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Perfect super edge-magic graphs

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Abstract

In this paper we introduce the concept of perfect super edge-magic graphs and we prove some classes of graphs to be perfect super edge-magic.

Key Words: super edge-magic, perfect super edge-magic, valence, $\bigotimes_h$-product.
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1 Introduction

For the graph theory terminology and notation not defined in this paper we refer the reader to the following sources [1, 2, 7, 11]. In 1970, Kotzig and Rosa [9] defined the concept of edge-magic graphs and edge-magic labelings as follows: let $G = (V, E)$ be a (p, q)-graph ($|V| = p$ and $|E| = q$). Then G is called *edge-magic* if there is a bijective function $f : V \cup E \longrightarrow \{i\}_{i=1}^{p+q}$ and an integer k such that $f(x) + f(xy) + f(y) = k$ for all $xy \in E$. If such a function exists, then f is called an edge-magic labeling of G and k is called the *valence* of f. Another common name for the valence of a super edge magic labeling is the *magic sum* (see [11]).

Motivated by the concept of edge-magic labelings, Enomoto *et al.* [3] introduced in 1998 the concepts of *super edge-magic* graph and labeling as follows: let $G = (V, E)$ be any (p, q)-graph and let $f : V \cup E \to \{i\}_{i=1}^{p+q}$ be an edge-magic labeling of G with the extra property that $f(V) = \{i\}_{i=1}^{p}$. Then G is called super edge-magic and f a super edge-magic labeling of G. In [4], Figueroa-Centento *et al.* provided the following useful characterization of super edge-magic labelings and graphs.

Lemma 1.1. *Let $G = (V, E)$ be a graph of order p and size q. Then G is super edge-magic if and only if there is a bijective function $\bar{f} : V \longrightarrow \{i\}_{i=1}^{p}$ such that the set $S(\bar{f}) = \{\bar{f}(u) + \bar{f}(v) : uv \in E\}$ is a set of q consecutive integers.*

In this case, $\bar{f}$ can be extended to a super edge-magic labeling f.

We remark that the valence of the labeling f is determined by the formula $p+q+\min S(\bar{f})$. Also from now on, we will call $\bar{f}$ the canonical form of the super edge-magic labeling f.

When we say that a digraph has a super edge-magic labeling we mean that its underlying graph has such a labeling, see [6]. We will use the notation $und(D)$ in order to denote the underlying graph of a digraph D.

Figueroa *et al.* defined, in [6], the following product: let $D = (V, E)$ be a digraph with adjacency matrix $A(D) = (a_{ij})$ and let $\Gamma = \{F_i\}_{i=1}^m$ be a family of m digraphs with the same set V' of p' vertices. Assume that $h : E \longrightarrow \Gamma$ is any function that assigns elements of Γ to the arcs of D. Then the digraph $D \otimes_h \Gamma$ is defined by (i) $V(D \otimes_h \Gamma) = V \times V'$ and (ii) $((a_1, b_1), (a_2, b_2)) \in E(D \otimes_h \Gamma) \Leftrightarrow [(a_1, a_2) \in E(D) \wedge (b_1, b_2) \in E(h(a_1, a_2))]$.

An alternative way of defining the same product is through adjacency matrices, since we can obtain the adjacency matrix of $D \otimes_h \Gamma$ as follows:

(i) If $a_{ij} = 0$ then a_{ij} is replaced by the $p' \times p'$ 0-square matrix.

(ii) If $a_{ij} = 1$ then a_{ij} is replaced by $A(h(i, j))$ where $A(h(i, j))$ is the adjacency matrix of the digraph $h(i, j)$.

Note that when h is constant, $D \otimes_h \Gamma$ is the Kronecker product. From now on, let S_n denote the set of all super edge-magic 1-regular labeled digraphs of order n where each vertex takes the name of the label that has been assigned to it. The following result was introduced in [6]:

Theorem 1.1. *Let D be a (super) edge-magic digraph and let $h : E(D) \longrightarrow S_n$ be any function. Then $und(D \otimes_h S_n)$ is (super) edge-magic.*

In [8] (see also research problem 25 in [11]) it was introduced the following interesting question, that as far as we know is still open, regarding edge-magic labelings of cycles.

