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Uniqueness for ultrametric analytic functions

by
Jacqueline Ojeda *

Abstract

Let $\mathbb{K}$ be a complete algebraically closed p-adic field of characteristic zero and let $f,\ g$ be two meromorphic functions inside an open disc of $\mathbb{K}$. We first study polynomials of uniqueness for such functions. Suppose now $f,\ g$ are entire functions on $\mathbb{K}$. Let $a \in \mathbb{K}\setminus\{0\}$ and $n, k \in \mathbb{N}$, with $k \geq 2$ and let α be a small entire function with respect to f and g. If $f^n(f-a)^k f'$ and $g^n(g-a)^k g'$ share α, counting multiplicities, with $n \geq \max\{6-k, k+1\}$ then $f = g$. If $\alpha \in \mathbb{K}^*$ and if $n \geq \max\{5-k, k+1\}$ then $f = g$.

Let $f,\ g$ be unbounded analytic functions inside an open disk of $\mathbb{K}$ and let α be a small function analytic inside in the same disk. If $f^n(f-a)^2 f'$ and $g^n(g-a)^2 g'$ share α counting multiplicities, with $n \geq 4$, then $f = g$. If $f^n(f-a)f'$ and $g^n(g-a)g'$ share α counting multiplicities, with $n \geq 5$, then $f = g$.

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1 Introduction and Main Results

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, complete for an ultrametric absolute value denoted by $|\cdot|$.

We denote by $\mathcal{A}(\mathbb{K})$ the $\mathbb{K}$-algebra of entire functions in $\mathbb{K}$, by $\mathcal{M}(\mathbb{K})$ the field of meromorphic functions in $\mathbb{K}$, i.e. the field of fractions of $\mathcal{A}(\mathbb{K})$ and by $\mathbb{K}(x)$ the field of rational functions.

Let $a \in \mathbb{K}$ and $R \in]0, +\infty[$. We denote by $d(a, R^-)$ the "open" disc $\{x \in \mathbb{K} : |x-a| < R\}$. Similarly, we denote by $\mathcal{A}(d(a, R^-))$ the set of analytic functions in $d(a, R^-)$, i.e. the $\mathbb{K}$-algebra of power series $\displaystyle\sum_{n=0}^{\infty} a_n(x-a)^n$ converging in

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$d(a,R^-)$ and by $\mathcal{M}(d(a,R^-))$ the field of meromorphic functions inside $d(a,R^-)$, i.e. the field of fractions of $\mathcal{A}(d(a,R^-))$. Moreover, we denote by $\mathcal{A}_b(d(a,R^-))$ the $\mathbb{K}$ - subalgebra of $\mathcal{A}(d(a,R^-))$ consisting of the bounded analytic functions in $d(a,R^-)$, i.e. which satisfy $\sup\limits_{n\in\mathbb{N}} |a_n|R^n < +\infty$ and by $\mathcal{M}_b(d(a,R^-))$ the field of fractions of $\mathcal{A}_b(d(a,R^-))$. Finally, we denote by $\mathcal{A}_u(d(a,R^-))$ the set of unbounded analytic functions in $d(a,R^-)$, i.e. $\mathcal{A}(d(a,R^-)) \setminus \mathcal{A}_b(d(a,R^-))$. Similarly, we set $\mathcal{M}_u(d(a,R^-)) = \mathcal{M}(d(a,R^-)) \setminus \mathcal{M}_b(d(a,R^-))$.

Functions of uniqueness for entire functions and for meromorphic functions were studied in several papers. Particularly, a sufficient condition for an analytic function P to be a function of uniqueness for analytic or meromorphic functions depends on the number of zeros a_k of the derivative such that $P(a_k) \neq P(a_j)\ \forall j \neq k$ that we can call *Fujimoto's points*. Indeed, this condition was first introduced by H. Fujimoto [9] and was particularly studied for p-adic meromorphic functions in [5] and [10]. In general, one has a conclusions for p-adic entire function when P admits at least 2 Fujimoto's points a_k. For functions in $\mathcal{A}_u(d(a,R^-))$, 3 Fujimoto's points are necessary to have a general conclusion. And for function in $\mathcal{M}_u(d(a,R^-))$, 4 Fujimoto's points are necesasry. On the other hand, polynomials with only 2 Fujimoto's points where used in several papers to study problems of uniqueness in $\mathbb{C}$ and in $\mathbb{K}$.

Here, we first aim at studying polynomials of a particular type, with only 2 Fujimoto's points where we can get conclusions for functions in both $\mathcal{A}_u(d(a,R^-))$ and $\mathcal{M}_u(d(a,R^-))$. We must first recall previous results.

Next, we will study the problem of value sharing a small function α for functions of the form $f^n(f-a)^k f'$, i.e. the derivative of $P(f)$ where P is a polynomial with 2 Fujimoto's points, again. That study follows a previous one in $\mathbb{C}$ [13].

