Uniqueness for ultrametric analytic functions
by
Jacqueline Ojeda

Abstract

Let $K$ be a complete algebraically closed $p$-adic field of characteristic zero and let $f, g$ be two meromorphic functions inside an open disc of $K$. We first study polynomials of uniqueness for such functions. Suppose now $f, g$ are entire functions on $K$. Let $a \in K \setminus \{0\}$ and $n, k \in \mathbb{N}$, with $k \geq 2$ and let $\alpha$ be a small entire function with respect to $f$ and $g$. If $f^n(f - a)^k f'$ and $g^n(g - a)^k g'$ share $\alpha$, counting multiplicities, with $n \geq \max\{6 - k, k + 1\}$ then $f = g$. If $\alpha \in K^*$ and if $n \geq \max\{5 - k, k + 1\}$ then $f = g$.

Let $f, g$ be unbounded analytic functions inside an open disk of $K$ and let $\alpha$ be a small function analytic inside the same disk. If $f^n(f - a)^2 f'$ and $g^n(g - a)^2 g'$ share $\alpha$ counting multiplicities, with $n \geq 4$, then $f = g$. If $f^n(f - a)^2 f'$ and $g^n(g - a)g'$ share $\alpha$ counting multiplicities, with $n \geq 5$, then $f = g$.

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1 Introduction and Main Results

Let $K$ be an algebraically closed field of characteristic zero, complete for an ultrametric absolute value denoted by $| \cdot |$.

We denote by $A(K)$ the $K$-algebra of entire functions in $K$, by $M(K)$ the field of meromorphic functions in $K$, i.e. the field of fractions of $A(K)$ and by $K(x)$ the field of rational functions.

Let $a \in K$ and $R \in ]0, +\infty[$. We denote by $d(a, R^-)$ the “open” disc $\{x \in K : |x - a| < R\}$. Similarly, we denote by $A(d(a, R^-))$ the set of analytic functions in $d(a, R^-)$, i.e. the $K$-algebra of power series $\sum_{n=0}^{\infty} a_n (x - a)^n$ converging in

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Let $P(x) = \frac{1}{n+k+1}x^{n+k+1} - \binom{k}{1} \frac{a}{n+k}x^{n+k} + \ldots + \left( \frac{k}{k-1} \right) \frac{(-a)^{k-1}}{n+2}x^{n+2} + \frac{(-a)^{k}}{n+1}x^{n+1}. \quad (1)$
Clearly
\[ P'(x) = x^{n+k} - \binom{k}{1} ax^{n+k-1} + \ldots + \binom{k}{k-1} (-a)^{k-1} x^{n+1} + (-a)^k x^n, \]
this is, \[ P'(x) = x^n (x - a)^k. \]

**Remark 1.** When \( \min\{c_1, c_2\} = 2 \), this kind of polynomials was introduced by Frank and Reinders in order to find the smallest URSCM ever found, for complex entire or meromorphic functions \([8]\). By \([4]\) and \([19]\), it is known that if \( \min\{c_1, c_2\} = 1 \), then \( P \) is not a polynomial of uniqueness for \( M(K) \). Indeed, up to an affine change of variable, \( P(y) \) may be reduced to \( y^n - y^{n-1} + c \) with \( c \in K \). Now, let \( h \in M_u(d(0, R^-)) \) and let us define \( f \) and \( g \) as: \( g = \frac{h^{n-1} - 1}{h^n - 1} \) and \( f = hg \). Then we can check that the polynomial \( P \) satisfies \( P(f) = P(g) \).

Now, we do not know whether Theorem A might be extended to \( M_u(d(0, R^-)) \). Indeed, the proof of Theorem A requires the use of algebraic curves theory on a p-adic field, particularly Picard-Berkovich Theory \([1]\) that does not apply to functions defined in \( M(d(0, R^-)) \). However, here we can state two theorems concerning \( A_u(d(0, R^-)) \) and \( M_u(d(0, R^-)) \) whose proof uses some methods similar to those of Theorem A in \([3]\).

**Theorem 1.** Let \( P \in K[x] \) of degree \( n \geq 3 \) be such that \( P' \) only has two distinct zeros, one of them being of order 1. Then \( P \) is a polynomial of uniqueness for \( A_u(d(0, R^-)) \).

