfrak{K})$ is a strict 2-functor.

Theorem 2.5 *The* 2*-functors* $\Lambda(\mathfrak{K})$ *and* $\Theta(\mathfrak{K})$ *are inverses each other, and they are natural in* $\mathfrak{K}$. *In particular, the* 2*-category* $\mathfrak{Bimon}$ *represents the functor*

$$\mathfrak{Bimon}(-) : 2\text{-CAT} \to 2\text{-CAT}.$$

Proof: Let us show that $\Theta(\mathfrak{K})\circ\Lambda(\mathfrak{K})$ is the identity functor of $\mathfrak{Bimon}(\mathfrak{K})$. If (X,H) is a bimonad on $\mathfrak{K}$, we have:

$$\left[\Theta\left(\mathfrak{K}\right)_0 \circ \Lambda(\mathfrak{K})_0\right](X,H) = \Theta\left(\mathfrak{K}\right)\left(\mathfrak{F}\right) = \left(\mathfrak{F}_0\left(X_0\right), \mathfrak{F}_1\left(H_0\right)\right),$$

where $\mathfrak{F} = \Lambda(\mathfrak{K})_0\left(X,H\right)$. But $\mathfrak{F}$ is the unique 2-functor with source $\mathfrak{Bimon}$ and target $\mathfrak{K}$ such that $\mathfrak{F}_0\left(X_0\right) = X$ and $\mathfrak{F}_1\left(H_0\right) = H$. If m is the multiplication of (X,H) then $\mathfrak{F}_2(\widehat{m}_0) = m$, by the definition of $\mathfrak{F} = \Lambda(\mathfrak{K})_0\left(X,H\right)$. Similar relation hold for the unit, comultiplication, counit and entwining structure. This shows that

$$\Theta\left(\mathfrak{K}\right)_0 \circ \Lambda(\mathfrak{K})_0 = Id_{\mathfrak{Bimon}(\mathfrak{K})_0}$$

We now take a morphism (F,σ) of bimonads.

$$\left[\Theta\left(\mathfrak{K}\right)_1 \circ \Lambda(\mathfrak{K})_1\right](F,\sigma) = \Theta\left(\mathfrak{K}\right)_1\left(F,\left\{\tau_{T_0^n}\right\}_n\right) = \left(F,\tau_{T_0}\right),$$

where $\left\{\tau_{T_0^n}\right\}_n$ is uniquely defined such that $\tau_{T_0} = \sigma$. Hence $\Theta\left(\mathfrak{K}\right)_1$ is a left inverse of $\Lambda(\mathfrak{K})_1$. The identity

$$\Theta\left(\mathfrak{K}\right)_2 \circ \Lambda(\mathfrak{K})_2 = Id_{\mathfrak{Bimon}(\mathfrak{K})_2}$$

is trivial, as both $\Theta\left(\mathfrak{K}\right)_2$ and $\Lambda(\mathfrak{K})_2$ map a 2-cell to itself. On the other hand, if $\mathfrak{F} : \mathfrak{Bimon} \to \mathfrak{K}$ is a 2-functor then:

$$\left[\Lambda(\mathfrak{K})_0 \circ \Theta\left(\mathfrak{K}\right)_0\right]\left(\mathfrak{F}\right) = \Lambda(\mathfrak{K})_0\left(X,H,m,u,\delta,\varepsilon,\lambda\right),$$

where the bimonad $X = \mathfrak{F}_0\left(X_0\right)$, $H = \mathfrak{F}_1\left(H_0\right)$, etc. Hence $\Lambda(\mathfrak{K})_0\left(X,H\right)$ is the unique 2-functor that maps $X_0 \to X$, $H_0 \to H$, $\widehat{m}_0 \to m$, etc. Since $\mathfrak{F}$ has this properties we deduce $\Theta\left(\mathfrak{K}\right)_0$ is a right inverse of $\Lambda(\mathfrak{K})_0$, so $\Theta\left(\mathfrak{K}\right)_0$ and $\Lambda(\mathfrak{K})_0$ are inverses each other. Let $\mathfrak{T} : \mathfrak{F} \to \mathfrak{G}$ be a transformation, where $\mathfrak{F},\mathfrak{G} : \mathfrak{Bimon} \to \mathfrak{K}$. If $\mathfrak{T} = \left(T,\left\{\tau_{T_0^n}\right\}_n\right)$, then

$$\left[\Lambda(\mathfrak{K})_1 \circ \Theta\left(\mathfrak{K}\right)_1\right]\left(\mathfrak{T}\right) = \Lambda(\mathfrak{K})_1\left(T,\tau_{T_0}\right) = \left(T,\left\{\tau_{T_0^n}\right\}_n\right).$$