The following Theorem A may be seen as Statement (b) when $l = 2$ in [19]. More recently, N. T. Hoa in [11] and separately A. Escassut in [5] gave the following theorem concerning polynomials of uniqueness for entire functions and meromorphic functions.

Definition. A polynomial $P(x)$ is called *a polynomial of uniqueness* for a family of functions $\mathcal{F}$, if for any $f, g \in \mathcal{F}$ such that $P(f) = P(g)$ we have $f = g$.

Theorem A. *Let $P \in \mathbb{K}[x]$ be such that P' has exactly two distinct zeros γ_1 of order c_1 and γ_2 of order c_2. Then P is a polynomial of uniqueness for $\mathcal{A}(\mathbb{K})$. Moreover, if $\min\{c_1, c_2\} \geq 2$, then P is a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$.*

Example: Let $n, k \in \mathbb{N}$ and $a \in \mathbb{K} \setminus \{0\}$. Let $P \in \mathbb{K}[x]$ be defined as

$$P(x) = \frac{1}{n+k+1}x^{n+k+1} - \binom{k}{1}\frac{a}{n+k}x^{n+k} + \ldots + \binom{k}{k-1}\frac{(-a)^{k-1}}{n+2}x^{n+2} + \frac{(-a)^k}{n+1}x^{n+1}. \tag{1}$$

Clearly

$$P'(x) = x^{n+k} - \binom{k}{1} a x^{n+k-1} + \ldots + \binom{k}{k-1} (-a)^{k-1} x^{n+1} + (-a)^k x^n,$$

this is, $P'(x) = x^n(x-a)^k$.

Remark 1. *When* $\min\{c_1, c_2\} = 2$*, this kind of polynomials was introduced by Frank and Reinders in order to find the smallest URSCM ever found, for complex entire or meromorphic functions* [8]. *By* [4] *and* [19]*, it is known that if* $\min\{c_1, c_2\} = 1$*, then* P *is not a polynomial of uniqueness for* $\mathcal{M}(\mathbb{K})$*. Indeed, up to an affine change of variable,* $P(y)$ *may be reduced to* $y^n - y^{n-1} + c$ *with* $c \in \mathbb{K}$*. Now, let* $h \in \mathcal{M}_u(d(0, R^-))$ *and let us define* f *and* g *as:* $g = \dfrac{h^{n-1} - 1}{h^n - 1}$ *and* $f = hg$*. Then we can check that the polynomial* P *satisfies* $P(f) = P(g)$*.*

Now, we do not know whether Theorem A might be extended to $\mathcal{M}_u(d(0, R^-))$. Indeed, the proof of Theorem A requires the use of algebraic curves theory on a p-adic field, particularly Picard-Berkovich Theory [1] that does no apply to functions defined in $\mathcal{M}(d(0, R^-))$. However, here we can state two theorems concerning $\mathcal{A}_u(d(0, R^-))$ and $\mathcal{M}_u(d(0, R^-))$ whose proof uses some methods similar to those of Theorem A in [3].

Theorem 1. *Let* $P \in \mathbb{K}[x]$ *of degree* $n \geq 3$ *be such that* P' *only has two distinct zeros, one of them being of order* 1*. Then* P *is a polynomial of uniqueness for* $\mathcal{A}_u(d(0, R^-))$*.*

Theorem 2. *Let* $P \in \mathbb{K}[x]$ *of degree* $n \geq 6$ *be such that* P' *only has two distinct zeros, one of them being of order* 2*. Then* P *is a polynomial of uniqueness for* $\mathcal{M}_u(d(0, R^-))$*.*

Example. Let $a \in \mathbb{K} \setminus \{0\}$ and $n \in \mathbb{N}$. Thanks to the previous theorems we can deduce that

(i) $P(x) = \dfrac{1}{n+2} x^{n+2} - \dfrac{a}{n+1} x^{n+1}$ is a polynomial of uniqueness for $\mathcal{A}_u(d(0, R^-))$.

(ii) $P(x) = \dfrac{1}{n+3} x^{n+3} - \dfrac{2a}{n+2} x^{n+2} + \dfrac{a^2}{n+1} x^{n+1}$ is a polynomial of uniqueness for $\mathcal{M}_u(d(0, R^-))$ whenever $n \geq 3$.