**Theorem 2.** Let \( P \in K[x] \) of degree \( n \geq 6 \) be such that \( P' \) only has two distinct zeros, one of them being of order 2. Then \( P \) is a polynomial of uniqueness for \( M_u(d(0, R^-)) \).

**Example.** Let \( a \in K \setminus \{0\} \) and \( n \in \mathbb{N} \). Thanks to the previous theorems we can deduce that

(i) \( P(x) = \frac{1}{n+2} x^{n+2} - \frac{a}{n+1} x^{n+1} \) is a polynomial of uniqueness for \( A_u(d(0, R^-)) \).

(ii) \( P(x) = \frac{1}{n+3} x^{n+3} - \frac{2a}{n+2} x^{n+2} + \frac{a^2}{n+1} x^{n+1} \) is a polynomial of uniqueness for \( M_u(d(0, R^-)) \) whenever \( n \geq 3 \).

In \([17]\), we studied the uniqueness of a pair \((f, g)\) of meromorphic functions in \( K \) (resp. unbounded meromorphic functions in \( d(0, R^-) \)) such that \( f^n f' \) and \( g^n g' \) share one value counting multiplicities (C.M.) or ignoring multiplicities (I.M.). We proved, for example, that if \( f^n f' \) and \( g^n g' \) share one value C.M. with \( n \geq 11 \),
then \( f = dq \) with \( d^{n+1} = 1 \) whenever \( f, g \in \mathcal{M}(K) \). And if \( f^n f' \) and \( g^n g' \) share one value I.M. with \( n \geq 9 \), then \( f = dq \) with \( d^{n+1} = 1 \) whenever \( f, g \in \mathcal{A}_u(d(0, R^-)) \). Here we will study the following problem:

Let \( f, g \in \mathcal{A}(K) \) be transcendental (resp. \( f, g \in \mathcal{A}_u(d(0, R^-)) \)) and let \( \alpha \) be a small analytic function with respect to \( f \) and \( g \) in \( K \) (resp. in \( d(0, R^-) \)) such that \( f^n(f-a)^k f' \) and \( g^n(g-a)^k g' \) share \( \alpha \), counting multiplicities, with \( n \in \mathbb{N}^* \) and \( a \in K \setminus \{0\} \). Can we conclude that \( f = g \) ?

This kind of questions was studied in complex analysis in many papers concerning meromorphic functions or entire functions in \( C \), and let \( \omega_\gamma(f) = n \). If \( f \) has a pole of order \( n \) at \( \gamma \), we put \( \omega_\gamma(f) = -n \) and finally, if \( f(\gamma) \neq 0, \infty \), we put \( \omega_\gamma(f) = 0 \).

We denote by \( Z(r, f) \) the counting function of zeros of \( f \) in \( d(0, r) \), counting multiplicities, i.e. we set

\[
Z(r, f) = \sum_{\omega_\gamma(f) > 0, |\gamma| \leq r} \omega_\gamma(f)(\log r - \log |\gamma|).
\]

Similarly, we denote by \( \overline{Z}(r, f) \) the counting function of zeros of \( f \) in \( d(0, r) \), ignoring multiplicities, and set

\[
\overline{Z}(r, f) = \sum_{\omega_\gamma(f) > 0, |\gamma| \leq r} (\log r - \log |\gamma|).
\]

In the same way, we set \( N(r, f) = Z(r, \frac{1}{f}) \) (resp. \( \overline{N}(r, f) = \overline{Z}(r, \frac{1}{f}) \)) to denote the counting function of poles of \( f \) in \( d(0, r) \), counting multiplicities (resp. ignoring multiplicities).

For \( f \in \mathcal{M}(d(0, R^-)) \) having no zero and no pole at 0, the Nevanlinna function is defined by \( T(r, f) = \max \{ Z(r, f) + \log |f(0)|, N(r, f) \} \).

In this paper we will show two results using the p-adic Nevanlinna theory together with a few specific properties of ultrametric analytic functions or ultrametric meromorphic functions. We begin introducing the following definitions.