Question 1.1. *Characterize the set of valences of edge-magic labelings for the cycle C_n, $n \in \mathbb{N} \setminus \{1, 2\}$.*

Following the same line, Figueroa-Centento *et al.* proved in [5] the following result.

Theorem 1.2. *The star $K_{1,n}$ is edge-magic. Furthermore, there are only three possible valences for edge-magic labelings of $K_{1,n}$. These valences are $2n+4, 3n+3$ and $4n + 2$. Moreover, the first two valences correspond to super edge-magic labelings of $K_{1,n}$.*

Motivated by the previous result and Question 1.1 we introduce the concept of perfect super edge-magic graphs. Let $G = (V, E)$ be a (p, q)-graph. We define the set

$$S_G = \left\{ \frac{\sum_{u\in V} \deg(u)g(u) + \sum_{i=p+1}^{p+q} i}{q} : \text{ the function } g : V \to \{i\}_{i=1}^{p} \text{ is bijective} \right\}.$$

If $\lceil \min S_G \rceil \leq \lfloor \max S_G \rfloor$ then the *super edge-magic interval* of G, I_G, is the set

$$I_G = [\lceil \min S_G \rceil, \lfloor \max S_G \rfloor] \cap \mathbb{N}.$$

The *super edge-magic set*, σ_G, of G is

$$\sigma_G = \{k \in I_G : \text{ there exists a super edge-magic labeling of } G \text{ with valence } k\}.$$

A graph G is said to be *perfect super edge-magic if* $I_G = \sigma_G$. From Theorem 1.2 we get immediately that the star $K_{1,n}$ is not perfect super edge-magic for $n \geq 3$. In this paper we prove that different graphs are perfect super edge-magic.

We conclude this introduction mentioning that in the last decade labelings of the additive type have experimented a fast development. The books by Bača and Miller [1] and by Wallis [11] as well as the very complete dynamic survey by Gallian [7] are a good proof of this fact. We suggest these three sources for any reader who is interested in obtaining a deeper understanding and information on labelings.

2 Perfect super edge-magic graphs

We begin with the following easy observation.

Proposition 2.1. *An r-regular graph G is perfect super edge-magic if and only if G is super edge-magic.*

Proof: The result is true since if G is an r-regular graph, then $|S_G| = 1$. Thus, we have $|I_G| = 1$ whenever S_G contains exactly one element which is an integer. □

Let P_n be the path on n vertices. Next we show that P_n is perfect super edge-magic.

Theorem 2.1. *The path P_n is perfect super edge-magic for every $n \in \mathbb{N}$.*

Proof: First of all consider the path P_n when n is odd. Then the maximum possible valence occurs when labels 1 and 2 are assigned to the leaves of P_n. Thus, the maximum possible valence is

$$\left\lfloor \frac{\sum_{i=1}^{2n-1} i + \sum_{i=1}^{n} i - 3}{n-1} \right\rfloor = \left\lfloor \frac{5n^2 - n - 6}{2(n-1)} \right\rfloor.$$

On the other hand, the minimum possible valence occurs when the labels $n-1$ and n are assigned to the leaves of P_n. Thus, the minimum possible valence is

$$\left\lceil \frac{\sum_{i=1}^{2n-1} i + \sum_{i=1}^{n} i - (2n-1)}{n-1} \right\rceil = \left\lceil \frac{5n^2-5n+2}{2(n-1)} \right\rceil .$$

Hence, we get that the difference between the maximum possible valence and the minimum possible valence is upper bounded by:

$$\frac{5n^2-n-6}{2(n-1)} - \frac{5n^2-5n+2}{2(n-1)} = \frac{2n-4}{n-1} < 2.$$

Therefore, if we show two super edge-magic labelings with distint valences we complete the case. Define the path P_n as follows: $V(P_n) = \{v_i\}_{i=1}^{n}$ and $E(P_n) = \{v_i v_{i+1}\}_{i=1}^{n-1}$.