In [17], we studied the uniqueness of a pair (f, g) of meromorphic functions in $\mathbb{K}$ (resp. unbounded meromorphic functions in $d(0, R^-)$) such that $f^n f'$ and $g^n g'$ share one value counting multiplicities (C.M.) or ignoring multiplicities (I.M.). We proved, for example, that if $f^n f'$ and $g^n g'$ share one value C.M. with $n \geq 11$, then $f = dg$ with $d^{n+1} = 1$ whenever $f, g \in \mathcal{M}(\mathbb{K})$. And if $f^n f'$ and $g^n g'$ share one value I.M. with $n \geq 9$, then $f = dg$ with $d^{n+1} = 1$ whenever $f, g \in \mathcal{A}_u(d(0, R^-))$. Here we will study the following problem:

Let $f, g \in \mathcal{A}(\mathbb{K})$ be transcendental $\big($resp. $f, g \in \mathcal{A}_u(d(0, R^-))\big)$ and let α be a small analytic function with respect to f and g in $\mathbb{K}$ $\big($resp. in $d(0, R^-)\big)$ such that $f^n(f-a)^k f'$ and $g^n(g-a)^k g'$ share α, counting multiplicities, with $n \in \mathbb{N}^*$ and $a \in \mathbb{K} \setminus \{0\}$. Can we conclude that $f = g$?

This kind of questions was studied in complex analysis in many papers concerning meromorphic functions or entire functions in $\mathbb{C}$ with various conclusions, see, for example, [13], [14], [15] and [16]. Actually, such a problem is deeply linked to the problem of polynomials of uniqueness.

Now, in order to define small functions, we have to briefly recall the definitions of the classical Nevanlinna theory in the field $\mathbb{K}$ and a few specific properties of ultrametric analytic or meromorphic functions.

Let log be the real logarithm function of base > 1 and let $f \in \mathcal{M}(\mathbb{K})$ $\big($resp. $f \in \mathcal{M}(d(0, R^-))\big)$ having no zero and no pole at 0. Let $r \in]0, +\infty[$ $\big($resp. $r \in]0, R[\big)$ and let $\gamma \in d(0, r)$. If f has a zero of order n at γ, we put $\omega_\gamma(f) = n$. If f has a pole of order n at γ, we put $\omega_\gamma(f) = -n$ and finally, if $f(\gamma) \neq 0, \infty$, we put $\omega_\gamma(f) = 0$

We denote by $Z(r, f)$ the *counting function of zeros of f* in $d(0, r)$, counting multiplicities, i.e. we set

$$Z(r, f) = \sum_{\omega_\gamma(f) > 0,\ |\gamma| \leq r} \omega_\gamma(f)(\log r - \log|\gamma|).$$

Similarly, we denote by $\overline{Z}(r, f)$ the counting function of zeros of f in $d(0, r)$, ignoring multiplicities, and set

$$\overline{Z}(r, f) = \sum_{\omega_\gamma(f) > 0,\ |\gamma| \leq r} (\log r - \log|\gamma|).$$

In the same way, we set $N(r, f) = Z\Big(r, \dfrac{1}{f}\Big)$ $\Big($resp. $\overline{N}(r, f) = \overline{Z}\Big(r, \dfrac{1}{f}\Big)\Big)$ to denote the *counting function of poles of f* in $d(0, r)$, counting multiplicities (resp. ignoring multiplicities).

For $f \in \mathcal{M}(d(0, R^-))$ having no zero and no pole at 0, the *Nevanlinna function* is defined by $T(r, f) = \max\big\{Z(r, f) + \log|f(0)|, N(r, f)\big\}$.

In this paper we will show two results using the p-adic Nevanlinna theory together with a few specific properties of ultrametric analytic functions or ultrametric meromorphic functions. We begin introducing the following definitions.

In order to go on, we must recall the definition of a *small function* with respect to a meromorphic function and some pertinent properties.

Definition. Let $f \in \mathcal{M}(\mathbb{K})$ $\big($resp. Let $f \in \mathcal{M}(d(0, R^-))\big)$ such that $f(0) \neq 0, \infty$. A function $\alpha \in \mathcal{M}(\mathbb{K})$ $\big($resp. $\alpha \in \mathcal{M}(d(0, R^-))\big)$ having no zero and no pole at 0 is called *a small function with respect to* f, if it satisfies $\lim_{r\to+\infty}\frac{T(r,\alpha)}{T(r,f)} = 0$ $\Big($resp. $\lim_{r\to R^-}\frac{T(r,\alpha)}{T(r,f)} = 0\Big)$.

If 0 is a zero or a pole of f or α, we can make a change of variable such that the new origin is not a zero or a pole for both f and α. Thus it is easily seen that the last relation do not really depend on the origin.

We denote by $\mathcal{M}_f(\mathbb{K})$ $\big($resp. $\mathcal{M}_f(d(0,R^-))\big)$ the set of small meromorphic functions with respect to f in $\mathbb{K}$ $\big($resp. in $d(0,R^-)\big)$.