In order to go on, we must recall the definition of a small function with respect to a meromorphic function and some pertinent properties.

**Definition.** Let \( f \in \mathcal{M}(K) \) (resp. \( f \in \mathcal{M}(d(0, R^-)) \)) such that \( f(0) \neq 0, \infty \). A function \( \alpha \in \mathcal{M}(K) \) (resp. \( \alpha \in \mathcal{M}(d(0, R^-)) \) having no zero and no pole
at 0 is called a small function with respect to \( f \), if it satisfies \( \lim_{r \to \infty} T(r, \alpha) = 0 \) (resp. \( \lim_{r \to 0} T(r, \alpha) = 0 \)).

If 0 is a zero or a pole of \( f \) or \( \alpha \), we can make a change of variable such that the new origin is not a zero or a pole for both \( f \) and \( \alpha \). Thus it is easily seen that the last relation do not really depend on the origin.

We denote by \( \mathcal{M}_f(K) \) (resp. \( \mathcal{M}_f(d(0, R^-)) \)) the set of small meromorphic functions with respect to \( f \) in \( K \) (resp. in \( d(0, R^-) \)).

**Remark 2.** Thanks to classical properties of the Nevanlinna function \( T(r, f) \) with respect to the operations in a field of meromorphic functions, such as \( T(r, f + g) \leq T(r, f) + T(r, g) \) and \( T(r, fg) \leq T(r, f) + T(r, g) \) for \( f, g \in M(K) \) and \( r > 0 \), we easily proved in [7] that \( \mathcal{M}_f(K) \) (resp. \( \mathcal{M}_f(d(0, R^-)) \)) is a subfield of \( \mathcal{M}(K) \) (resp. \( \mathcal{M}(d(0, R^-)) \)) and \( \mathcal{M}(K) \) (resp. \( \mathcal{M}(d(0, R^-)) \)) is a transcendental extension of \( \mathcal{M}_f(K) \) (resp. \( \mathcal{M}_f(d(0, R^-)) \)).

Now, we can give some sufficient conditions to get a positive answer to our question. Let us remember the following definition.

**Definition.** Let \( f, g, \alpha \in M(K) \) (resp. \( f, g, \alpha \in M(d(0, R^-)) \)). We say that \( f \) and \( g \) share the function \( \alpha \) C.M., if \( f - \alpha \) and \( g - \alpha \) have the same zeros with the same multiplicities in \( K \) (resp. in \( d(0, R^-) \)).

**Theorem 3.** Let \( f, g \in A(K) \) be transcendental such that \( f^n(f - a)^k f' \) and \( g^n(g - a)^k g' \) share the function \( \alpha \in A_f(K) \cap A_g(K) \) C.M. when \( n, k \in \mathbb{N} \) and \( a \in K \setminus \{0\} \). If \( n \geq \max\{6 - k, k + 1\} \), then \( f = g \). Moreover, if \( \alpha \in K \setminus \{0\} \) and \( n \geq \max\{5 - k, k + 1\} \), then \( f = g \).

**Theorem 4.** Let \( f, g \in A_c(d(0, R^-)) \), \( \alpha \in A_f(d(0, R^-)) \cap A_g(d(0, R^-)) \) and let \( a \in K \setminus \{0\} \). If \( f^n(f - a)^2 f' \) and \( g^n(g - a)^2 g' \) share the function \( \alpha \) C.M. and \( n \geq 4 \), then \( f = g \). Moreover, if \( f^n(f - a)^2 f' \) and \( g^n(g - a)^2 g' \) share the function \( \alpha \) C.M. and \( n \geq 5 \), then again \( f = g \).

2 Basic Results and Proofs of Theorems

We have to recall the ultrametric Nevanlinna second main Theorem in a basic form which we will frequently use.

