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# On star partition dimension of the generalized gear graph

by
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*Dedicated to the memory of Laurenţiu Panaitopol (1940-2008) on the occasion of his 70th anniversary*

**Abstract**

For a connected graph $G$ and any two vertices $u$ and $v$ in $G$, let $d(u,v)$ denote the distance between $u$ and $v$. For a subset $S$ of $V(G)$, the distance between a vertex $v$ and $S$ is $d(v,S) = \min\{d(v,x) \mid x \in S\}$. For an ordered $k$-partition of $V(G)$ $\Pi = \{S_1, S_2, \ldots, S_k\}$ and a vertex $v$, the representation of $v$ with respect to $\Pi$ is the $k$-vector $r(v \mid \Pi) = (d(v,S_1), d(v,S_2), \ldots d(v,S_k))$. $\Pi$ is a resolving partition for $G$ if the $k$-vectors $r(v \mid \Pi)$, $v \in V(G)$ are distinct. The minimum $k$ for which there exists a resolving $k$-partition of $V(G)$ is the partition dimension of $G$, denoted by $\mathrm{pd}(G)$. $\Pi = \{S_1, S_2, \ldots, S_k\}$ is a star resolving $k$-partition for $G$ if it is a resolving partition and each subgraph induced by $S_i$, $1 \leq i \leq k$, is a star. The minimum $k$ for which there exists a star resolving $k$-partition of $V(G)$ is the star partition dimension of $G$, denoted by $\mathrm{spd}(G)$.

Let $J_{k,n}$ be the graph obtained from the wheel $W_{kn}$ by keeping spokes with step $k$, for $k \geq 2$ and $n \geq 2$. In this paper the star partition dimension for this family of graphs is determined.

**Key Words**: Distance, resolving partition, star resolving partition, partition dimension, star partition dimension, gear graph.
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## 1 Introduction

As described in [1] and [5], dividing the vertex set of a graph into classes according to some prescribed rule is a fundamental process in graph theory. Perhaps the best known example of this process is graph coloring. In [2], the vertices of a connected graph are represented by other criterion, namely through partitions of vertex set and distances between each vertex and the subsets in the partition. Thus a new concept was introduced - resolving partition for a graph.

Let $G$ be a connected graph with vertex set $V(G)$ and edge set $E(G)$. For any two vertices $u$ and $v$ in $G$, let $d(u,v)$ be the distance between $u$ and $v$. For a subset $S$ of $V(G)$ and a vertex $v$ of $G$, the distance $d(v,S)$ between $v$ and $S$ is defined as $d(v,S) = \min\{d(v,x) \mid x \in S\}$.

For an ordered $k$-partition $\Pi = \{S_1, S_2, \ldots, S_k\}$ of $V(G)$ and a vertex $v$ of $G$, the representation of $v$ with respect to $\Pi$ is the $k$-vector

$$r(v \mid \Pi) = (d(v,S_1), d(v,S_2), \ldots, d(v,S_k)).$$

$\Pi$ is called a resolving $k$-partition for $G$ if the $k$-vectors $r(v \mid \Pi)$, $v \in V(G)$ are distinct. The minimum $k$ for which there is a resolving $k$-partition of $V(G)$ is the partition dimension of $G$ and is denoted by $\mathrm{pd}(G)$. A resolving partition of $V(G)$ containing $\mathrm{pd}(G)$ classes is called a minimum resolving partition [2].

In [4] a particular case of resolving partitions - connected resolving partitions was considered. $\Pi = \{S_1, S_2, \ldots, S_k\}$ is said to be a connected resolving $k$-partition if it is a resolving partition and each subgraph induced by $S_i$, $1 \le i \le k$, is connected in $G$. The minimum $k$ for which there is a connected resolving $k$-partition of $V(G)$ is the connected partition dimension of $G$, denoted by $\mathrm{cpd}(G)$.

In this paper we consider a particular case of resolving partitions - star resolving partitions, mentioned in [4] as topic for study. $\Pi = \{S_1, S_2, \ldots, S_k\}$ is called a star resolving $k$-partition if it is a resolving partition and each subgraph induced by $S_i$ is a star, for $1 \le i \le k$. The minimum $k$ for which there is a star resolving $k$-partition of $V(G)$ is the star partition dimension of $G$, denoted by $\mathrm{spd}(G)$. A star resolving partition of $V(G)$ containing $\mathrm{spd}(G)$ classes is called a minimum star resolving partition.