Remark 2. *Thanks to classical properties of the Nevanlinna function $T(r,f)$ with respect to the operations in a field of meromorphic functions, such as $T(r,f+g) \le T(r,f)+T(r,g)$ and $T(r,fg) \le T(r,f)+T(r,g)$, for $f,g \in \mathcal{M}(\mathbb{K})$ and $r>0$, we easily proved in [7] that $\mathcal{M}_f(\mathbb{K})$ $\big($resp. $\mathcal{M}_f(d(0,R^-))\big)$ is a subfield of $\mathcal{M}(\mathbb{K})$ $\big($resp. $\mathcal{M}(d(0,R^-))\big)$ and $\mathcal{M}(\mathbb{K})$ $\big($resp. $\mathcal{M}(d(0,R))\big)$ is a transcendental extension of $\mathcal{M}_f(\mathbb{K})$ $\big($resp. of $\mathcal{M}_f(d(0,R^-))\big)$.*

Now, we can give some sufficient conditions to get a positive answer to our question. Let us remember the following definition.

Definition. Let $f,g,\alpha \in \mathcal{M}(\mathbb{K})$ $\big($resp. $f,g,\alpha \in \mathcal{M}(d(0,R^-))\big)$. We say that f and g *share the function α C.M.*, if $f-\alpha$ and $g-\alpha$ have the same zeros with the same multiplicities in $\mathbb{K}$ $\big($resp. in $d(0,R^-)\big)$.

Theorem 3. *Let $f,g \in \mathcal{A}(\mathbb{K})$ be transcendental such that $f^n(f-a)^k f'$ and $g^n(g-a)^k g'$ share the function $\alpha \in \mathcal{A}_f(\mathbb{K}) \cap \mathcal{A}_g(\mathbb{K})$ C.M. when $n,k \in \mathbb{N}$ and $a \in \mathbb{K}\setminus\{0\}$. If $n \ge \max\{6-k,\ k+1\}$, then $f=g$. Moreover, if $\alpha \in \mathbb{K}\setminus\{0\}$ and $n \ge \max\{5-k,\ k+1\}$, then $f=g$.*

Theorem 4. *Let $f,g \in \mathcal{A}_u(d(0,R^-))$, let $\alpha \in \mathcal{A}_f(d(0,R^-)) \cap \mathcal{A}_g(d(0,R^-))$ and let $a \in \mathbb{K}\setminus\{0\}$. If $f^n(f-a)^2 f'$ and $g^n(g-a)^2 g'$ share the function α C.M. and $n \ge 4$, then $f=g$. Moreover, if $f^n(f-a)f'$ and $g^n(g-a)g'$ share the function α C.M. and $n \ge 5$, then again $f=g$.*

2 Basic Results and Proofs of Theorems

We have to recall the *ultrametric Nevanlinna second main Theorem* in a basic form which we will frequently use.

Let $f \in \mathcal{M}(\mathbb{K})$ $\big($resp. $f \in \mathcal{M}(d(0,R^-))\big)$ satisfy $f'(0) \ne 0,\infty$. Let S be a finite subset of $\mathbb{K}$ and $r \in]0,+\infty[$ $\big($resp. $r \in]0,R[\big)$. We denote by $Z_0^S(r,f')$ *the counting function of zeros of f' in $d(0,r)$ which are not zeros of any $f-s$ for $s \in S$*. This is, if $(\gamma_n)_{n\in\mathbb{N}}$ is the finite or infinite sequence of zeros of f' in $d(0,r)$ that are not zeros of $f-s$ for $s \in S$, with multiplicy order q_n respectively, we set

$$Z_0^S(r,f') = \sum_{|\gamma_n|\le r} q_n(\log r - \log|\gamma_n|).$$

Theorem N. $\big($[2], [6]$\big)$ *Let $a_1, ..., a_n \in \mathbb{K}$ with $n \geq 2$ an entire, and let $f \in \mathcal{M}(\mathbb{K})$ (resp. let $f \in \mathcal{M}(d(0, R^-))$). Let $S = \{a_1, ..., a_n\}$. Assume that none of f, f' and $f - a_j$ with $1 \leq j \leq n$, equals 0 or ∞ at the origin. Then, for $r > 0$ $\big($resp. for $r \in]0, R[\big)$, we have*

$$(n-1)T(r, f) \leq \sum_{j=1}^{n} \overline{Z}(r, f - a_j) + \overline{N}(r, f) - Z_0^S(r, f') - \log r + O(1).$$

A special Nevanlinna Theorem is known deriving from the Nevanlinna Theorem on 3 small functions [18]:

Lemma 1. *Let $f \in \mathcal{M}(\mathbb{K}) \setminus \{0\}$ non identically zero $\big($resp. $f \in \mathcal{M}(d(0, R^-))\big)$ and let $\alpha \in \mathcal{M}_f(\mathbb{K})$ $\big($resp. $\alpha \in \mathcal{M}_f(d(0, R^-))\big)$ have no zero and no pole at 0. Then for $r > 0$ $\big($resp. for $r \in]0, R[\big)$, we have $T(r, f) \leq \overline{Z}(r, f) + \overline{Z}(r, f - \alpha) + \overline{N}(r, f) + S_f(r)$.*

