On the location of zeros of a polynomial

by

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Abstract

Observing that for the zeros of polynomial \( p(z) = z^n + \sum_{j=0}^{n-1} a_j z^j \), Cauchy’s bound

\[ |z| < 1 + A, \quad A = \max_{0 \leq j \leq n-1} |a_j| \]

does not reflect the fact that for \( A \to 0 \), all zeros approach the origin \( z = 0 \), Boese and Luther suggested the proper bound

\[ |z| < R', \]

\[ R' = \begin{cases} 
A(1 - nA)/(1 - (nA)^{1/n}), & A \leq 1/n, \\
\min \{ (1 + A)(1 - (A/(1 + A)^{p+1} - nA)), \\
1 + 2((nA - 1)/(n + 1)) \}, & A \geq 1/n.
\end{cases} \]

We have obtained a generalization of Boese and Luther’s bound by considering the polynomial

\[ z^n + a_p z^{n-p} + a_{p+1} z^{n-p-1} + \ldots + a_n, 1 \leq p < n \]

and have also suggested certain related results.

Key Words: Cauchy’s bound, angle-independent bound, angle-independent zero free bound.

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1 Introduction and statement of results

Concerning the zeros of a polynomial of degree \( n \) we have the following well known result due to Cauchy [2].

Theorem A. All zeros of the polynomial

\[ f(z) = z^n + a_{n-1} z^{n-1} + \ldots + a_0, \tag{1.1} \]
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with

\[ A = \max_{0 \leq j \leq n-1} |a_j|, \]

(1.2)

lie in the disc

\[ |z| < 1 + A. \]

Boese and Luther [1] observed that Cauchy’s bound does not reflect the fact that for

\[ A \to 0, \]

all zeros approach

\[ z = 0. \]

Accordingly they obtained the appropriate bound and proved

**Theorem B.** All zeros of the polynomial \( f(z) \) lie in the disc

\[ |z| < R', \]

(1.3)

where

\[
R' = \begin{cases} 
A(1-nA)/(1-(nA)^{1/n}) \cdot (1/n), & A \leq 1/n, \\
\min (1+A)(1-(A/((1+A)^{p+1}-nA)), \\
1+2(nA-1)/(n+1)), & A \geq 1/n.
\end{cases}
\]

In this paper we have firstly obtained a generalization of Theorem B. More precisely we have proved

**Theorem 1.** All the zeros of the polynomial

\[ q(z) = z^p + apz^{p-1} + \ldots + a_n, 1 \leq p < n, \]

(1.4)

with

\[ D = \max_{p \geq n} |a_t| \]

(1.5)

and

\[
R = \begin{cases} 
\min((1+2((D(n-p+1)-1)/(p+n))], \\
|D|((n-p+1)/n - 1)/((n-p+1)D)^{1/n}/n - 1)^{1/p}, & (n-p+1)D < 1 \\
\min[D + ((n-p+1)D)^{p-1}]^{1/p}, \\
1 - D(pD + ((n-p+1)D)^{p+1}/p - (n-p+1)D^{-1}), \\
1 + 2((D(n-p+1)-1)/(p+n))], & (n-p+1)D > 1, \\
1, & (n-p+1)D = 1,
\end{cases}
\]

lie in the disc

\[ |z| \leq R. \]
Remark 1. In many cases Theorem 1 gives better bound than that obtained by Theorem B (e.g. for the polynomial
\[ q(z) = z^5 + a_4z + a_5; \quad |a_4| = 1, |a_5| = 4; \quad p = 4, \]
all the zeros lie in

(i) \[ |z| < 4.9985, \quad (\text{by Theorem B}), \]
(ii) \[ |z| < 2.6, \quad (\text{by Theorem 1}). \]