If $\Pi = \{S_1, S_2, \ldots, S_k\}$ is an ordered partition of $V(G)$ and $u_1, u_2, \ldots, u_r$ are $r$ distinct vertices, we say that $u_1, u_2, \ldots, u_r$ are separated by classes $S_{i_1}$, ..., $S_{i_q}$ of partition $\Pi$ if the $q$-vectors

$$(d(u_p, S_{i_1}), d(u_p, S_{i_2}), \ldots, d(u_p, S_{i_q})), \text{ for } 1 \le p \le r$$

are distinct.

Let $J_{k,n}$ be the graph obtained from the wheel $W_{kn}$ by keeping spokes with step $k$, for $k \ge 2$ and $n \ge 2$. This is a generalization of gear graph [3], also known as Jahangir graph $J_{2n}$ [6]. More precisely, $J_{k,n}$ is obtained from the cycle $C_{kn}$ and a new vertex, as follows. Denote by $0, 1, \ldots, kn-1$ the vertices of the cycle $C_{kn}$ and by $c$ the new vertex. We join by edges the vertex $c$ with vertices $0, k, 2k, \ldots, (n-1)k$ of the cycle $C_{kn}$ (see fig. 1 for $k = 3$ and $n = 5$).

In the next section we will determine the star partition dimension of $J_{k,n}$.

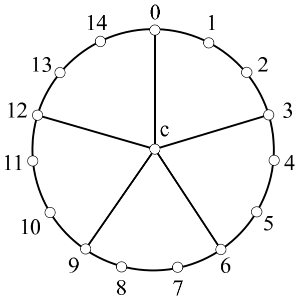


Figure 1: $J_{3,5}$

## 2 Star partition dimension of $J_{k,n}$

We call a partition $\Pi = \{S_0, S_1, \ldots, S_q\}$ of $V(J_{k,n})$ a star partition of $J_{k,n}$ if each subgraph induced by $S_i$, $0 \le i \le q$, is a star. Hence, a star resolving partition is a star partition which is also a resolving partition.

Moreover, for a star partition of $J_{k,n}$ $\Pi = \{S_0, S_1, \ldots, S_q\}$, we assume that classes are numbered such that center $c$ is in $S_0$. We denote by $i_1, i_2, \ldots, i_r$ $(0 \le i_1 < i_2 < \ldots < i_r \le kn-1)$ the vertices of $S_0 \cap V(C_{kn})$. Hence we have

$$S_0 = \{c, i_1, i_2, \ldots, i_r\}.$$

Every two consecutive vertices from $S_0 \cap V(C_{kn})$ determine sets of vertices which induce paths on $C_{kn}$ called gaps [6]. More precisely, we say that the pair of vertices $\{i_t, i_{t+1}\}$, $1 \le t \le r-1$ generates the gap $\{j | i_t < j < i_{t+1}\}$ and the pair of vertices $\{i_r, i_1\}$ generates the gap $\{j | i_r < j \le kn-1\} \cup \{j | 0 \le j < i_1\}$. It is clear that some gaps may be empty.

**Theorem 2.1.** *For any $k \ge 2$ and $n \ge 2$ we have*

$$spd(J_{k,n}) = \begin{cases} 3, & \text{if } k = 2 \text{ or } 3 \text{ and } n = 2; \\ \frac{kn}{3}, & \text{if } k \equiv 0 (\text{mod } 3) \text{ and } (k,n) \ne (3,2); \\ \left\lfloor \frac{k}{3} \right\rfloor n + 1, & \text{if } k \equiv 1 (\text{mod } 3) \text{ and } (n \ge 3 \text{ or} \\ & (n = 2 \text{ and } k \ge 4)); \\ \left\lfloor \frac{k}{3} \right\rfloor n + 1 + \left\lceil \frac{n}{2} \right\rceil, & \text{if } k \equiv 2 (\text{mod } 3) \text{ and } (n \ge 4 \text{ or} \\ & (n = 2 \text{ and } k \ge 5)); \\ 3 \left\lfloor \frac{k}{3} \right\rfloor + 2, & \text{if } k \equiv 2 (\text{mod } 3) \text{ and } n = 3. \end{cases}$$

**Proof**: We will determine the minimum number of classes for a star partition of $J_{k,n}$, which is not necessarily a resolving partition. Let denote this number by $N_s$.

We consider three cases, in accordance with the residue class modulo 3 to which $k$ belongs.