Let \( f \in M(K) \) (resp. \( f \in M(d(0, R^-)) \)) satisfy \( f'(0) \neq 0, \infty \). Let \( S \) be a finite subset of \( K \) and \( r \in [0, +\infty) \) (resp. \( r \in [0, R]\)). We denote by \( Z^s_0(r, f') \) the counting function of zeros of \( f' \) in \( d(0, r) \) which are not zeros of any \( f - s \) for \( s \in S \). This is, if \( (\gamma_n)_{n \in \mathbb{N}} \) is the finite or infinite sequence of zeros of \( f' \) in \( d(0, r) \) that are not zeros of \( f - s \) for \( s \in S \), with multiplicity order \( q_n \), respectively, we set

\[
Z^s_0(r, f') = \sum_{|\gamma_n| \leq r} q_n (\log r - \log |\gamma_n|).
\]
Theorem N. ([2], [6]) Let $a_1, ..., a_n \in \mathbb{K}$ with $n \geq 2$ an entire, and let $f \in \mathcal{M}(\mathbb{K})$ (resp. let $f \in \mathcal{M}(d(0, R^-))$). Let $S = \{a_1, ..., a_n\}$. Assume that none of $f$, $f'$ and $f - a_j$ with $1 \leq j \leq n$, equals 0 or $\infty$ at the origin. Then, for $r > 0$ (resp. for $r \in (0, R]$), we have

$$(n - 1)T(r, f) \leq \sum_{j=1}^{n} Z(r, f - a_j) + \mathcal{N}(r, f) - Z_0^S(r, f') - \log r + O(1).$$

A special Nevanlinna Theorem is known deriving from the Nevanlinna Theorem on 3 small functions [18]:

**Lemma 1.** Let $f \in \mathcal{M}(\mathbb{K}) \setminus \{0\}$ non identically zero (resp. $f \in \mathcal{M}(d(0, R^-))$) and let $\alpha \in \mathcal{M}_f(\mathbb{K})$ (resp. $\alpha \in \mathcal{M}_f(d(0, R^-))$) have no zero and no pole at 0. Then for $r > 0$ (resp. for $r \in (0, R]$), we have $T(r, f) \leq Z(r, f) + Z(r, f - \alpha) + \mathcal{N}(r, f) + S_f(r)$.

Lemma 2 is well known (see for instance Theorem 2.4.5 [6])

**Lemma 2.** Let $f \in \mathcal{M}(\mathbb{K})$ not constant (resp. $f \in \mathcal{M}_u(d(0, R^-))$). There exists at most one value $b \in \mathbb{K}$ such that $f(x) \neq b \forall x \in \mathbb{K}$ (resp. $f(x) \neq b \forall x \in d(0, R^-)$).

In order to prove Theorem 2, we will use the following lemma that is known in complex analysis [8] and that, in general, is valid for an algebraically closed field of characteristic zero such as $\mathbb{K}$.

**Lemma 3.** Let $n \geq 3$ be an integer and let

$$Q(X) = (n - 1)^2(X^n - 1)(X^{n-2} - 1) - n(n - 2)(X^{n-1} - 1)^2$$

be a polynomial with coefficient in $\mathbb{K}$ of degree $2n - 2$. Then 1 is a zero of $Q$ of multiplicity order 4 and all the other zeros of $Q$ are simple.

**Proof of Theorems 1 and 2.** Let $f, g \in \mathcal{M}_u(d(0, R^-))$ be such that $P(f) = P(g)$. Let $f = hg$.

Suppose first we are in the hypothesis of Theorem 1. By Remark 1, we have $g = h^{n-1} - 1$. Each pole of $h^{n-1} - 1$ is a zero of $g$ of same order, hence $h^{n-1} - 1$ belongs to $\mathcal{M}_u(d(0, R^-))$ and so does $h$. Consequently, by Lemma 2, $h$ avoids at most one value. Now we know that the $n$-th roots of unity and the $n - 1$-th roots of unity have no common element but 1. Consequently, since $n \geq 3$, $h^{n-1} - 1$ admits at least one pole, a contradiction because $g$ is analytic in $d(0, R^-)$.
We now place ourselves in the hypothesis of Theorem 2. From the hypothesis \( P(f) = P(g) \) we can derive

\[
(n - 1)(n - 2)(h^n - 1)g^2 - 2an(n - 2)(h^{n-1} - 1)g + a^2n(n - 1)(h^{n-2} - 1) = 0. \tag{2}
\]

Suppose that \( h \) is not a constant. Let \( r \in ]0, R[ \). Considering the previous expression we can easily deduce that \( h \) is unbounded in \( d(0, R -) \), because if \( h \) is bounded we have

\[
T(r, (n - 1)(n - 2)(h^n - 1)g^2) \geq 2T(r, g) + O(1)
\]

and

\[
T(r, 2an(n - 2)(h^{n-1} - 1)g - a^2n(n - 1)(h^{n-2} - 1)) \leq T(r, g) + O(1),
\]

a contradiction to (2).