By applying Theorem 1 on the ray \( \theta = \text{constant}, \) we obtain

Corollary 1. Let
\[ \cos_t, t = \max(0, \cos t), \quad \text{for real } t \]
and
\[ -a_k = A_k e^{i\alpha_k}, \quad p \leq k \leq n. \]
Then the angle-independent bound \( R(D), \) for all zeros of \( q(z), \) of Theorem 1 can be replaced by \( R(D(\theta)), \) where

\[ D(\theta) = \max_{p \leq k \leq n} \{ A_k \cos_t (\alpha_k - k\theta) \}, \quad \theta \in [0, 2\pi). \]

In the same paper [1] Boese and Luther obtained a zero free region around origin also, for an \( n \)th degree polynomial and proved

Theorem C. The polynomial
\[ f(z) := a_0 + a_1z + \ldots + a_n z^n, \quad a_0 a_n \neq 0, n \geq 2 \]
is zero-free in the open disk
\[ |z| < R_0, \]

\[ R_0 := \begin{cases} A/(1 + A - \rho_0^n), & 0 \leq A \leq n, \\ [1 + A - A(1 + A - (A/\rho_0))^{-1/n}]^{1/n}, & n \leq A, \end{cases} \]
\[ \rho_0 := \begin{cases} A/(1 + A), & 0 \leq A \leq A_0, \\ \max \left\{ \frac{A}{(1 + A)}, 1 + \sqrt{D - \left( \frac{3}{2(n - 1)^2} \right)} \right\}, & A_0 \leq A \leq n, \\ \left\{ A \max \left\{ 1/n, 1 - (n/A)^{1/n} \right\} \right\}^{1/n}, & n \leq A, \end{cases} \]
\[ A_0 := n(5/8 - (3/(4n - 4))), \]
\[ A := \max \{|a_0|, |a_1|, |a_2|, \ldots, |a_n|\}, \]
\[ D := (9/(4(n - 1)^2)) - (6(1 - (A/n))/(n^2 - 1)). \]

In this paper we have obtained a partial generalization of Theorem C also, by proving
Theorem 2. The polynomial

\[ s(z) = a_n + a_{n-1}z + a_{n-2}z^{n-2} + \ldots + a_{p}z^{n-p} + a_0z^n, \ a_0a_n \neq 0; \ 1 \leq p < n, (1.16) \]

with

\[ E = \frac{|a_0|}{\max(|a_{n-1}|, |a_{n-2}|, \ldots, |a_p|, |a_0|)}, (1.17) \]

\[ \rho = \begin{cases} \max(E/(n-p+1), E/(1+E)), & E < n-p+1 \\ (E \max(1/(n-p+1), 1/[1+(E/(n-p+1))^{-p/n}])^{1/n}, & E > n-p+1 \end{cases} \]

\[ R = \begin{cases} E/(1+E-\rho^{p-p}+\rho^{n-1}-\rho^n), & E < n-p+1 \\ 1 + E - (1+E-\rho^{p-p}+\rho^{n-1}-(E/\rho)^{(n-p)/n})^{1/n}, & E > n-p+1 \\ E(1+E-\rho^{p-p}+\rho^{n-1}-(E/\rho)^{-1/n} - 1, & E = n-p+1 \end{cases} \]

is zero free in the disc

\[ |z| < R. \]

Remark 2. Theorem 2, with the possibility

\[ E \geq n-p+1 \quad \& \quad p = 1 \]

reduces to the corresponding possibility of Theorem C. But Theorem 2, with the possibility

\[ E < n-p+1 \quad \& \quad p = 1 \]

reduces partly to the corresponding possibility of Theorem C, (to be precise, only for \(0 \leq A \leq A_0\), (and to be more precise, Theorem 2 is a refinement of Theorem C, for \(0 \leq A \leq A_0\)).

Remark 3. To be more precise, for the possibility

\[ E \neq n-p+1, \]

the zero free disc is

\[ |z| \leq R. \]

By applying Theorem 2 on the ray \( \theta = \text{constant} \), we obtain
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**Corollary 2.** Let

\[
\frac{a_k}{a_n} = A_k e^{t_1}, \quad k = 0, p, p + 1, \ldots, n - 1; 1 \leq p < n \quad a_0a_n \neq 0, \\
\cos t = \max(0, \cos t), \text{ for real } t
\]

and

\[
E(\theta) = \frac{1}{\max \left\{ A_0 \cos(\alpha_0 + n\theta), A_p \cos(\alpha_p + (n - p)\theta), \ldots, A_{n-1} \cos(\alpha_{n-1} + \theta) \right\}}.
\]

Then the angle-independent zero free bound \( R(\theta) \) for \( s(z) \), of Theorem 2 can be replaced by \( R(E(\theta)) \).