Labeling 1: The function $f : V(P_n) \to \{i\}_{i=1}^{n}$ defined by the rule

$$f(v_i) = \begin{cases} \frac{i+1}{2}, & \text{if } i \text{ is odd,} \\ \frac{i+1+n}{2}, & \text{if } i \text{ is even,} \end{cases}$$

is the canonical form of a super edge-magic labeling of P_n with valence $(5n+3)/2$.

Labeling 2: The function $g : V(P_n) \to \{i\}_{i=1}^{n}$ defined by the rule

$$g(v_i) = \begin{cases} \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i+n}{2}, & \text{if } i \text{ is odd,} \end{cases}$$

is the canonical form of a super edge-magic labeling of P_n with valence $(5n+1)/2$.

Next, we consider the path P_n when n is even. In this case, again we get that the maximum possible valence is $\lfloor (5n^2-n-6)/(2(n-1)) \rfloor$ and the minimum possible valence is $\lceil (5n^2-5n+2)/(2(n-1)) \rceil$, or equivalently, $\lfloor 5n/2+2-1/(n-1) \rfloor$ and $\lceil 5n/2+1/(n-1) \rceil$. If $n = 2k$ we get, respectively, $\lfloor 5k+2-1/(2k-1) \rfloor$ and $\lceil 5k+1/(2k-1) \rceil$. Therefore, any valence, val, is in the interval $[5k+1/(2k-1), 5k+2-1/(2k-1)]$. Notice that, $[5k+1/(2k-1), 5k+2-1/(2k-1)] \cap \mathbb{N} = \{5k+1\}$. Therefore, since P_n is super edge-magic, val$= 5k+1$ is attained. That is, P_n is perfect super edge-magic.

□

Although P_n is perfect super edge-magic for all $n \in \mathbb{N}$, we have that $|I_{P_n}| = 2$ if n is odd and $|I_{P_n}| = 1$ if n is even. It would be interesting to find an infinite family of perfect super edge-magic simple graphs $\mathfrak{F} = \{F_1, F_2, \ldots\}$ such that $\lim_{i\to+\infty} |I_{F_i}| = +\infty$. The next result allows us to construct many such families.

Recall that the crown product of two graphs G_1 and G_2 is the graph $G = G_1 \bigodot G_2$ obtained by placing a copy of G_1 and $|V(G_1)|$ copies of G_2 and then joining each vertex of G_1 with all vertices in one copy of G_2 in such a way that all vertices in the same copy of G_2 are joined with exactly one vertex of G_1. See Figure 1 for an example.

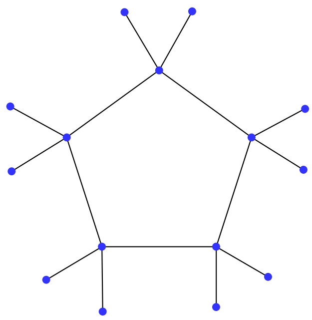

Figure 1: The graph $G = C_5 \bigodot \overline{K_2}$, where $\overline{K_2}$ is the complement of K_2.

Theorem 2.2. *Let C_m be a cycle of order $m = p^k$, where $p > 2$ is a prime number. Then the graph $G \cong C_m \bigodot \overline{K_n}$ is perfect super edge-magic.*

In order to prove the previous theorem we need the next easy lemma.

Lemma 2.1. *Let $p \neq 2$ be a prime. For each $x, n \in \mathbb{N}$ with $1 < x \leq p^n$ then, either $gcd(p^n, x) = 1$, or $gcd(p^n, x-1) = 1$.*

Proof: Assume to the contrary that there exist $x, n \in \mathbb{N}$ with $1 < x \leq p^n$ such that $gcd(p^n, x) = p^{k_1}$ and $gcd(p^n, x-1) = p^{k_2}$, for some $k_1, k_2 > 0$. Thus, there exist $a, b \in \mathbb{N}$ such that $x = ap^{k_1}$ and $x - 1 = bp^{k_2}$. Hence, we obtain that $1 = ap^{k_1} - bp^{k_2}$. Therefore, p divides 1 which is a contradiction. □

Proof of Theorem 2.2.