Lemma 2 is well known (see for instance Theorem 2.4.5 [6])

Lemma 2. *Let $f \in \mathcal{M}(\mathbb{K})$ not constant $\big($resp. $f \in \mathcal{M}_u(d(0, R^-))\big)$. There exists at most one value $b \in \mathbb{K}$ such that $f(x) \neq b$ $\forall x \in \mathbb{K}$ (resp. $f(x) \neq b$ $\forall x \in d(0, R^-)$).*

In order to prove Theorem 2, we will use the following lemma that is known in complex analysis [8] and that, in general, is valid for an algebraically closed field of characteristic zero such as $\mathbb{K}$.

Lemma 3. *Let $n \geq 3$ be an integer and let*

$$Q(X) = (n-1)^2(X^n - 1)(X^{n-2} - 1) - n(n-2)(X^{n-1} - 1)^2$$

be a polynomial with coefficient in $\mathbb{K}$ of degree $2n - 2$. Then 1 is a zero of Q of multiplicity order 4 and all the other zeros of Q are simple.

Proof of Theorems 1 and 2. Let $f, g \in \mathcal{M}_u(d(0, R^-))$ be such that $P(f) = P(g)$. Let $f = hg$.

Suppose first we are in the hypothesis of Theorem 1. By Remark 1, we have $g = \dfrac{h^{n-1} - 1}{h^n - 1}$. Each pole of $h^{n-1} - 1$ is a zero of g of same order, hence $h^{n-1} - 1$ belongs to $\mathcal{M}_u(d(0, R^-))$ and so does h. Consequently, by Lemma 2, h avoids at most one value. Now we know that the n-th roots of unity and the $n-1$-th roots of unity have no common element but 1. Consequently, since $n \geq 3$, $\dfrac{h^{n-1} - 1}{h^n - 1}$ admits at least one pole, a contradiction because g is analytic in $d(0, R^-)$.

We now place ourselves in the hypothesis of Theorem 2. From the hypothesis $P(f) = P(g)$ we can derive

$$(n-1)(n-2)(h^n-1)g^2 - 2an(n-2)(h^{n-1}-1)g+$$

$$+a^2n(n-1)(h^{n-2}-1) = 0. \tag{2}$$

Suppose that h is not a constant. Let $r \in]0, R[$. Considering the previous expression we can easily deduce that h is unbounded in $d(0, R^-)$, because if h is bounded we have

$$T\big(r, (n-1)(n-2)(h^n-1)g^2\big) \geq 2T(r,g) + O(1)$$

and $T\big(r, 2an(n-2)(h^{n-1}-1)g - a^2n(n-1)(h^{n-2}-1)\big) \leq T(r,g) + O(1)$, a contradiction to (2).

On the other hand, by simple calculations, we can write (2) as

$$\Big((n-1)(n-2)(h^n-1)g - an(n-2)(h^{n-1}-1)\Big)^2 = -a^2n(n-2)Q(h) \tag{3}$$

where $Q(h) = (n-1)^2(h^n-1)(h^{n-2}-1) - n(n-2)(h^{n-1}-1)^2$ is a polynomial of degree $2n-2$. Since $n \geq 6$, by Lemma 3, we deduce that $Q(h)$ is of the form $(h-1)^4(h-\gamma_1)(h-\gamma_2)...(h-\gamma_{2n-6})$, where every $\gamma_i \in \mathbb{K} \setminus \{0,1\}$ $\big(i = 1, ..., 2n-6\big)$, is a simple zero of Q. Now, from (3), every zero of $h - \gamma_i$ $\big(i = 1, ..., 2n-6\big)$, has multiplicity at least 2. Assume, without loss of generality, that 0 is neither a zero nor a pole of $h - \gamma_i$ $\big(i = 1, ..., 2n-6\big)$. Then,

$$\sum_{i=1}^{2n-6} \overline{Z}(r, h-\gamma_i) \leq \frac{1}{2} \sum_{i=1}^{2n-6} Z(r, h-\gamma_i) \leq (n-3)T(r,h) + O(1).$$

Thereby, applying Theorem N to h at the points γ_i $\big(i = 1, ..., 2n-6\big)$, and considering that $\overline{N}(r,h) \leq T(r,h)$, we obtain

$$\begin{aligned}(2n-7)T(r,h) &\leq \sum_{i=1}^{2n-6} \overline{Z}(r, h-\gamma_i) + \overline{N}(r,h) + O(1) \\ &\leq (n-2)T(r,h) + O(1).\end{aligned}$$

Since $T(r,h)$ is unbounded in $]0, R[$, we have a contradiction whenever $n \geq 6$. Hence, h is a constant. Therefore, by (3), we have $h^n - 1 = 0$ and $h^{n-1} - 1 = 0$. It follows that $h = 1$ and hence $f = g$. $\square$