Finally, in this paper we have obtained a result, which again gives a zero free region around origin for an \( n^{th} \) degree polynomial, but better than those obtainable by many other known results. More precisely we have proved

**Theorem 3.** The polynomial

\[
p(z) = a_0 + a_1z + \ldots + a_n z^n, \quad a_0a_n \neq 0, n \geq 2,
\]

with

\[
|a_1/a_0| = A_1,
\]

\[
\max_{2 \leq j \leq n} |a_j/a_0| = \delta,
\]

\[
A_1/\left\{1 - (1/(1+A_1)^{n-1})\right\} = A_0, 0 < A_1,
\]

\[
G(A_1, \delta) = \begin{cases} 
(A_1 + 1) - \sqrt{(A_1 - 1)^2 + 4\delta}/(2(A_1 - \delta)), & A_1 \neq \delta, \\
1/(A_1 + 1), & A_1 = \delta,
\end{cases}
\]

\[
\rho = \begin{cases} 
\max \left\{1/(n-1)\delta + A_1\right\}, & (n-1)\delta + A_1 > 1 \& A_1 \geq \delta \text{ or } \delta > \max(A_1, A_0) \text{ or } A_1 = 0 \\
\max \left\{(\delta - A_1)/(n\delta)\right\}^{1/(n-1)}, & (n-1)\delta + A_1 > 1 \& A_1 < \delta \leq A_0, \\
\max \left\{1/[((n-1)\delta + A_1)^{1/n}]\right\}, & (n-1)\delta + A_1 > 1 \& A_1 < \delta \leq A_0, \\
1/(\delta(1 - ((n-1)\delta + A_1)^{1/n})^{-1} + (A_1 - \delta)((n-1)\delta + A_1)^{(n-1)/n})^{1/n}], & (n-1)\delta + A_1 < 1
\end{cases}
\]
\[ R = \begin{cases} 
(1 + A_1 + (\delta - A_1) \rho - \delta \rho^n)^{-1}, & (n-1)\delta + A_1 > 1 \& [A_1 \geq \delta \text{ or } A_1 < \delta \leq A_0] \\
\delta, & (n-1)\delta + A_1 > 1 \& \delta > \max(A_1, A_0) \text{ or } A_1 = 0 \\
(1 + A_1) / (1 + (\delta - A_1) \rho - \delta \rho^n)^{1/n}, & (n-1)\delta + A_1 < 1 \& A_1 \leq \delta, \\
\rho, & (n-1)\delta + A_1 < 1 \& A_1 > \delta, \\
1, & (n-1)\delta + A_1 = 1, 
\end{cases} \]

is zero free in the disc \(|z| < R\).

**Remark 4.** The polynomial 
\[ p(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3; |a_0| = 1, |a_1| = 6, |a_2| = 12, |a_3| = 8, \]
is zero free in

(i) \(|z| < .077, \text{ by Theorem C}\),  
(ii) \(|z| < .077, \text{ by [3, Exercise no. 2, p. 126]}\),  
(iii) \(|z| < .125, \text{ by Theorem 3}\).

