On the fractal dimension of a nowhere differentiable basin boundary

by

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Abstract

In this work, we study the behavior of a two-dimensional endomorphism which admits only two attractors located at infinite distance. Using the analytical expression of their common basin boundary, its fractal dimension is numerically calculated by a variant of the Box-counting algorithm. The log-log plots are presented and the obtained numerical results are compared with the literature.

Key Words: Discrete dynamical system, Basin boundary, Fractal dimension.

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1 Introduction

The properties of the basin boundaries of the attractors have been studied since a long time in the holomorphic case (see for instance [17], [3], [2], and references therein).

In the area of dynamics of non-invertible maps, many authors ([18], [12], [13], [14], [15]) have investigated and emphasized the great importance of attractors, fractal dimensions and basin boundaries. When a discrete dynamical system is governed by a non-invertible two-dimensional map which depends on one (or more) parameter(s), Mira et al ([16]) and Lopez et al ([10]) have used the concept of critical curve to identify the relations between the attractive basins, the basin boundaries and the appearance/disappearance of their attractors by global bifurcations.

In this paper, we consider an endomorphism, called "Kaplan-Yorke map" ([7]), which depends continuously on two parameters. The particularity of this system is that it admits only two attractors at infinite distance. For different
values of the parameters, we shall try to highlight the importance and the structure of the boundary $F$ of both attractors. In each case, the fractal dimension of $F$, in fact the capacity dimension ($d_{cap}$) and the information dimension ($d_{inf}$), is numerically calculated using a variant of the Box-counting algorithm ([5], [1]). The numerical results are compared with those proposed by other authors ([11], [8]).

2 The KAPLAN-YORKE Map

Let $E = [0, 1] \times \mathbb{R}$, the system is generated by the continuous map (proposed by Kaplan and Yorke ([7])):

$$f : E \rightarrow E$$

$$(x, y) \mapsto (a \cdot x \mod 1, b \cdot y + \cos(2 \cdot \pi \cdot x))$$

where $(a \in \mathbb{N}) > (b \in \mathbb{R}) > 1$.

2.1 Stability of the fixed and periodic points of $f$

To study the stability of a fixed point $(x, y)$ of the transformation $f$, we must consider the eigenvalues of its Jacobian matrix. This latter has the lower-triangular form

$$J(x, y) = \begin{pmatrix} a & 0 \\ -2 \cdot \pi \cdot \sin(2 \cdot \pi \cdot x) & b \end{pmatrix}$$

We see that all the stationary points have the same eigenvalues $a$ and $b$. Both values are greater than one, therefore all the fixed points of our system are repulsive nodes. By the chain rule of differentials, it follows that the periodic points (or cycles) are also repulsive. We conclude that the Kaplan-Yorke map does not possess attractors, at least at finite distance.

2.2 Other attractors of $f$

Let $X(0) = (x_0, y_0) \in E$ with large $|y_0|$, the iterates $X^{(k+1)} = f(X^{(k)}) = (x_k, y_k)$, $k = 0, 1, 2, ..., $ take the general form

$$X^{(k+1)} = (u_k, b^{k+1} \cdot y_0 + b^k \cdot \cos(2 \cdot \pi \cdot x_k) + ...)$$

where $u_k \in [0, 1]$. We remark that $X^{(k+1)}$ converge towards $\pm \infty$ in the $y$-direction, depending on the sign of $y_0$. This observation allows us to establish that the discrete dynamical system $(E, N, f)$ admits only the two points $(y^- = -\infty)$ and $(y^+ = +\infty)$ as attractors. In the rest of this study, $F$ will denote the boundary of their basins.

By definition, the set $F$ attracts the inverse orbits of all the points which lie in its neighborhood; Mc Donald et al ([11]) have used this important property to
exhibit the analytical expression of the curve $F$, which is given by the following function $g$

$$g : [0, 1] \rightarrow \mathbb{R} \quad (2.3)$$

$$x \mapsto -\sum_{j=1}^{\infty} \frac{\cos(2 \cdot \pi \cdot a^{j-1} \cdot x)}{b^j}$$

This function verifies clearly the equality $g(t) = g(1-t)$ for all $t \in [0, 1]$, then the curve $F$ is symmetric with respect to $x = \frac{1}{2}$.