Let us first determine the super edge-magic interval of $G = (V, E)$. Let $g : V \to \{i\}_{i=1}^{m+mn}$ be a bijective function. Then, the corresponding element in S_G is given by:

$$\frac{\sum_{u \in V} \deg(u) g(u) + \sum_{i=m+mn+1}^{2(m+mn)} i}{m+mn} =$$

$$= \frac{\sum_{u \in C} (2+n) g(u) + \sum_{u \in L} g(u) + \sum_{i=m+mn+1}^{2(m+mn)} i}{m+mn},$$

where L denotes the set of leaves of G and $C = V \setminus L$. Thus, the maximum of S_G occurs when $\{g(u) : u \in L\} = \{1, 2, \ldots, mn\}$ and the minimum when $\{g(u) : u \in L\} = \{m+1, 2, \ldots, m+mn\}$.

In what follows, we will prove that $\sigma_G = [\min\ S_G, \max\ S_G]$. Assume that $C = \{v_0, v_2, \ldots, v_{m-1}\}$, $L = \{v_i^j : i = 0, \ldots, m-1; j = 1, \ldots, n\}$ and $E(G) = \{v_i v_{i+_m 1}, v_i v_i^j : i = 0, \ldots, m-1; j = 1, \ldots, n\}$, where $+_m$ denotes the sum modulo m. Let $\overrightarrow{G}$ be an orientation of G such that, the subdigraph induced by C is strongly connected and all leaves have indegree 1. Notice that, $\overrightarrow{G} \cong \overrightarrow{K}^l_{1,n} \otimes_h \{\overrightarrow{C_m}\}$, where h is constant, $\overrightarrow{C_m}$ is the strong orientation of C_m with arcs of the form $(v_i, v_{i+_m 1})$ (we identify $V(\overrightarrow{C_m}) = C$) and $\overrightarrow{K}^l_{1,n}$ is the digraph obtained by orienting $K_{1,n}$ in such a way that all leaves have indegree 1 and a loop is attached to the central vertex. Thus, by Theorem 1.1 G is super edge-magic when m is an odd natural number. Moreover, by construction, if we consider the super edge-magic labeling of the cycle $\overrightarrow{C_m}$ defined by:

$$g(v) = \begin{cases} i+1, & \text{if } v = v_{2i}, \\ i + \frac{m+1}{2}, & \text{if } v = v_{2i-1}, \end{cases}$$

and a super edge-magic labeling of $\overrightarrow{K}^l_{1,n}$ that assigns to the central vertex the label 1, then the labeling f induced by the product (see proof of Theorem 1.1 in [10]) assigns the same labels to the corresponding vertices of the cycle of G. Whereas the labels of the leaves are defined by the rule $f(v_i^j) = mj + f(v_{i+_n 1})$. (See the digraph on the left that appears in Figure 2.)

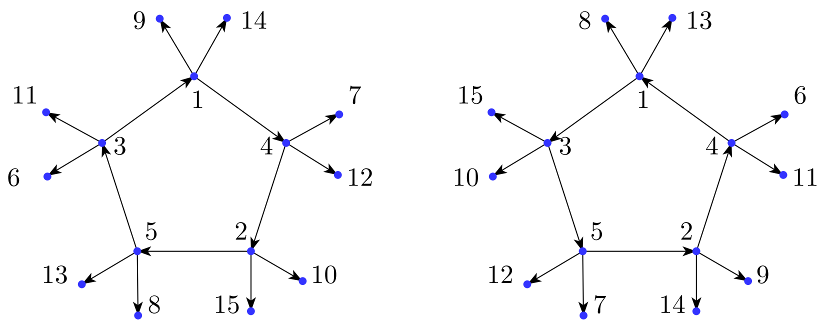

Figure 2: Two orientations of $G = C_5 \bigodot \overline{K_2}$ with the super edge-magic labelings induced by the product.