**Remark 5.** To be more precise, for the possibility 
\((n-1)\delta + A_1 > 1, \)
the zero free disc is 
\(|z| \leq R.\)

By applying Theorem 3 on the ray 
\(\theta = \text{ constant}, \)
we obtain

**Corollary 3.** Let 
\[ -\frac{a_k}{a_0} = B_k e^{i \theta_k}, k = 1, 2, \ldots, n \& a_0a_n \neq 0, \quad (1.32) \]
\[ \cos \theta = \max(0, \cos t), \text{ for real } t, \quad (1.33) \]
\[ A_1(\theta) = B_1 \cos e(\alpha_1 + \theta), \quad (1.34) \]
and 
\[ \delta(\theta) = \max_{2 \pi / n} B_j \cos e(\alpha_j + j\theta). \quad (1.35) \]

Then the angle-independent zero free bound \(R(A_1, \delta)\) for \(p(z)\), of Theorem 3 can be replaced by \(R(A_1(\theta), \delta(\theta))\), except when \(\delta(\theta) = 0\), in which case \(R(A_1, \delta)\) will be replaced by \(1/A_1(\theta)\).
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2 Proofs of the theorems

Proof of Theorem 1. For each zero
\[ z(\neq 0) = re^{i\theta}, \]  
(2.1)
of \( q(z) \), we have
\[ -z^n = a_p z^{n-p} + a_{p+1} z^{n-p-1} + \ldots + a_n, \]  
(by (1.4)),
and therefore
\[ r^n \leq D(r^{n-p} + r^{n-p-1} + \ldots + 1), \]  
(by (2.1) and (1.5)),
(2.2)
which implies that
\[ D > 0. \]
Further let
\[ \phi(r) = r^n - D(r^{n-p} + r^{n-p-1} + \ldots + 1). \]  
(2.3)
Then the equation
\[ \phi(r) = 0 \]  
(2.4)
has a unique positive root \( \beta \), with
\[ \beta = 1, \text{ if } (n + p)D = 1, \]  
(2.5)
\[ \beta < 1, \text{ if } (n + p)D < 1, \]  
(2.6)
\[ \beta > 1, \text{ if } (n + p)D > 1, \]  
(2.7)
thereby helping us to write
\[ \phi(r) = (r - \beta)\psi(r), \]  
(2.8)
with
\[ \psi(r) > 0, r > 0. \]  
(2.9)
Now by (2.2), (2.3), (2.8) and (2.9) we can say that
\[ r \leq \beta. \]  
(2.10)
But as \( \beta \) is a root of (2.4), we also have
\[ \beta^n = D(\beta^{n-p} + \beta^{n-p-1} + \ldots + \beta + 1), \]  
(by (2.3)),
(2.11)
Therefore if
\[ (n - p + 1)D < 1 \]
then by using (2.6) in (2.11), we get
\[ \beta < |D(n - p + 1)|^{1/n}. \]
which on being used in (2.11), gives us
\[
\beta^n < D \left( (n - p + 1)D^{(n-p)/n} + ((n - p + 1)D)^{(n-p-1)/n} + \ldots + 1 \right),
\]
\[\beta \leq \left[ D \left( \frac{(n - p + 1)D^{(n-p+1)/n} - 1}{(n - p + 1)D^{1/n} - 1} \right)^{1/n} \right]. \tag{2.12}
\]

Now we assume that
\[D(n - p + 1) > 1.\]

Therefore on using (2.7) in (2.11), we get firstly
\[
\beta > \left( D\left( (n - p + 1)D^{p-1} \right)^{1/(n+1)} \right)^{1/n},
\]
which implies
\[\beta = \left( D(n - p + 1) \right)^{1/(n+1)} s, \text{ for certain } s(> 1) \tag{2.13}\]
and secondly
\[
\beta' < D\left( \beta^{p-1} + \ldots + \beta^{p-p} \right),
\]
i.e.
\[\beta \leq \beta^p < (n - p + 1)D. \tag{2.14}\]

Further by (2.11) we get
\[
\beta^{p+1} + D = \left( D + \beta^{p-1} \right)^{p} - 1, \tag{2.15}
\]
which by (2.13), implies
\[
s^p + \left( s^{-(p+1)/(n-p+1)} \right) < \left( D + ((n - p + 1)D)^{p-1} \right)^{(n-p+1)/p}, \text{ (say)}. \tag{2.16}
\]