On the other hand, by simple calculations, we can write (2) as

\[
\left( (n - 1)(n - 2)(h^n - 1)g - an(n - 2)(h^{n-1} - 1) \right)^2 = -a^2n(n - 2)Q(h) \tag{3}
\]

where \( Q(h) = (n - 1)^2(h^n - 1)(h^{n-2} - 1) - n(n - 2)(h^{n-1} - 1)^2 \) is a polynomial of degree \( 2n - 2 \). Since \( n \geq 6 \), by Lemma 3, we deduce that \( Q(h) \) is of the form \( (h - 1)^2(h - \gamma_1)(h - \gamma_2)...(h - \gamma_{2n-6}) \), where every \( \gamma_i \in K \setminus \{0, 1\} \) \((i = 1, ..., 2n - 6)\), is a simple zero of \( Q \). Now, from (3), every zero of \( h - \gamma_i \) \((i = 1, ..., 2n - 6)\), has multiplicity at least 2. Assume, without loss of generality, that 0 is neither a zero nor a pole of \( h - \gamma_i \) \((i = 1, ..., 2n - 6)\). Then,

\[
\sum_{i=1}^{2n-6} Z(r, h - \gamma_i) \leq \frac{1}{2} \sum_{i=1}^{2n-6} Z(r, h - \gamma_i) \leq (n - 3)T(r, h) + O(1).
\]

Thereby, applying Theorem N to \( h \) at the points \( \gamma_i \) \((i = 1, ..., 2n - 6)\), and considering that \( N(r, h) \leq T(r, h) \), we obtain

\[
(2n - 7)T(r, h) \leq \sum_{i=1}^{2n-6} Z(r, h - \gamma_i) + N(r, h) + O(1)
\]

\[
\leq (n - 2)T(r, h) + O(1).
\]

Since \( T(r, h) \) is unbounded in \([0, R[\), we have a contradiction whenever \( n \geq 6 \). Hence, \( h \) is a constant. Therefore, by (3), we have \( h^n - 1 = 0 \) and \( h^{n-1} - 1 = 0 \). It follows that \( h = 1 \) and hence \( f = g \). \( \square \)

Let us recall this classical lemma [12]:

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Lemma 4. Let $f \in M(\mathbb{K})$ and $\alpha_i \in M_f(\mathbb{K})$ (resp. $f \in M(d(0,R^-))$ and $\alpha_i \in M_f(d(0,R^-))$) $(i = 1,\ldots,n)$, be such that $f(0) \neq 0, \infty$ and $\alpha_i(0) \neq 0, \infty$ $(i = 1,\ldots,n)$. If $P(X) = \sum_{i=0}^{n} \alpha_i X^i \in M_f(\mathbb{K})[X]$ (resp. $P(X) = \sum_{i=0}^{n} \alpha_i X^i \in M_f(d(0,R^-))[X]$), then for $r > 0$ (resp. $r \in [0,R]$), we have $T(r, P(f)) = nT(r,f) + S_f(r)$.

Moreover, if $\alpha_i \in \mathbb{K}$ $(i = 1,\ldots,n)$, then for $r > 0$ (resp. $r \in [0,R]$), we have $T(r, P(f)) = nT(r,f) + O(1)$.

Lemma 5. Let $f, g \in M(\mathbb{K})$ be transcendental (resp. $f, g \in M_n(d(0,R^-))$), $a \in \mathbb{K} \setminus \{0\}$ and $n, k \in \mathbb{N}$ with $n \geq k + 2$ (resp. $n \geq k + 3$). Let

$$F = \frac{1}{n + k + 1} f^{n+k+1} + \binom{k}{1} \frac{(-a)}{n+k} f^{n+k} + \ldots +$$

$$+ \binom{k}{k-1} \frac{(-a)^{k-1}}{n+2} f^{n+2} + \binom{k}{n+1} \frac{(-a)^{k}}{f^{n+1}}$$

and

$$G = \frac{1}{n + k + 1} g^{n+k+1} + \binom{k}{1} \frac{(-a)}{n+k} g^{n+k} + \ldots +$$

$$+ \binom{k}{k-1} \frac{(-a)^{k-1}}{n+2} g^{n+2} + \binom{k}{n+1} \frac{(-a)^{k}}{g^{n+1}}.$$ 

If $F' = G'$, then $F = G$.