Let $\mathcal{M}_m$ be the set of all matrices of order $m \times m$ and let us denote f by f_1 from now on. By identifying each vertex of $\overrightarrow{G}$ with the label assigned by f_1, we can construct the adjacency matrix of the digraph $\overrightarrow{G}$, which is of the form: $A^1 = (\ A^1_{ij}\ )$, where each $A^1_{ij} \in \mathcal{M}_m$, $A^1_{ij} = (0)$ for $i > 1$. Whereas A^1_{1j} has the structure

$$\begin{pmatrix} M & Id_{(m-1)/2} \\ Id_{(m+1)/2} & N \end{pmatrix},$$

where M and N are two null matrices of size respectively, $(m-1)/2 \times (m+1)/2$ and $(m+1)/2 \times (m-1)/2$, and $Id_k = diag(\overbrace{1, \ldots, 1}^{k})$. An example of this structure can be observed in the first 5 rows of the adjacency matrix, A^1, of the digraph (on the left) that appears in Figure 2. See the example next.

$$\left(\begin{array}{ccccc|ccccc|ccccc} 0&0&0&1&0&0&0&0&1&0&0&0&0&1&0\\ 0&0&0&0&1&0&0&0&0&1&0&0&0&0&1\\ 1&0&0&0&0&1&0&0&0&0&1&0&0&0&0\\ 0&1&0&0&0&0&1&0&0&0&0&1&0&0&0\\ 0&0&1&0&0&0&0&1&0&0&0&0&1&0&0 \end{array}\right).$$

Finally, notice that in this process we can consider the opposite strong orientation of the cycle that we denote by $\overleftarrow{C_m}$. In that case, if we identify each vertex of $\overrightarrow{G}$ with the labels induced by the product we obtain an adjacency matrix of $\overrightarrow{G}$ with the same structure of A^1. Let us denote this matrix by B^1. Then, $B^1 = (\ B^1_{ij}\ )$, where each $B^1_{ij} \in \mathcal{M}_m$, $B^1_{ij} = (0)$ for $i > 1$. Whereas B^1_{1j} has the structure

$$\left(\begin{array}{ll} N & Id_{(m+1)/2} \\ Id_{(m-1)/2} & M \end{array}\right).$$

The next matrix corresponds to the first 5 rows of the adjacency matrix B^1 of the digraph (on the right) that appears in Figure 2.

$$\left(\begin{array}{ccccc|ccccc|ccccc} 0&0&1&0&0&0&0&1&0&0&0&0&1&0&0\\ 0&0&0&1&0&0&0&0&1&0&0&0&0&1&0\\ 0&0&0&0&1&0&0&0&0&1&0&0&0&0&1\\ 1&0&0&0&0&1&0&0&0&0&1&0&0&0&0\\ 0&1&0&0&0&0&1&0&0&0&0&1&0&0&0 \end{array}\right).$$

Let A^r be the matrix obtained from A^1 by translating each row $r-1$ units, for $r \leq mn+1$. Thus, if $A^r = (a^r_{ij})$ then

$$a^r_{ij} = \begin{cases} a^1_{(i-r+1)j}, & \text{if } i \geq r, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Similarly, define B^r to be the matrix obtained from B^1 by translating each row $r-1$ units, for $r \leq mn+1$.

Claim 1. *For each r, $1 \leq r \leq mn+1$, either A^r or B^r is the adjacency matrix of a super edge-magic labeled digraph, whose underlying graph is G. Moreover, if f_r is the induced super edge-magic labeling of G, then $val(f_r) = val(f_1) + r - 1$.*

We denote by $G(A^r)$ and $G(B^r)$ respectively, the digraphs with adjacency matrices A^r and B^r. We denote also by $S(A^r)$ and $S(B^r)$ respectively, the subgraphs of $G(A^r)$ and $G(B^r)$ with vertex set $\{r, \ldots, r-1+m\}$. Looking at the adjacency matrices, it is not difficult to check that: the digraphs $S(A^r)$ and $S(B^r)$ are 1-regular and (ii), the graphs $und(G(A^r))$ and $und(G(B^r))$ are of the form $H \bigodot \overline{K_n}$ where H is a 2-regular graph.