The function
\[\psi(s) = s^p + \left( s^{-p+1}/(n - p + 1) \right), s > 1\]
has positive and strictly increasing derivative and as
\[s < B^{1/p}, \text{ (by (2.15))},\]
we have
\[\psi(B^{1/p}) - \psi(s) < (B^{1/p} - s)\psi'(B^{1/p}),\]
i.e.
\[
(s - B^{1/p})\psi'(B^{1/p}) + \psi(B^{1/p}) < \psi(s), \tag{2.17}
\]
\[
< B, \text{ (by (2.16) and (2.15))}, \]
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i.e. \( s < B^{1/p} \left[ 1 - \left( \frac{(pB^{\alpha+1}/p - 1)}{(n - p + 1)} \right) \right], \) (by (2.16)),

which by (2.13) and (2.15) implies that

\[
\beta < D + ((n - p + 1)D)^{p-1}/\left[ 1 - D(p(n - p + 1)D)^{(\alpha+1)/p} - (n - p + 1)D \right]. \quad (2.17)
\]

For the function

\[
g(r) = \frac{1}{(r^p + r^{p-1} + \ldots + r^n)}, \quad r > 0,
\]

we have

\[
g'(r) = \frac{(pr^p + (p + 1)r^{p-2} + \ldots + nr^{n-1})/(r^p + r^{p-1} + \ldots + r^n)^2,}{r > 0,}
\]

\[
> 0, \text{ (and strictly increasing), } r > 0
\]

and therefore

\[
g(s) + (r - s)g'(s) < g(r); \quad r > 0 \text{ and } s > 0,
\]

i.e

\[
g(s) + (r - s)g'(s) \leq g(r), \quad r > 0 \text{ and } s > 0,
\]

\[
\leq D, \text{ (by (2.18) and (2.2)).} \quad (2.21)
\]

By taking

\[
s = 1,
\]

in (2.21) and using (2.18), (2.19) and (2.20), we get

\[
r \leq \left[ 1 + 2((D(n - p + 1) - 1)/(p + n)) \right]
\]

and now, on combining (2.10) with (2.12), (2.17) and (2.5), Theorem 1 follows.

**Proof of Corollary 1.** For each zero

\[
z(\neq 0) = re^{i\theta},
\]

of \( q(z) \), we have

\[
r^n \leq \sum_{k=p}^{n} A_k r^{n-k} \cos_k(\alpha_k - k\theta), \text{ (by (1.7), (2.22) and (1.6)),}
\]

\[
\leq D(\theta) \left\{ r^{n-p} + r^{n-p-1} + \ldots + 1 \right\}.
\]

Now Corollary 1 follows, by following the line of proof of Theorem 1.

**Proof of Theorem 2.** For each zero

\[
z = re^{i\theta},
\]

(2.23)
of \( s(z) \), we have
\[
E \leq r + r^2 + \ldots + r^{n-p} + r^n, \text{ (by (2.23) and (1.17))}.
\] (2.24)

Further let
\[
\chi(r) = r^n + r^{n-p} + \ldots + r^2 + r - E.
\] (2.25)

Then the equation
\[
\chi(r) = 0
\] (2.26)
has a unique positive root \( \gamma \), with
\[
\gamma = 1, \text{ if } n - p + 1 = E,
\] (2.27)
\[
\gamma < 1, \text{ if } n - p + 1 > E,
\] (2.28)
\[
\gamma > 1, \text{ if } n - p + 1 < E,
\] (2.29)
thereby helping us to write
\[
\chi(r) = (r - \gamma) g(r),
\] (2.30)
with
\[
g(r) > 0, r > 0.
\] (2.31)

By (2.24), (2.25), (2.30) and (2.31) we can say that
\[
r \geq \gamma.
\] (2.32)

But as \( \gamma \) is a root of (2.26) we also have
\[
E = \gamma + \gamma^2 + \ldots + \gamma^{n-p} + \gamma^n, \text{ (by (2.25))}.
\] (2.33)