Proof: Note that $F \equiv f^{n+1} P(f)$ with $P \in \mathbb{K}[x]$ and $deg(P) = k$. Let $r > 0$ (resp. $r \in [0,R]$). We have $\mathcal{Z}(r,F) \leq \mathcal{Z}(r,f) + \mathcal{Z}(r,P(f)) \leq T(r,f) + T(r,P(f))$ and $\mathcal{N}(r,F) = \mathcal{N}(r,f) \leq T(r,f)$. Moreover, since $F' = G'$, $F - G$ is a constant $c$.

Suppose $c \neq 0$. Then $\mathcal{Z}(r,F - c) = \mathcal{Z}(r,G) \leq \mathcal{Z}(r,g) + \mathcal{Z}(r,P(g)) \leq T(r,g) + T(r,P(g))$. But, by Lemma 4, we have $T(r,P(f)) = k \cdot T(r,f) + O(1)$ and $T(r,P(g)) = k \cdot T(r,g) + O(1)$. Consequently $\mathcal{Z}(r,F) \leq (k + 1)T(r,f) + O(1)$ and $\mathcal{Z}(r,F - c) \leq (k + 1)T(r,g) + O(1)$. Moreover, by the same Lemma 4, we have $T(r,F) = (n + k + 1)T(r,f) + O(1)$.

Considering the previous expressions and applying Theorem N to $F$, we obtain

$$(n + k + 1)T(r,f) \leq T(r,F) \leq \mathcal{Z}(r,F) + \mathcal{Z}(r,F - c) + \mathcal{N}(r,F) - \log r + O(1)$$

$$\leq (k + 2)T(r,f) + (k + 1)T(r,g) - \log r + O(1),$$
this is,

\[(n - 1)T(r, f) \leq (k + 1)T(r, g) - \log r + O(1).\] (4)

Since \(G\) satisfies the same hypothesis as \(F\), similarly we have

\[(n - 1)T(r, g) \leq (k + 1)T(r, f) - \log r + O(1).\] (5)

Thus, adding (4) and (5), we have

\[\[(n - 1)\left[T(r, f) + T(r, g)\right] \leq (k + 1)\left[T(r, f) + T(r, g)\right] - 2\log r + O(1),\]

a contradiction when \(r \to +\infty\) and \(n \geq k + 2\) (resp. when \(r \to R^-\) and \(n > k + 2\)) because \(f\) and \(g\) are transcendental meromorphic functions in \(K\) (resp. are unbounded meromorphic functions in \(d(0, R^-)\)). Consequently, \(c = 0\). \qed

Now, when analytic functions are concerned, since \(N(r, F) = N(r, G) = 0\), we derive Lemma 6.

**Lemma 6.** Let \(f, g \in A(K)\) be transcendental (resp. Let \(f, g \in A_a(d(0, R^-))\)), \(a \in K \setminus \{0\}\) and \(n, k \in \mathbb{N}\) with \(n \geq k + 1\) (resp. \(n \geq k + 2\)). Let

\[F = \frac{1}{n + k + 1}f^{n+k+1} + \binom{k}{1} \frac{(-a)}{n+k}f^{n+k} + \ldots + \]

\[+ \left(\frac{k}{k-1}\right) \frac{(-a)^{k-1}}{n+2}f^{n+2} + \frac{(-a)^k}{n+1}f^{n+1}\]

and

\[G = \frac{1}{n + k + 1}g^{n+k+1} + \binom{k}{1} \frac{(-a)}{n+k}g^{n+k} + \ldots + \]

\[+ \left(\frac{k}{k-1}\right) \frac{(-a)^{k-1}}{n+2}g^{n+2} + \frac{(-a)^k}{n+1}g^{n+1}.\]

If \(F' = G'\), then \(F = G\).