By construction it is clear that: $(a, b) \in E(S(A^1)) \Leftrightarrow b-a = (m+1)/2\ (mod\ m)$ and $(a, b) \in E(S(B^1)) \Leftrightarrow b - a = (m - 1)/2\ (mod\ m)$. Which implies, using (1), that $(a, b) \in E(S(A^r)) \Leftrightarrow (a - (r - 1), b) \in E(S(A^1))$ and $(a, b) \in E(S(B^r)) \Leftrightarrow (a - (r - 1), b) \in E(S(B^1))$. That is to say

$$(a, b) \in E(S(A^r)) \Leftrightarrow b - a = \frac{m+1}{2} - (r - 1)\ mod(\ m),$$

$$(a, b) \in E(S(B^r)) \Leftrightarrow b - a = \frac{m-1}{2} - (r - 1)\ mod(\ m).$$

If $und(S(A^r))$ is not isomorphic to a cycle, then we have that $\gcd((m+1)/2-(r-1), m) \neq 1$. Thus, by Lemma 2.1 we obtain that $\gcd((m-1)2-(r-1), m) = 1$ and hence, $und(S(B^r))$ is a cycle. By (1), it follows that $\{a + b\ :\ (a, b) \in E(G(A^r))\} = r - 1 + \{a - (r - 1) + b\ :\ (a - (r - 1), b) \in E(G(A^1))\}$. Therefore, since $G(A^1)$ is super edge magic, we obtain that $G(A^r)$ is super edge-magic, and in particular, that $\text{val}(f_r) = \text{val}(f_1) + r - 1$. □

Next we prove a technical result that will show that the hypothesis $m = p^k$ cannot be removed.

Lemma 2.2. *Let p, q be odd coprime numbers and let $m = pq$. Then there exist integers α, β with $1 = \alpha p + \beta q$ and $max\{|\alpha p|, |\beta q|\} \leq (m + 1)/2$.*

Proof: By Bézout's identity, we know that there exist integers α, β such that $1 = \alpha p+\beta q$, with $\alpha p > |\beta q|$. Thus, the following identity $1 = (\alpha-kq)p+(\beta+kp)q$ also holds for any $k \in \mathbb{R}$. Assume that $\alpha p > (m + 1)/2$ (otherwise we are done). Let k be an integer such that $|\alpha - kq| < q/2$ (it exists since q is odd). Hence with such a choice of k, we have $|\alpha - kq|p \leq pq/2$ and $|\beta + kp|q = |1 - (\alpha - kq)p| \leq 1 + |\alpha - kq|p \leq 1 + pq/2$. Thus, since $2 + pq$ is odd, we have that $|\beta + kp|q \leq (m + 1)/2$. □

Observation 2.3. *The hypothesis that $m = p^k$ in the statement of Theorem 2.2 cannot be improved using our approach. By Lemma 2.2, if $m = pq$ is odd with p and q being coprime numbers, then there exist integers α, β such that $1 = \alpha p+\beta q$ with $\alpha p > \beta q$ and $\alpha p \leq (pq + 1)/2$. Taking $r = 1 + (m + 1)/2 - |\alpha p|$, we have that:*

$$gcd(\frac{m+1}{2} - (r - 1), m) \neq 1,$$

$$gcd(\frac{m-1}{2} - (r - 1), m) \neq 1.$$

Therefore, in the proof of Theorem 2.2 neither $und(S(A^r))$ nor $und(S(B^r))$ are cycles, but the edge disjoint union of isomorphic cycles. That is to say, the process described in the proof of Theorem 2.2 does not provided a super edge-magic labeling of G for all possible valences. The magic interval may contain some holes.

3 Conclusions

In this paper we have introduced the concept of perfect super edge-magic graphs, and we have shown that there are infinitely many graphs that are perfect super edge-magic. We would like to encourage researchers to persuade in finding more such families, or in proving that some graphs are not perfect super edge-magic. Similar concepts can be defined for edge-magicness. We feel that the problem for edge-magicness is considerably harder, and again we would like to encourage researchers to try to make progress in this field.

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