Now firstly we assume that
\[
E < n - p + 1.
\]
Therefore on using (2.28) in (2.33), we get
\[
\gamma > \max(E/(n - p + 1), E/(1 + E), (= \rho), \text{ (by (1.18))},
\] (2.34)
which, by (2.28) and (2.31), implies
\[
(\rho - 1)(\rho - \gamma) g(\rho) > 0,
\]
i.e.
\[
E/(1 + E - \rho^{n-p} + \rho^{n-1} - \rho^n) > \rho, \text{ (by (2.30) and (2.25))}.
\]

Further for the function
\[
f(x) = -x^n + x^{n-1} - x^{n-p}, 1 < p < n,
\] (2.35)
we have
\[
\dot{f}(x) = -x^{n-p-1} \left\{ nx^p - (n - 1)x^{p-1} + (n - p) \right\},
\]
\[
= -x^{n-p-1} \{ g(x) \}, \text{ say},
\] (2.36)
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with
\[ g(0) > 0, \]
(2.37)
\[ g'(x) = npx^{p-2} \left( x - \frac{(n-1)(p-1)/(np)}{n} \right), \]
(2.38)
and
\[ g \left( \frac{(n-1)(p-1)}{np} \right) > 0. \]
(2.39)
By (2.38), (2.39) and (2.37) we can say that \( g(x) \) is positive in \([0, \infty)\) and therefore by (2.36) and (2.35), \( f(x) \) is strictly decreasing in \([0, \infty)\). Now by (2.34) and (2.28) we get
\[ E/(1 + E - \rho^{n-p} + \rho^{p-1} - \rho^n) < E/(1 + E - \gamma^{n-p} + \gamma^{p-1} - \gamma^n), \]
(2.40)
Secondly we assume that
\[ E > n - p + 1. \]
(2.41)
Therefore on using (2.29) in (2.33), we get
\[ \gamma > \{E/(n - p + 1)\}^{1/n}. \]
(2.42)
Again by (2.33), we have
\[ \gamma^n = \frac{E}{\left(1 + (1/\gamma)^p + \ldots + (1/\gamma)^{p-1}\right)}, \]
\[ > E/\left(1 + (1/\gamma^p)/(1 - (1/\gamma))\right), \quad \text{(by (2.29))}, \]
\[ > E/\left(1 + ((E/(n - p + 1))^{1/p}/(1 - (E/(n - p + 1))^{-1/p}\))\right), \]
thereby implying by (2.42) and (1.19) that
\[ \gamma > \rho, \]
(2.43)
with
\[ \rho > 1, \quad \text{(by (2.41) and (1.19))}. \]
(2.44)
Now by (2.43), (2.44) and (2.31), we have
\[ (\rho - 1)(\rho - \gamma)g(\rho) < 0, \]
i.e.
\[ (1 + E - \rho^{n-p} + \rho^{p-1} - (E/\rho))^{1/n} > \rho, \quad \text{(by (2.30) and (2.25))}. \]
(2.45)
Further by (2.43) and (2.44), we get
\[ \rho^{n-p} \left(\gamma/\rho\right)^{p-1} - 1 \leq \rho^{n-1} \left(\gamma/\rho\right)^{p-1} - 1, \]
which implies
\[ (1 + E - \rho^{n-p} + \rho^{p-1} - (E/\rho))^{1/n} < (1 + E - \gamma^{n-p} + \gamma^{p-1} - (E/\gamma))^{1/n}, \quad \text{(by (2.43))}, \]
\[ = \gamma, \quad \text{(by (2.33))}. \]
(2.46)
Now let
\[ \delta = \left(1 + E - \rho^{n-p} + \rho^{n-1} - \frac{E}{\rho}\right)^{1/n}. \]
Then
\[ \gamma > \delta, \quad \text{by (2.46)} \]
and
\[ \delta > 1, \quad \text{by (2.45) and (2.44)}. \]
Hence on repeating all steps, after (2.44) and upto (2.46), we will get
\[ \left(1 + E - \delta^{n-p} + \delta^{n-1} - \frac{E}{\delta}\right)^{1/n} < \gamma, \]
i.e.
\[
\begin{align*}
1 + E - (1 + E - \rho^{n-p} + \rho^{n-1}) & - \frac{E}{(n-\rho)/\rho^2} + \\
(1 + E - \rho^{n-p} + \rho^{n-1} - \frac{E}{\rho})^{(n-1)/n} & - E(1 + E - \rho^{n-p} + \rho^{n-1} - \frac{E}{\rho})^{-1/n} \\
< & \gamma.
\end{align*}
\]
Finally by using (2.32), (2.40), (2.47) and (2.27), Theorem 2 follows.