Lemma 7 is immediate:

**Lemma 7.** Let \(f \in M(K)\) and \(\alpha \in M_f(K)\) (resp. Let \(f \in M(d(0, R^-))\) and \(\alpha \in M_f(d(0, R^-))\)). Then \(\alpha\) is a small function with respect to \(f'(f - a)^k f^n\).
Proof. Indeed, on one hand, we have $N(r, f'(f - a)^k f^n) > N(r, f)$ and on the other hand, each zero of $f$ is not a pole of $f'$ hence $Z(r, f'(f - a)^k f^n) \geq Z(r, f)$.

Proof of Theorem 3. Since $f, g \in \mathcal{A}(\mathbb{K})$ and $f^n(f - a)^k f'$ and $g^n(g - a)^k g'$ share $\alpha$ C.M., then $\frac{f^n(f - a)^k f' - \alpha}{g^n(g - a)^k g' - \alpha}$ is a meromorphic function having no zeros and no poles in $\mathbb{K}$, hence it is a constant $c$ in $\mathbb{K} \setminus \{0\}$.

Put $F = f^n(f - a)^k f'$ and suppose $c \neq 1$. Then,

$$F = c(g^n(g - a)^k g') + \alpha(1 - c). \quad (6)$$

Let $r > 0$. Since $\alpha(1 - c) \in \mathcal{A}_f(\mathbb{K})$, by Lemma 7, we deduce that $\alpha(1 - c) \in \mathcal{A}_f(\mathbb{K})$. So, applying Lemma 1 to $F$, we obtain

$$T(r, F) \leq Z(r, F) + Z(r, F - \alpha(1 - c)) + S_f(r) \quad (7)$$

$$= Z(r, f^n) + Z(r, (f - a)^k) + Z(r, f') + Z(r, g^n) + Z(r, g - a) + Z(r, g') + S_f(r)$$

$$\leq T(r, f) + T(r, f - a) + T(r, f') + 3T(r, g) + S_f(r).$$

But $f$ is entire. So, $T(r, F) = nT(r, f) + kT(r, f - a) + T(r, f') + O(1)$. Thus, considering the above equality in Inequality (7), we have

$$(n + k - 2)T(r, f) \leq 3T(r, g) + S_f(r). \quad (8)$$

Similarly, since $g$ satisfies the same hypothesis as $f$, we can deduce that

$$(n + k - 2)T(r, g) \leq 3T(r, f) + S_g(r). \quad (9)$$

Thereby, adding (8) and (9), we obtain

$$(n + k - 2)[T(r, f) + T(r, g)] \leq 3[T(r, f) + T(r, g)] + S_f(r) + S_g(r),$$

a contradiction whenever $n + k \geq 6$ and $f, g \in \mathcal{A}(\mathbb{K})$ are transcendental. Thus $c = 1$. Consequently, by (6) and Lemma 6, we have

$$\frac{1}{n + k + 1} f^{n+k+1} + \left(\frac{k}{1}\right) \frac{(-a)^n}{n+k} f^{n+k} + \ldots + \frac{(-a)^k}{n+1} f^{n+1} =$$

$$= \frac{1}{n + k + 1} g^{n+k+1} + \left(\frac{k}{1}\right) \frac{(-a)^n}{n+k} g^{n+k} + \ldots + \frac{(-a)^k}{n+1} g^{n+1},$$

because $n \geq \max\{6 - k, k + 1\}$. Thus, the conclusion comes from Theorem 1.