Let us recall this classical lemma [12]:

Lemma 4. *Let $f \in \mathcal{M}(\mathbb{K})$ and $\alpha_i \in \mathcal{M}_f(\mathbb{K})$ $\big($resp. $f \in \mathcal{M}(d(0,R^-))$ and $\alpha_i \in \mathcal{M}_f(d(0,R^-))\big)$ $\big(i = 1,...,n\big)$, be such that $f(0) \neq 0,\infty$ and $\alpha_i(0) \neq 0,\infty$ $\big(i = 1,...,n\big)$. If $P(X) = \sum_{i=0}^{n} \alpha_i X^i \in \mathcal{M}_f(\mathbb{K})[X]$ $\Big($resp. $P(X) = \sum_{i=0}^{n} \alpha_i X^i \in \mathcal{M}_f(d(0,R^-))[X]\Big)$, then for $r > 0$ $\big($resp. $r \in]0,R[\big)$, we have $T(r,P(f)) = nT(r,f) + S_f(r)$.*

Moreover, if $\alpha_i \in \mathbb{K}$ $\big(i = 1,...,n\big)$, then for $r > 0$ $\big($resp. $r \in]0,R[\big)$, we have $T(r,P(f)) = nT(r,f) + O(1)$.

Lemma 5. *Let $f,g \in \mathcal{M}(\mathbb{K})$ be transcendental $\big($resp. $f,g \in \mathcal{M}_u(d(0,R^-))\big)$, $a \in \mathbb{K}\setminus\{0\}$ and $n,k \in \mathbb{N}$ with $n \geq k+2$ $\big($resp. $n \geq k+3\big)$. Let*

$$F = \frac{1}{n+k+1} f^{n+k+1} + \binom{k}{1}\frac{(-a)}{n+k} f^{n+k} + \ldots +$$

$$+ \binom{k}{k-1}\frac{(-a)^{k-1}}{n+2} f^{n+2} + \frac{(-a)^k}{n+1} f^{n+1}$$

and

$$G = \frac{1}{n+k+1} g^{n+k+1} + \binom{k}{1}\frac{(-a)}{n+k} g^{n+k} + \ldots +$$

$$+ \binom{k}{k-1}\frac{(-a)^{k-1}}{n+2} g^{n+2} + \frac{(-a)^k}{n+1} g^{n+1}.$$

If $F' = G'$, then $F = G$.

Proof: Note that $F \equiv f^{n+1}P(f)$ with $P \in \mathbb{K}[x]$ and $\deg(P) = k$. Let $r > 0$ $\big($resp. $r \in]0,R[\big)$. We have $\overline{Z}(r,F) \leq \overline{Z}(r,f) + \overline{Z}(r,P(f)) \leq T(r,f) + T(r,P(f))$ and $\overline{N}(r,F) = \overline{N}(r,f) \leq T(r,f)$. Moreover, since $F' = G'$, $F - G$ is a contant c.

Suppose $c \neq 0$. Then $\overline{Z}(r,F-c) = \overline{Z}(r,G) \leq \overline{Z}(r,g) + \overline{Z}(r,P(g)) \leq T(r,g) + T(r,P(g))$. But, by Lemma 4, we have $T(r,P(f)) = k\ T(r,f) + O(1)$ and $T(r,P(g)) = k\ T(r,g) + O(1)$. Consequently $\overline{Z}(r,F) \leq (k+1)T(r,f) + O(1)$ and $\overline{Z}(r,F-c) \leq (k+1)T(r,g) + O(1)$. Moreover, by the same Lemma 4, we have $T(r,F) = (n+k+1)T(r,f) + O(1)$.

Considering the previous expressions and applying Theorem N to F, we obtain

$$\begin{aligned}(n+k+1)T(r,f) \leq T(r,F) &\leq \overline{Z}(r,F) + \overline{Z}(r,F-c) + \overline{N}(r,F) - \log r + O(1)\\ &\leq (k+2)T(r,f) + (k+1)T(r,g) - \log r + O(1),\end{aligned}$$

this is,

$$(n-1)T(r,f) \leq (k+1)T(r,g) - \log r + O(1). \tag{4}$$

Since G satisfies the same hypothesis as F, similarly we have

$$(n-1)T(r,g) \leq (k+1)T(r,f) - \log r + O(1). \tag{5}$$

Thus, adding (4) and (5), we have

$$(n-1)\big[T(r,f)+T(r,g)\big] \leq (k+1)\big[T(r,f)+T(r,g)\big] - 2\log r + O(1),$$

a contradiction when $r \to +\infty$ and $n \geq k+2$ $\big($resp. when $r \to R^-$ and $n > k+2\big)$ because f and g are transcendental meromorphic functions in $\mathbb{K}$ $\big($resp. are unbounded meromorphic functions in $d(0,R^-)\big)$. Consequently, $c = 0$. $\square$

Now, when analytic functions are concerned, since $N(r,F) = N(r,G) = 0$, we derive Lemma 6.