**Proof of Corollary 2.** For each zero
\[ z = re^{i\theta} \]
of \( s(z) \), we have
\[
1 \leq \sum_{k=p}^{n-1} A_k r^{n-k} \cos (\alpha_k + (n-k)\theta) + A_0 r^n \cos (\alpha_0 + n\theta), \quad \text{by (1.20), (2.48) and (1.21)},
\]
which, by (1.22), implies
\[ E(\theta) \leq r + r^2 + \ldots + r^{n-p} + r^n. \]
Now Corollary 2 follows, by following the line of proof of Theorem 2.

**Proof of Theorem 3.** For each zero
\[ z = re^{i\theta}, \]
of \( p(z) \), we have
\[
1 \leq A_1 r + \delta (r^2 + r^3 + \ldots + r^n), \quad \text{by (2.49), (1.24) and (1.25)}. \]
Further let
\[ \psi_1(r) = \delta (r^n + r^{n-1} + \ldots + r^2) + A_1 r - 1. \]
Then the equation
\[ \psi_1(r) = 0 \]
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has a unique positive root $\alpha$, with

$$
\alpha = 1, \text{ if } (n-1)\delta + A_1 = 1, \\
\alpha < 1, \text{ if } (n-1)\delta + A_1 > 1, \\
\alpha > 1, \text{ if } (n-1)\delta + A_1 < 1,
$$

thereby helping us to write

$$
\psi_1(r) = (r-\alpha)\phi_1(r),
$$

with

$$
\phi_1(r) > 0, \ r > 0.
$$

By (2.50), (2.51), (2.56) and (2.57) we can say that

$$
r \geq \alpha.
$$

But as $\alpha$ is a root of (2.52), we also have

$$
1 = \delta(a^n + a^{n-1} + \ldots + a^2) + A_1\alpha, \text{ (by (2.51)).}
$$

Now firstly we assume that

$$
(n-1)\delta + A_1 > 1.
$$

Therefore on using (2.54) in (2.59), we get firstly

$$
\alpha > 1/((n-1)\delta + A_1)
$$

and secondly

$$
(\delta - A_1)\alpha^2 + (A_1 + 1)\alpha - 1 > 0,
$$

which implies

$$
\alpha > G(A_1, \delta), \text{ (by (1.27) and (1.28)).}
$$

On combining (2.61) and (2.62) we get

$$
\alpha > \max\{1/((n-1)\delta + A_1), G(A_1, \delta)\},
$$

$$
= \rho, A_1 \geq \delta \text{ or } \delta > \max(A_1, A_0) \text{ or } A_1 = 0, \text{ (by (1.29)).}
$$

And if

$$
A_1 < \delta \leq A_0 \& \text{ so } A_1 > 0, \text{ by (1.26)) and } t = ((\delta - A_1)/\delta)^{1/(n-1)}
$$

then

$$
t \leq 1/(1 + A_1), \text{ (by (1.26)),}
$$

i.e.

$$
A_1 \leq (1 - A_1)t(1 - t)/t^2,
$$

i.e.

$$
\delta(1 - t^{n-1}) \leq (1 - A_1)t(1 - t)/t^2,
$$
i.e. 
\( \psi_1(t) \leq 0 \), (by (2.56) and (2.57)),

which implies

\[ \alpha \geq \frac{((\delta - A_1)/\delta)^{1/(n-1)},}{(\delta - A_1)/(n\delta))^{1/(n-1)}} \]

and therefore

\[ \alpha > \max \left\{ \left( \frac{((\delta - A_1)/(n\delta))^{1/(n-1)}, 1/((n - 1)\delta + A_1), G(A_1, \delta) }{1^{1/(n-1)}}, \right) \right\}, \text{ (by (2.63)), (2.67)} \]
\[ = \rho, \text{ (by (1.30)).} \]