**Case 1:** $k \equiv 1 (\mathrm{mod}\ 3)$.
Let $k = 3p+1$, $p \geq 1$. It is easy to see that if a class of a star partion of $J_{k,n}$ contains only vertices of the cycle $C_{kn}$, then this class has maximum three elements. It follows that for $n \geq 3$ or ($n = 2$ and $p \geq 2$) there exists a unique star partition with minimum number of classes, and that partition is $\Pi = \{S_0, S_1, \ldots, S_{pn}\}$ where

$$\begin{array}{l} S_0 = \{c\} \cup \{kt | t \in \{0, 1, 2, \ldots, n-1\}\} \\ S_{tp+1} = \{kt+1, kt+2, kt+3\}, S_{tp+2} = \{kt+4, kt+5, kt+6\}, \ldots, \\ S_{(t+1)p} = \{k(t+1)-3, k(t+1)-2, k(t+1)-1\}, \text{ for every } 0 \leq t \leq n-1 \end{array}$$

(see fig. 2 for $k = 4$ and $n = 5$).

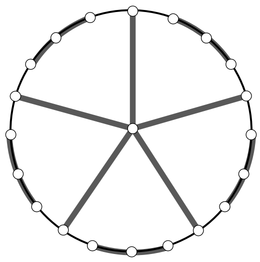

Figure 2: $J_{4,5}$

$\Pi$ has $N_s = pn + 1 = \left\lfloor \frac{k}{3} \right\rfloor n + 1$ classes, hence

$$\mathrm{spd}(J_{k,n}) \geq N_s = \left\lfloor \frac{k}{3} \right\rfloor n + 1.$$

But we can easily prove that $\Pi$ is a resolving partition, which implies

$$\mathrm{spd}(J_{k,n}) = \left\lfloor \frac{k}{3} \right\rfloor n + 1.$$

Indeed, the vertices from class $S_i$, for $1 \leq i \leq pn$ are separated by classes $S_{i-1}$ or $S_{i+1}$.

For $n \geq 3$, vertices from $S_0$ are separated by classes $S_1$, $S_{p+1}$, $S_{2p+1}$, $\ldots$, $S_{(n-1)p+1}$ because we have

$$\begin{array}{l} d(kt, S_{tp+1}) = 1, \text{ for every } 0 \leq t \leq n-1 \\ d(kt, S_{jp+1}) = 3, \text{ for every } 0 \leq t < j \leq n-2 \\ d(k(n-1), S_1) = 3 \\ d(c, S_{tp+1}) = 2, \text{ for every } 0 \leq t \leq n-1. \end{array}$$

For $n = 2$ and $p \geq 2$ (or, equivalently, $k \geq 7$) we have $S_0 = \{c, 0, k\}$ and vertices $c$, $0$ and $k$ are separated by classes $S_1$ and $S_{(n-1)p+1}$.

If $n = 2$ and $p = 1$ ($k = 4$), $kn = 8$ there exists a star resolving partition with three classes

$$\Pi = \{\{c, 0, 1, 7\}, \{2, 3, 4\}, \{5, 6\}\}$$

hence $spd(J_{4,2}) = 3$.
**Case 2:** $k \equiv 0 (\text{mod } 3)$.
Let $k = 3p$, $p \geq 1$. Let $\Pi = \{S_0, S_1, \ldots, S_{N_s-1}\}$ be a star partition of $J_{k,n}$ with minimum number of classes. We assume $c \in S_0$.

If $|S_0| = 1$, then the minimum number of classes in a star partition of $J_{k,n}$ is equal to $pn+1$, each of the classes $S_1, \ldots, S_{pn}$ containing exactly three consecutive vertices of the cycle $C_{kn}$.

Assume $|S_0| > 1$.

If $\Pi$ induces on $C_{kn}$ exactly one nonempty gap, we have $2 \leq |S_0| \leq 4$, which implies that the minimum number of classes in a star partition of $J_{k,n}$ is at least

$$1 + \left\lceil \frac{kn-3}{3} \right\rceil = \frac{kn}{3} = pn.$$

If $\Pi$ induces at least 2 nonempty gaps, denote by $w = |S_0| - 1$ the number of these gaps.

It is not difficult to see that in this case the number of vertices from any gap is congruent with 2 modulo 3. It follows that in a minimum star partition of $J_{k,n}$ the vertices of every gap must be partitioned in paths with 3 vertices and one path with 2 vertices, hence the number of classes with 2 vertices is $w$.