### 2.3 Properties of $g$

As the parameter $b$ is greater than one, the sum (2.3) is absolutely and uniformly convergent, then the function $g$ is continuous and the boundary $F$ is a continuous curve. On the other hand, we have

$$g'(x) = \frac{dg(x)}{dx} = \frac{2 \cdot \pi}{a} \cdot \sum_{j=1}^{\infty} \left(\frac{a}{b}\right)^j \cdot \sin(2 \cdot \pi \cdot a^{j-1} \cdot x)$$

since $a > b > 1$, so the formal sum of $g'$ diverges. As a consequence, the set $F$ is a nowhere differentiable basin boundary of the system governed by the map $f$.

Note that the Takagi functions, studied by Dubuc et al ([4]), present the same geometrical properties as the curve $F$.

### 2.4 Estimation of fractal dimensions : a variant of the Box-counting algorithm

#### 2.4.1 Definitions of the capacity and the information dimensions

Let $U$ be a bounded subset of the plane (chaotic attractor of a two dimensional dissipative dynamical system, Julia set of a polynomial, etc.). In this subsection, we briefly recall the definitions of the capacity dimension and the information dimension of $U$.

By covering this set with a grid of boxes of length $\epsilon$, the capacity dimension $d_{\text{cap}}$ is defined by

$$d_{\text{cap}}(U) = \lim_{\epsilon \to 0} \frac{-\ln(T(\epsilon))}{\ln(\epsilon)}$$

where $T(\epsilon)$ is the number of non-empty boxes needed to completely cover the set $U$.

If we note by $P_i(\epsilon)$, $1 \leq i \leq T(\epsilon)$, the probability that the $i^{th}$ box of the partitioning contains at least one point of $U$, then the information dimension is given by
\[ d_{\text{inf}}(U) = \lim_{\epsilon \to 0} \frac{H(\epsilon)}{\ln(\epsilon)} \]

where

\[ H(\epsilon) = -\sum_{i=1}^{T(\epsilon)} P_i(\epsilon) \ln(P_i(\epsilon)) \]

In numerical experiments, \( d_{\text{cap}} \) (resp. \( d_{\text{inf}} \)) corresponds to the slope of the plot \(-\ln(T(\epsilon))\) (resp. \( H(\epsilon) \)) versus \( \ln(\epsilon) \) (see [6] for further details).

### 2.4.2 A new approach of the Box-counting algorithm

This part is devoted to the presentation of a new approach of the Box-counting algorithm in order to estimate \( d_{\text{cap}}(U) \) and \( d_{\text{inf}}(U) \). Hereafter, we assume that the set \( U \) consists of \( N \) points.

#### 1- The preprocessing step

Consider a square domain \( D = [X_1, X_2] \times [Y_1, Y_2] \) which contains the \( N \) points of \( U \). We begin by dividing \( D \) into \( n^2 \) boxes of side \( \epsilon = (X_2 - X_1)/n \). Preprocess the \( N \) previous points according to this fixed grid is a procedure which allows us to report “rapidly” the points of \( (U \cap r) \) for any box \( r \) of the grid ([9]). Therefore, one obtains the \( n^2 \) integers \( \text{card}(U \cap r_{ij}(\epsilon)) = |r_{ij}(\epsilon)|, (i, j) \in ([1, n])^2 \), where \( (i, j) \) is the position of a box \( r(\epsilon) \) on the grid.

#### 2- A variant of the Box-counting algorithm

We choose a number \( n = n_1 \) of subdivisions along the interval \([X_1, X_2]\) (and \([Y_1, Y_2]\)) such that \( n_1 \) admits a maximal number \( p \) of dividers \( (n_s)_{s=1..p} \), and we associate them the sides \( \epsilon_s = (X_2 - X_1)/n_s \).

The main idea of our method can be summarized as follows : the preprocessing operation is carried out only for the grid of side \( \epsilon_1 \), this gives the quantities \( (|r_{ij}(\epsilon_1)|, (i, j) \in ([1, n_1])^2 \) and, consequently, \( T(\epsilon_1) \) and \( H(\epsilon_1) \) (see the above notations). The particular choice of the sides \( (\epsilon_s)_{s=1..p} \) allows us to calculate, without any other test on the \( N \) points of the set \( U \), all the remainder quantities \( (|r_{ij}(\epsilon_s)|), (i, j) \in ([1, n_s])^2, T(\epsilon_s) \) and \( H(\epsilon_s) \) \( s = 2..p \).

With these results, we can now construct the two log \( - \) log plots to determine the numerical values \( d_{\text{cap}}(U) \) and \( d_{\text{inf}}(U) \).