Further for the function

\[ h(x) = 1/(1 + A_1 + (\delta - A_1)x - \delta x^n), x \in (0, \alpha], \]

we have

\[ h(\rho) > \rho, A_1 \geq \delta \text{ or } A_1 < \delta \leq A_0, \text{ (by (2.64) and (2.68)),} \]
\[ h'(x) > 0, x^{n-1} > (\delta - A_1)/(n\delta) \]

and therefore, if

\[ A_1 \geq \delta \]

then \( h(x) \) will be strictly increasing in \((0, \alpha]\), thereby implying

\[ h(\rho) < 1/(1 + A_1 + (\delta - A_1)\alpha - \delta\alpha^n) = \alpha, \text{ (by (2.59)),} \]

and if

\[ A_1 < \delta \leq A_0 \]

then \( h(x) \) will be strictly increasing in \([((\delta - A_1)/(n\delta))^{1/(n-1)}, \alpha], \) (by (2.66)), thereby implying

\[ h(\rho) < 1/(1 + A_1 + (\delta - A_1)\alpha - \delta\alpha^n) = \alpha, \text{ (by (2.59)).} \]

Secondly we assume that

\[ (n - 1)\delta + A_1 < 1. \]

Therefore on using (2.53) in (2.59), we get firstly

\[ \alpha \geq \frac{1/((n - 1)\delta + A_1)^{1/n,}}{1^{1/n}}, \text{ (by (2.73)),} \]
\[ > 1 \]

and secondly

\[ 1 < \alpha^n \left\{ \delta(1 - (1/\alpha))^{-1} + (A_1 - \delta)(1/\alpha)^{n-1} \right\}, \]

which, by (2.74), implies

\[ \alpha > 1/((\delta(1 - ((n - 1)\delta + A_1)^{1/(n-1)} + (A_1 - \delta)(n - 1)\delta + A_1)^{1/(n-1)/n})^{1/n}. \]
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On combining (2.74) and (2.76) we get

\[ \alpha \geq \rho, \text{ (by (1.31))}, \]
\[ > 1, \text{ (by (2.75))}. \]

Further if

\[ A_1 \leq \delta \]

then by (2.77) and (2.78), we have

\[ \rho \leq \left\{ \frac{(A_1/\delta) + (1/\delta)(1 - (1/\rho)) + (1 - (A_1/\delta))\rho^{1/n}}{1}, \right\} \]
\[ \leq \left\{ \frac{(A_1/\delta) + (1/\delta)(1 - (1/\alpha)) + (1 - (A_1/\delta))\alpha^{1/n}}{1}, \right\} \]
\[ = \alpha, \text{ (by (2.59))}. \]

(2.79)

(2.80)

Finally on combining (2.58) along with ((2.64), (2.68), (2.70), (2.71), (2.72)), ((2.77), (2.79), (2.80)) and (2.53), Theorem 3 follows.

**Proof of Corollary 3.** For each zero

\[ z = re^{i\theta} \]

of \( p(z) \), we have

\[ 1 \leq \sum_{j=1}^{n} B_j r^j \cos(r_j + j\theta), \text{ (by (1.32), (2.81) and (1.33))}, \]

which, by (1.34) and (1.35), implies

\[ 1 \leq A_1(\theta) r + \delta(\theta) \left[ r^2 + r^3 + \ldots + r^n \right]. \]

(2.82)

Now if

\[ \delta(\theta) > 0 \]

then Corollary 3 follows, by following the line of proof of Theorem 3, and if

\[ \delta(\theta) = 0 \]

then

\[ r \geq 1/A_1(\theta), \text{ (by (2.82))}, \]

thereby completing the proof of Corollary 3.

**References**


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