Consequently, the number of classes in a minimum star partition equals

$$1 + \left\lceil \frac{kn-w-2w}{3} \right\rceil + w = 1 + \frac{kn}{3}.$$

We have obtained that $spd(J_{k,n}) \geq N_s \geq \frac{kn}{3}$.

Moreover, for $n \geq 3$ or ($n = 2$ and $p \geq 2$), partition $\Pi = \{S_0, S_1, \ldots, S_{pn-1}\}$ having classes $S_0 = \{c, 0, 1, kn-1\}$ and $S_t = \{3t-1, 3t, 3t+1\}$, for $1 \leq t \leq pn-1$ is a star resolving partition with $pn$ classes of $J_{k,n}$ (see fig. 3 for $k = 3$ and $n = 5$).

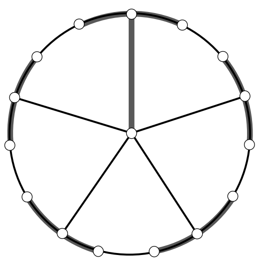

Figure 3: $J_{3,5}$

For $n = 2$ and $p = 1$ partition $\Pi = \{\{c, 3\}, \{0, 1, 2\}, \{4, 5\}\}$ is a minimum star resolving partition, so in this case $spd(J_{3,2}) = 3$.

It follows that $spd(J_{k,n}) = \frac{kn}{3}$ for every $(k, n) \neq (3, 2)$ and $spd(J_{3,2}) = 3$.

**Case 3:** $k \equiv 2(\text{mod } 3)$.
Let $k = 3p + 2$, $p \geq 0$. Let $\Pi = \{S_0, S_1, \ldots, S_{N_s-1}\}$ be a star partition of $J_{k,n}$ having minimum number of classes such that $c \in S_0$.

If the number of nonempty gaps induced on $C_{kn}$ by vertices from $S_0 - \{c\}$ is at least 2, then we can suppose that $\Pi$ has no gap containing exactly $k - 1 = 3p + 1$ vertices, bounded by 2 consecutive vertices of degree 3, $kt$ and $k(t + 1)$ of $C_{kn}$. Indeed, if $\Pi$ has such a gap, let $W$, then $\Pi$ induces a minimum star partition $\Pi|_W$ of $W$ having $p + 1$ classes (one class being a singleton or 2 classes inducing each a path with 2 vertices and the remaining classes induce each a path with 3 vertices).

We shall delete $kt$ or $k(t+1)$ from $S_0$; suppose that this choice is $kt$, hence we define $S_0' = S_0 - \{kt\}$. Then we shall add vertex $kt$ to $W$ and we shall consider a star partition of $W \cup \{kt\}$ into $p + 1$ classes.

In this way a new star partition of $J_{k,n}$ having minimum number of classes has been produced with no induced gap of cardinality equal to $3p + 1$.

Consider now a minimum star partition $\Pi = \{S_0, S_1, \ldots, S_{N_s-1}\}$ of $J_{k,n}$ such that $c \in S_0$ having no gap of cardinality $3p + 1$.

If $|S_0| = 1$, then the minimum number of classes in a star partition of $J_{k,n}$ is equal to

$$1 + \left\lceil \frac{kn}{3} \right\rceil = 1 + pn + \left\lceil \frac{2n}{3} \right\rceil \geq 1 + pn + \left\lceil \frac{n}{2} \right\rceil .$$

If $\Pi$ induces on $C_{kn}$ exactly one gap, then $2 \leq |S_0| \leq 4$.

It follows that this minimum number of classes in a star partition is at least

$$1 + \left\lceil \frac{kn - 3}{3} \right\rceil = 1 + pn + \left\lceil \frac{2n - 3}{3} \right\rceil .$$

For $n = 2$ or $n \geq 4$ we have

$$1 + pn + \left\lceil \frac{2n - 3}{3} \right\rceil \geq 1 + pn + \left\lceil \frac{n}{2} \right\rceil ,$$

but for $n = 3$ we have $\left\lceil \frac{2n-3}{3} \right\rceil = \left\lceil \frac{n}{2} \right\rceil - 1$, so this lower bound becomes

$$3p + 2 = 3 \left\lfloor \frac{k}{3} \right\rfloor + 2 = pn + \left\lceil \frac{n}{2} \right\rceil .$$