In the above seqvence of equations, $\left\{\gamma_{T_0^n}\right\}_n$ are constructed inductively, using the relation (25). Since, by definition $\gamma_{T_0} = \tau_{T_0}$, and $\left\{\tau_{T_0^n}\right\}_n$ also satisfy (25) we deduce that $\gamma_{T_0^n} = \tau_{T_0^n}$ for any $n \in \mathbb{N}$. Thus $\Theta\left(\mathfrak{K}\right)_1$ is a right inverse of $\Lambda(\mathfrak{K})_1$. Finally, $\Theta\left(\mathfrak{K}\right)_2$ is a right inverse of $\Lambda(\mathfrak{K})_2$, as they map a 2-cell to itself. We have just concluded that $\Theta\left(\mathfrak{K}\right)$ and $\Lambda(\mathfrak{K})$ are inverses each other.

It remains to prove that the diagram

$$\begin{array}{ccc}
[\mathfrak{Bimon},\mathfrak{K}] & \xrightarrow{\Theta(\mathfrak{K})} & \mathfrak{Bimon}(\mathfrak{K}) \\
{\scriptstyle [\mathfrak{Bimon},\mathfrak{F}]}\downarrow & & \downarrow{\scriptstyle \mathfrak{Bimon}(\mathfrak{F})} \\
[\mathfrak{Bimon},\mathfrak{L}] & \xrightarrow[\Theta(\mathfrak{L})]{} & \mathfrak{Bimon}(\mathfrak{L})
\end{array}$$

is commutative, for every 2-functor $\mathfrak{F} : \mathfrak{K} \to \mathfrak{L}$. By definition $[\mathfrak{Bimon}, \mathfrak{F}]_0 (\mathfrak{F}') = \mathfrak{F}' \circ \mathfrak{F}$, for every 2-functor $\mathfrak{F}' : \mathfrak{Bimon} \to \mathfrak{F}$. Hence,

$$(\Theta(\mathfrak{L})_0 \circ [\mathfrak{Bimon}, \mathfrak{F}]_0)(\mathfrak{F}') = \Theta(\mathfrak{L})_0 (\mathfrak{F}' \circ \mathfrak{F}) = (X, H, m, u, \delta, \varepsilon, \lambda),$$

where $X = (\mathfrak{F}' \circ \mathfrak{F})_0 (X_0)$, $H = (\mathfrak{F}' \circ \mathfrak{F})_1 (H_0)$, $m = (\mathfrak{F}' \circ \mathfrak{F})(\widehat{m}_0)$, etc. On the other hand,

$$\begin{aligned}[\mathfrak{Bimon}(\mathfrak{F})_0 \circ \Theta(\mathfrak{K})_0](\mathfrak{F}') &= \mathfrak{Bimon}(\mathfrak{F})_0 (\Theta(\mathfrak{K})_0 (\mathfrak{F}')) \\ &= \mathfrak{Bimon}(\mathfrak{F})(\mathfrak{F}'_0 (X_0), \mathfrak{F}'_1 (H_0), \mathfrak{F}'_2(\widehat{m}_0), \mathfrak{F}'_2 (\widehat{u}_0), \mathfrak{F}'_2(\widehat{\delta}_0), \mathfrak{F}'_2 (\widehat{\varepsilon}_0), \mathfrak{F}'_2(\widehat{\lambda}_0)) \\ &= (X, H, m, u, \delta, \varepsilon, \lambda).\end{aligned}$$

Note that for the last identity we used the definition of the functor $\mathfrak{Bimon}(\mathfrak{F})$ and the relations that define the bimonad $(X, H, m, u, \delta, \varepsilon, \lambda)$.
If $\mathfrak{T} : \mathfrak{F} \to \mathfrak{G}$ is a transformation, with $\mathfrak{T} = \left(T, \{\tau_{T_0^n}\}_n\right)$ then

$$\begin{aligned}(\mathfrak{Bimon}(\mathfrak{F})_1 \circ \Theta(\mathfrak{K})_1)(\mathfrak{T}) &= \mathfrak{Bimon}(\mathfrak{F})_1 (T, \tau_{T_0}) = (\mathfrak{F}_1 (T), \mathfrak{F}_2 (\tau_{T_0})) \\ &= \Theta(\mathfrak{L})_1 \left(\mathfrak{F}_1 (T), \{\mathfrak{F}_2(\tau_{T_0^n})\}_n\right) = (\Theta(\mathfrak{L})_1 \circ [\mathfrak{Bimon}, \mathfrak{F}]_1)(\mathfrak{T}).\end{aligned}$$