**Remark 1.** Kaplan et al ([8]) show that the theoretical capacity dimension \( d_{\text{cap}} \) of the boundary \( F \) is given by the relation

\[ d_{\text{cap}} = 2 - \frac{\ln(b)}{\ln(a)} \quad (2.4) \]
This last formula is derived from the application of the well-known Kaplan-Yorke’s conjecture ([7]). In this context, we recall that $a$ and $b$ are both positive (see eq. 2.1) and are the eigenvalues of the Jacobian matrix of $f$ (see eq. 2.2).

### 3 Numerical and graphical results

#### 3.1 Use of a variant of the Box-counting algorithm

In this section, we apply the variant of the Box-counting algorithm (presented above and detailed in ([1])) to estimate the capacity dimension $d_{cap}$ and the information dimension $d_{inf}$ of a set which approach the boundary $F$.

The procedure is as follows: Let $N$ a large integer. For a fixed pair $(a, b)$ of the parameters (see eq. 2.1), we calculate $(N + 1)$ points $(x, g(x))$ (see eq. 2.3) where $x$ starts from 0 and is increased with a step $(1/N)$. We assume that the set $\{(x, g(x)), x \in [0, 1]\}$ is a reasonable approximation of the boundary $F$ and the associated picture is shown. The previous algorithm is then applied to this set and the numerical values of $d_{cap}$ and $d_{inf}$ are reported in a table and compared to the theoretical value of the capacity dimension (see eq. 2.4). According to the literature, the associated log–log plots ([5], [6]) are also depicted for one pair $(a, b)$.

#### 3.2 Numerical results

For $N = 2 \times 10^5$ points $(x, g(x))$ - (see eq. 2.3) - and six distinct pairs $(a, b)$, the numerical values of $d_{cap}$ and $d_{inf}$ given by our algorithm ([1]) are listed in the table below. The fifth column contains the theoretical value $d_{cap}$ (see eq. 2.4) proposed by Kaplan et al ([8]), and in the last column we refer to the corresponding figure.

<table>
<thead>
<tr>
<th>$a$</th>
<th>$b$</th>
<th>$d_{cap}$</th>
<th>$d_{inf}$</th>
<th>$d_{cap}$ (eq. 2.4)</th>
<th>the boundary $F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.4</td>
<td>1.49</td>
<td>1.46</td>
<td>1.51</td>
<td>Figure 1</td>
</tr>
<tr>
<td>2</td>
<td>1.6</td>
<td>1.28</td>
<td>1.27</td>
<td>1.32</td>
<td>Figure 2</td>
</tr>
<tr>
<td>2</td>
<td>1.8</td>
<td>1.16</td>
<td>1.13</td>
<td>1.15</td>
<td>Figure 3</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1.31</td>
<td>1.30</td>
<td>1.36</td>
<td>Figure 4</td>
</tr>
<tr>
<td>3</td>
<td>2.5</td>
<td>1.16</td>
<td>1.13</td>
<td>1.16</td>
<td>Figure 5</td>
</tr>
<tr>
<td>3</td>
<td>2.8</td>
<td>1.08</td>
<td>1.07</td>
<td>1.06</td>
<td>Figure 6</td>
</tr>
</tbody>
</table>

In general, the numerical results are quite close to the corresponding theoretical value, however the observed differences may come from the relative smallness of size $N$ (the number of points $(x, g(x))$).
3.3 Graphical results

We present below, the continuous and non differentiable curve $F$ which corresponds to every pair $(a, b)$ listed in the above table. In each picture, the top (respectively, the bottom) region is the attractive basin of the attractor ($y^+ = +\infty$) (respectively, $y^- = -\infty$). A magnification in (Fig. 1), depicted in (Fig. 7), reveals the fractal and striated structure ([11]) of the basin boundary. For the pair $(a = 3, b = 2.8)$ - see the last line of the previous table -, the log-log plots, presented in (Fig. 8), attest the relative accuracy of our approximation of the two dimensions.

4 Conclusion

The basin boundary of the Kaplan-Yorke map was built for different values of the parameters. We have observed its high fractal aspect and we have calculated both capacity and information dimensions by a modified Box-counting algorithm. The numerical results are quite close to those proposed in the literature.
On the fractal dimension

(2)

(3)
On the fractal dimension

\textbf{Figure 7}: Magnification of a rectangular region $R$ in (Fig. 1) given by

$R = [0.35, 0.77] \times [0.5, 1.5]$
Figure 8: The log-log plots obtained for \((a, b) = (3, 2.8)\).

References


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