Lemma 6. *Let $f, g \in \mathcal{A}(\mathbb{K})$ be transcendental $\big($resp. Let $f, g \in \mathcal{A}_u(d(0,R^-))\big)$, $a \in \mathbb{K}\setminus\{0\}$ and $n, k \in \mathbb{N}$ with $n \geq k+1$ $\big($ resp. $n \geq k+2\big)$. Let*

$$F = \frac{1}{n+k+1}f^{n+k+1} + \binom{k}{1}\frac{(-a)}{n+k}f^{n+k} + \ldots +$$

$$+ \binom{k}{k-1}\frac{(-a)^{k-1}}{n+2}f^{n+2} + \frac{(-a)^k}{n+1}f^{n+1}$$

and

$$G = \frac{1}{n+k+1}g^{n+k+1} + \binom{k}{1}\frac{(-a)}{n+k}g^{n+k} + \ldots +$$

$$+ \binom{k}{k-1}\frac{(-a)^{k-1}}{n+2}g^{n+2} + \frac{(-a)^k}{n+1}g^{n+1}.$$

If $F' = G'$, then $F = G$.

Lemma 7 is immediate:

Lemma 7. *Let $f \in \mathcal{M}(\mathbb{K})$ and $\alpha \in \mathcal{M}_f(\mathbb{K})$ $\big($resp. Let $f \in \mathcal{M}(d(0,R^-))$ and $\alpha \in \mathcal{M}_f(d(0,R^-))\big)$. Then α is a small function with respect to $f'(f-a)^k f^n$.*

Proof Indeed, on one hand, we have $N(r, f'(f-a)^k f^n) > N(r, f)$ and on the other hand, each zero of f is not a pole of f' hence $Z(r, f'(f-a)^k f^n) \geq Z(r, f)$.

Proof of Theorem 3. Since $f, g \in \mathcal{A}(\mathbb{K})$ and $f^n(f-a)^k f'$ and $g^n(g-a)^k g'$ share α C.M., then $\dfrac{f^n(f-a)^k f' - \alpha}{g^n(g-a)^k g' - \alpha}$ is a meromorphic function having no zeros and no poles in $\mathbb{K}$, hence it is a constant c in $\mathbb{K} \setminus \{0\}$.

Put $F = f^n(f-a)^k f'$ and suppose $c \neq 1$. Then,

$$F = c\big(g^n(g-a)^k g'\big) + \alpha(1-c). \tag{6}$$

Let $r > 0$. Since $\alpha(1-c) \in \mathcal{A}_f(\mathbb{K})$, by Lemma 7, we deduce that $\alpha(1-c) \in \mathcal{A}_F(\mathbb{K})$. So, applying Lemma 1 to F, we obtain

$$\begin{aligned} T(r, F) &\leq \overline{Z}(r, F) + \overline{Z}\big(r, F - \alpha(1-c)\big) + S_F(r) \qquad (7)\\ &= \overline{Z}(r, f^n) + \overline{Z}\big(r, (f-a)^k\big) + \overline{Z}(r, f') + \overline{Z}(r, g^n) + \overline{Z}(r, g-a) + \\ &+ \overline{Z}(r, g') + S_f(r) \\ &\leq T(r, f) + T(r, f-a) + T(r, f') + 3T(r, g) + S_f(r). \end{aligned}$$

But f is entire. So, $T(r, F) = nT(r, f) + kT(r, f-a) + T(r, f') + O(1)$. Thus, considering the above equality in Inequality (7), we have

$$(n+k-2)T(r, f) \leq 3T(r, g) + S_f(r). \tag{8}$$

Similarly, since g satisfies the same hypothesis as f, we can deduce that

$$(n+k-2)T(r, g) \leq 3T(r, f) + S_g(r). \tag{9}$$

Thereby, adding (8) and (9), we obtain

$$(n+k-2)\big[T(r, f) + T(r, g)\big] \leq 3\big[T(r, f) + T(r, g)\big] + S_f(r) + S_g(r),$$

a contradiction whenever $n + k \geq 6$ and $f, g \in \mathcal{A}(\mathbb{K})$ are transcendental. Thus $c = 1$. Consequently, by (6) and Lemma 6, we have

$$\begin{aligned} &\frac{1}{n+k+1} f^{n+k+1} + \binom{k}{1} \frac{(-a)}{n+k} f^{n+k} + \ldots + \frac{(-a)^k}{n+1} f^{n+1} = \\ &= \frac{1}{n+k+1} g^{n+k+1} + \binom{k}{1} \frac{(-a)}{n+k} g^{n+k} + \ldots + \frac{(-a)^k}{n+1} g^{n+1}, \end{aligned}$$

because $n \geq \max\{6-k,\ k+1\}$. Thus, the conclusion comes from Theorem 1.