Finally, if $\Gamma : \mathfrak{T} \to \mathfrak{S}$ is a modification, then

$$\begin{aligned}(\mathfrak{Bimon}(\mathfrak{F})_2 \circ \Theta(\mathfrak{K})_2)(\Gamma) &= \mathfrak{Bimon}(\mathfrak{F})_2 (\Gamma_{X_0}) = \mathfrak{F}_2 (\Gamma_{X_0}) \\ &= \Theta(\mathfrak{K})_2 (\mathfrak{F}_2 (\Gamma_{X_0})) = (\Theta(\mathfrak{K})_2 \circ [\mathfrak{Bimon}, \mathfrak{F}])_2 (\Gamma).\end{aligned}$$

Thus the theorem is proven. □

Corollary 2.6 *If $(X, H, m, u, \delta, \varepsilon, \lambda)$ is a bimonad in $\mathfrak{K}$, the there is a unique 2-functor $\mathfrak{F} : \mathfrak{Bimon} \to \mathfrak{K}$ such that $\mathfrak{F}_0(X_0) = X$, $\mathfrak{F}_1(H_0) = H$, $\mathfrak{F}_2(\widehat{m}_0) = m$, etc.*

Proof: See the proof of the preceding theorem. □

2.7 (Hopf monads in $\mathfrak{K}$.) A bimonad $(X, H, m, u, \delta, \varepsilon, \lambda)$ in a 2-category $\mathfrak{K}$ is a Hopf monad if there is a 2-cell $\pi : H^2 \to H^2$ such that

$$m \bullet (\pi H) \bullet \delta = u \bullet \varepsilon = m \bullet (H\pi) \bullet \delta.$$

The 2-cell π is called the antipode of the Hopf monad (X, H).

Hopf bimonads in $\mathfrak{K}$ may be regarded as 0-cells in a 2-category $\mathfrak{Hopfmon}(\mathfrak{K})$. The 1-cells in this 2-category are the morphisms of Hopf monads. Let (X, H) and (X', H') be Hopf monads with the antipodes π and π', respectively. A morphism of Hopf monads from (X, H) to (X', H') is a morphism (F, σ) between the underlying is bimonads that commutes with the antipodes, in the sense that

$$\sigma \bullet \pi' F = F\pi \bullet \sigma.$$

The composition of two morphisms in $\mathfrak{Hopfmon}(\mathfrak{K})$ is defined as in the 2-category of bimonads in $\mathfrak{K}$. A 2-cell in $\mathfrak{Hopfmon}(\mathfrak{K})$ between two morphisms of Hopf monads is a 2-cell between the underlying bimonad morphisms. The vertical and horizontal composition in $\mathfrak{Hopfmon}(\mathfrak{K})$ are defined as in $\mathfrak{Bimon}$.

Proceeding as in the case of bimonads, one defines a new 2-category $\mathfrak{Hopfmon}$ as the quotient 2-category $\mathbb{F}\mathbf{\Gamma}_1/R'$, where $\mathbf{\Gamma}_1$ is the computad obtained from $\mathbf{\Gamma}$ by adding the 2-cell $\pi_0 : H_0^2 \to H_0^2$, and R' is the congruence generated by $\equiv$ and the extra relations:

$$m_0 \bullet (\sigma_0 H_0) \bullet \delta_0 \equiv u_0 \bullet \varepsilon_0 \equiv m_0 \bullet (H_0 \sigma_0) \bullet \delta_0.$$

Theorem 2.8 *The* 2*-category* $\mathfrak{Hopfmon}$ *represents the functor*

$$\mathfrak{Hopfmon}(-) : \text{2-CAT} \to \text{2-CAT}.$$

Proof: One argues as in the proof of Theorem 2.5. Details are omitted. □

Corollary 2.9 *If* $(X, H, m, u, \delta, \varepsilon, \lambda, \pi)$ *is a Hopf monad in* $\mathfrak{K}$*, the there is a unique* 2*-functor* $\mathfrak{F} : \mathfrak{Hopfmon} \to \mathfrak{K}$ *such that* $\mathfrak{F}_0(X_0) = X$, $\mathfrak{F}_1(H_0) = H$, $\mathfrak{F}_2(\widehat{m}_0) = m$, *etc.*

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