If $\Pi$ induces at least 2 nonempty gaps, let $w = |S_0| - 1$ denote the number of these gaps. Since every gap has cardinality at least $6p + 3$, one deduces

$$w \leq \left\lceil \frac{n}{2} \right\rceil .$$

It follows that in this case the minimum number of classes in a star partition equals

$$1+\left\lceil\frac{kn-w}{3}\right\rceil=1+pn+\left\lceil\frac{2n-w}{3}\right\rceil\geq 1+pn+\left\lceil\frac{n}{2}\right\rceil.$$

If $n=3$ in all cases we deduced

$$spd(J_{k,n})\geq N_s\geq pn+\left\lceil\frac{n}{2}\right\rceil=3p+2.$$

But for $n=3$ a resolving star partition with $3p+2$ classes is built by choosing $S_0=\{c,0,1,kn-1\}$, and other $3p+1$ classes are obtained by partitioning the vertices $\{2,3,\ldots,kn-2=9p+4\}$ into classes containing 3 consecutive vertices each: $\{2,3,4\}$, $\{5,6,7\}$, $\ldots$, $\{9p+2,9p+3,9p+4\}$ (see fig. 4 for $k=5$ and $n=3$). Consequently, $spd(J_{k,3})=3p+2$.

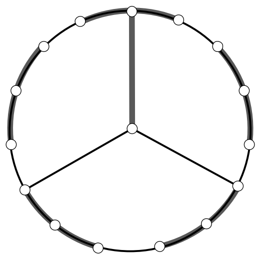

Figure 4: $J_{5,3}$

If $n\neq 3$ in all cases we have

$$\mathrm{spd}(J_{k,n})\geq N_s\geq 1+pn+\left\lceil\frac{n}{2}\right\rceil.$$

We consider a star partition $\Pi=\{S_0,S_1,\ldots,S_r\}$ defined as follows:

$$S_0=\{c\}\cup\{2kt|0\leq t\leq\left\lceil\frac{n}{2}\right\rceil-1\}$$

and every gap bounded by vertices from $S_0$ is divided into $2p+1$ classes with 3 consecutive vertices each, with at most one exception (for $n$ odd) - the last gap (bounded by $2k\left(\left\lceil\frac{n}{2}\right\rceil-1\right)$ and 0) has a class with one element (see fig. 5 for $k=5$ and $n=5$). We denote the classes obtained, in the clockwise sense, by $S_1,\ldots,S_r$. The number of classes in this partition is

$$r=1+\left\lceil\frac{kn-\lceil\frac{n}{2}\rceil}{3}\right\rceil=1+pn+\left\lceil\frac{2n-\lceil\frac{n}{2}\rceil}{3}\right\rceil=1+pn+\left\lceil\frac{n}{2}\right\rceil.$$

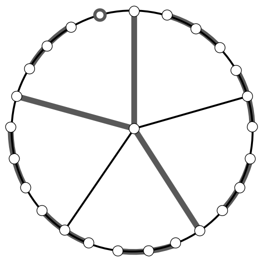

Figure 5: $J_{5,5}$

It is easy to see that for $n \geq 3$ or ($n = 2$ and $p \geq 1$) $\Pi$ is a resolving partition, this implying that

$$spd(J_{k,n}) = 1 + pn + \left\lceil \frac{n}{2} \right\rceil .$$

For $n = 2$ and $p = 0$ ($k = 2$) we have $\mathrm{spd}(J_{k,n}) = 3$, a minimum star resolving partition being $\{\{c\}, \{0, 1\}, \{2, 3\}\}$. □

## References


[1] G. Chartrand, T. W. Haynes, M. A. Henning, P. Zhang, Stratification and domination in graphs, Discrete Math. 272 (2003), no. 2-3, 171-185.

[2] G. Chartrand, E. Salehi, P. Zhang, The partition dimension of a graph, Aequationes Math. 59(2000), 97-108.

[3] J. A. Gallian, A dynamic survery of graph labeling, Elect. J. Combin. 15(2008) DS6.

[4] V. Saenpholphat, P. Zhang, Connected partition dimensions of graphs, Discussiones Mathematicae Graph Theory 22 (2002), 305-323.

[5] V. Saenpholphat, P. Zhang, Conditional resolvability in graphs: A survey, Int. J. Math. Math. Sci. 37-40(2004), 1997-2017.

[6] I. Tomescu, I. Javaid, On the metric dimension of the Jahangir graph, Bull. Math. Soc. Sci. Math. Roumanie 50(98), no.4(2007), 371-376.


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