In the case when $\alpha \in \mathbb{K} \setminus \{0\}$ and $f, g \in \mathcal{A}(\mathbb{K})$ are transcendental, we define F as in (6) and we suppose $c \in \mathbb{K} \setminus \{0, 1\}$. Since $\alpha(1-c) \in \mathbb{K} \setminus \{0\}$, we apply Theorem N to F. So, with a similar process to this in (7), we obtain

$$(n+k-2)T(r, f) \leq 3T(r, g) - \log r + O(1).$$

And, considering the function g, we obtain

$$(n+k-2)T(r,g) \leq 3T(r,f) - \log r + O(1).$$

Therefore, adding the two last inequalities, we have

$$(n+k-2)\big[T(r,f)+T(r,g)\big] \leq 3\big[T(r,f)+T(r,g)\big] - 2\log r + O(1),$$

a contradiction whenever $n \geq 5-k$ and $r \to +\infty$. Thus, the conclusion is obtained by considering $n \geq \max\{5-k, k+1\}$ in Lemma 6 and Theorem 1. $\square$

Proof of Theorem 4. Since $f, g \in \mathcal{A}(d(0,R^-))$ and $f^n(f-a)^k f'$ and $g^n(g-a)^k g'$ share α C.M., then $\dfrac{f^n(f-a)^k f' - \alpha}{g^n(g-a)^k g' - \alpha} = u(x)$ is a meromorphic function that has no zeros and no poles in $d(0,R^-)$ hence, $u(x)$ belongs to $\mathcal{M}_b(d(0,R^-))$.

Suppose $u \neq 1$. Then,

$$f^n(f-a)^k f' = u\big(g^n(g-a)^k g'\big) + \alpha(1-u). \tag{10}$$

Let $r \in]0,R[$. By Lemma 7, $\alpha(1-u)$ is a small function with respect to $f^n(f-a)^k f'$. Applying Lemma 1 to $f^n(f-a)^k f'$, we have

$$\begin{aligned} T(r, f^n(f-a)^k f') &\leq \overline{Z}(r, f^n(f-a)^k f') + \overline{Z}\big(r, f^n(f-a)^k f' - \alpha(1-u)\big) + S_F(r) \\ &\leq T(r,f) + T(r,f-a) + T(r,f') + 3T(r,g) + S_f(r). \end{aligned} \tag{11}$$

Since f and g are analytic functions, we have $T\big(r, f^n(f-a)^k f'\big) = nT(r,f) + kT(r,f-a) + T(r,f') + O(1)$.

Therefore, for $k=1$, Inequality (11) is reduced to

$$(n-1)T(r,f) \leq 3T(r,g) + S_f(r) \tag{12}$$

and, for $k=2$, is reduced to

$$nT(r,f) \leq 3T(r,g) + S_f(r). \tag{13}$$

Since f and g satisfies the same hypothesis, for $k=1$, we have again

$$(n-1)T(r,g) \leq 3T(r,f) + S_g(r) \tag{14}$$

and, for $k=2$,

$$nT(r,g) \leq 3T(r,f) + S_g(r). \tag{15}$$

Thereby, adding (12) and (14), we obtain

$$(n-1)\big[T(r,f)+T(r,g)\big] \leq 3\big[T(r,f)+T(r,g)\big] + S_f(r) + S_g(r),$$

and, adding (13) and (15), we have

$$n\big[T(r,f)+T(r,g)\big] \le 3\big[T(r,f)+T(r,g)\big] + S_f(r) + S_g(r).$$

So, for $k=1$, we have a contradiction when $n \ge 5$ and $r \to +\infty$. And, for $k=2$, we have a contradiction when $n \ge 4$ and $r \to +\infty$. Consequently $u=1$ and so, from (10), we have $f^n(f-a)^k f' = g^n(g-a)^k g'$. By Lemma 6, we have $\frac{1}{n+2}f^{n+2} + \frac{a}{n+1}f^{n+1} = \frac{1}{n+2}g^{n+2} + \frac{a}{n+1}g^{n+1}$ when $k=1$ and $\frac{1}{n+3}f^{n+3} + \frac{2a}{n+2}f^{n+2} + \frac{a^2}{n+1}f^{n+1} = \frac{1}{n+3}g^{n+3} + \frac{2a}{n+2}g^{n+2} + \frac{a^2}{n+1}g^{n+1}$ when $k=2$. Therefore, if $k=1$ or $k=2$, we can conclude that $f=g$ thanks to Theorem 1 or thanks to Theorem 2. $\square$

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Departamento de Matemática
Facultad de Ciencias Físicas y Matemáticas
Universidad de Concepción
Conception, Chile.
E-mail: `jacqojeda@udec.cl`