In the case when $a \in \mathbb{K} \setminus \{0\}$ and $f, g \in \mathcal{A}(\mathbb{K})$ are transcendental, we define $F$ as in (6) and we suppose $c \in \mathbb{K} \setminus \{0, 1\}$. Since $\alpha(1 - c) \in \mathbb{K} \setminus \{0, 1\}$, we apply Theorem N to $F$. So, with a similar process to this in (7), we obtain

$$(n + k - 2)T(r, f) \leq 3T(r, g) - \log r + O(1).$$
And, considering the function $g$, we obtain
\[(n + k - 2)T(r, g) \leq 3T(r, f) - \log r + O(1).\]

Therefore, adding the two last inequalities, we have
\[(n + k - 2)[T(r, f) + T(r, g)] \leq 3[T(r, f) + T(r, g)] - 2\log r + O(1),\]
a contradiction whenever $n \geq 5 - k$ and $r \to +\infty$. Thus, the conclusion is obtained by considering $n \geq \max\{5 - k, k + 1\}$ in Lemma 6 and Theorem 1. \(
\square
\)

**Proof of Theorem 4.** Since $f, g \in \mathcal{A}(d(0, R^-))$ and $f^\alpha(f - a)^k f'$ and $g^\alpha(g - a)^k g'$ share $\alpha$ C.M., then $f^\alpha(f - a)^k f' - \alpha = u(x)$ is a meromorphic function that has no zeros and no poles in $d(0, R^-)$ hence, $u(x)$ belongs to $\mathcal{M}_b(d(0, R^-))$.

Suppose $u \neq 1$. Then, 
\[f^\alpha(f - a)^k f' = u(g^\alpha(g - a)^k g') + \alpha(1 - u).\] (10)

Let $r \in [0, R]$. By Lemma 7, $\alpha(1 - u)$ is a small function with respect to $f^\alpha(f - a)^k f'$. Applying Lemma 1 to $f^\alpha(f - a)^k f'$, we have
\[T(r, f^\alpha(f - a)^k f') \leq Z(r, f^\alpha(f - a)^k f') + Z(r, g^\alpha(g - a)^k g' - \alpha(1 - u)) + S_F(r)\] (11)
\[\leq T(r, f) + T(r, f - a) + T(r, f') + 3T(r, g) + S_f(r).\]

Since $f$ and $g$ are analytic functions, we have $T(r, f^\alpha(f - a)^k f') = nT(r, f) + kT(r, f - a) + T(r, f') + O(1)$.

Therefore, for $k = 1$, Inequality (11) is reduced to
\[(n - 1)T(r, f) \leq 3T(r, g) + S_f(r)\] (12)
and, for $k = 2$, is reduced to
\[nT(r, f) \leq 3T(r, g) + S_f(r)\] (13)

Since $f$ and $g$ satisfies the same hypothesis, for $k = 1$, we have again
\[(n - 1)T(r, g) \leq 3T(r, f) + S_g(r)\] (14)
and, for $k = 2$,
\[nT(r, g) \leq 3T(r, f) + S_g(r)\] (15)

Thereby, adding (12) and (14), we obtain
\[(n - 1)[T(r, f) + T(r, g)] \leq 3[T(r, f) + T(r, g)] + S_f(r) + S_g(r),\]
and, adding (13) and (15), we have
\[ n[T(r, f) + T(r, g)] \leq 3[T(r, f) + T(r, g)] + S_f(r) + S_g(r). \]

So, for \( k = 1 \), we have a contradiction when \( n \geq 5 \) and \( r \to +\infty \). And, for \( k = 2 \), we have a contradiction when \( n \geq 4 \) and \( r \to +\infty \). Consequently \( u = 1 \) and so, from (10), we have \( f^n(f - a)^k f' = g^n(g - a)^k g' \). By Lemma 6, we have \( \frac{1}{n+2} f^{n+2} + \frac{a}{n+1} f^{n+1} = \frac{1}{n+2} g^{n+2} + \frac{a}{n+1} g^{n+1} \) when \( k = 1 \) and \( \frac{1}{n+3} f^{n+3} + \frac{2a}{n+2} f^{n+2} + \frac{a^2}{n+1} f^{n+1} = \frac{1}{n+3} g^{n+3} + \frac{2a}{n+2} g^{n+2} + \frac{a^2}{n+1} g^{n+1} \) when \( k = 2 \). Therefore, if \( k = 1 \) or \( k = 2 \), we can conclude that \( f = g \) thanks to Theorem 1 or thanks to Theorem 2. \( \square \)

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**References**


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Departamento de Matemática
Facultad de Ciencias Físicas y Matemáticas
Universidad de Concepción
Conception, Chile.
E-mail: jacqojeda@udec.cl