

On Metric and Partition Dimensions of Some Infinite Regular Graphs

by

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Abstract

In this paper some infinite regular graphs generated by tilings of the plane by regular triangles and hexagons are considered. These graphs have no finite metric bases but their partition dimension is finite and is evaluated in some cases. Also, it is proved that for every $n \geq 2$ there exists finite induced subgraphs of these graphs having metric dimension equal to n as well as infinite induced subgraphs with metric dimension equal to three.

Key Words: Metric dimension, partition dimension, plane tiling, infinite regular graph, induced subgraph.

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1 Notation and auxiliary results

If G is a connected graph, the distance $d(u, v)$ between two vertices $u, v \in V(G)$ is the length of a shortest path between them. Let $W = \{w_1, w_2, \dots, w_k\}$ be an ordered set of vertices of G and let v be a vertex of G . The representation $r(v|W)$ of v with respect to W is the k -tuple $(d(v, w_1), d(v, w_2), \dots, d(v, w_k))$. If distinct vertices of G have distinct representations with respect to W , then W is called a resolving set for G [1]. A resolving set of minimum cardinality is called a basis for G and this cardinality is the metric dimension of G , denoted by $\dim(G)$. The concepts of resolving set and metric basis have previously appeared in the literature (see [1], [2], [5] – [10]).

For a given ordered set of vertices $W = \{w_1, w_2, \dots, w_k\}$ of a graph G , the i -th component of $r(v|W)$ is 0 if and only if $v = w_i$. Thus, to show that W is a resolving set it suffices to verify that $d(x|W) \neq d(y|W)$ for each pair of distinct vertices $x, y \in V(G) \setminus W$.

A useful property in finding $\dim(G)$ is the following:

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Lemma 1.1. *Let W be a resolving set for a connected graph (G) and $u, v \in V(G)$. If $d(u, w) = d(v, w)$ for all vertices $w \in V(G) \setminus \{u, v\}$, then $\{u, v\} \cap W \neq \emptyset$.*

Another kind of dimension of a connected graph, called partition dimension was introduced in [3, 4] as follows: For a subset $S \subset V(G)$ and a vertex v of a connected graph G , the distance $d(v, S)$ between v and S is defined as usually, by $d(v, S) = \min\{d(v, x) : x \in S\}$. If $\Pi = (S_1, S_2, \dots, S_k)$ is an ordered k -partition of $V(G)$, the representation of v with respect to Π is the k -tuple $r(v|\Pi) = (d(v, S_1), d(v, S_2), \dots, d(v, S_k))$. If the k -tuples $r(v|\Pi)$ for $v \in V(G)$ are all distinct, then the partition Π is called a resolving partition and the minimum cardinality of a resolving partition of $V(G)$ is called the partition dimension of G and is denoted by $pd(G)$. Let $\Pi = \{S_1, S_2, \dots, S_k\}$ be an ordered partition of $V(G)$. If $u \in S_i, v \in S_j$ where $1 \leq i, j \leq k$ and $i \neq j$, then $r(u|\Pi) \neq r(v|\Pi)$ since $d(u, S_i) = 0$ but $d(v, S_i) \neq 0$. Thus, when determining whether a given partition Π of $V(G)$ is a resolving partition for $V(G)$, we need only to verify if the vertices of G belonging to the same class of Π have distinct representations with respect to Π . When $d(u, S_i) \neq d(v, S_i)$ we shall say that the class S_i distinguishes vertices u and v . Another useful property in determining $pd(G)$ is the following lemma [4].

Lemma 1.2. *Let Π be a resolving partition of $V(G)$ and $u, v \in V(G)$. If $d(u, w) = d(v, w)$ for all vertices $w \in V(G) \setminus \{u, v\}$, then u and v belong to different classes of Π .*

It is natural to think that the partition dimension and metric dimension are related; in [4] it was shown that for any nontrivial connected graph G we have $pd(G) \leq \dim(G) + 1$.

However, the partition dimension may be much smaller than the metric dimension.

These concepts have some applications in chemistry for representing chemical compounds [2, 7] or to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures [8].

Let (i, j) and (i', j') be two points with integral coordinates in $\mathbb{Z}^2$. It is well known that the following definitions yield metrics for $\mathbb{Z}^2$: $d_4((i, j), (i', j')) = |i - i'| + |j - j'|$ (city block distance) and $d_8((i, j), (i', j')) = \max(|i - i'|, |j - j'|)$ (chessboard distance). The indices 4 and 8 are appropriate because they represent the number of points at distance one (the neighbors) from a given point with respect to these two metrics. These two metrics on $\mathbb{Z}^2$ generate two infinite graphs $(\mathbb{Z}^2, \mathcal{E}_4)$ and $(\mathbb{Z}^2, \mathcal{E}_8)$ having the same vertex set $\mathbb{Z}^2$ and the set of edges consisting of all pairs of vertices whose city block and chessboard distances are 1. $(\mathbb{Z}^2, \mathcal{E}_4)$ is a planar 4-regular graph whose regions are unit squares and it is also known as the square lattice graph. $(\mathbb{Z}^2, \mathcal{E}_8)$ is 8-regular and can be obtained from $(\mathbb{Z}^2, \mathcal{E}_4)$ by drawing all diagonals of unit squares. In [8] it was proved that these two graphs have no finite metric bases and for any natural number $n \geq 3$, there exist induced subgraphs of $(\mathbb{Z}^2, \mathcal{E}_4)$ and $(\mathbb{Z}^2, \mathcal{E}_8)$, respectively having metric dimension equal to n and partition dimension equal to three. Also, in [11] it was

proved that $pd(\mathbb{Z}^2, \mathcal{E}_4) = 3$ and $pd(\mathbb{Z}^2, \mathcal{E}_8) = 4$.

In what follows we shall consider some infinite regular graphs generated by tilings of the plane by regular hexagons or equilateral triangles.

The graph G_3 is a planar 3-regular infinite graph whose regions are regular

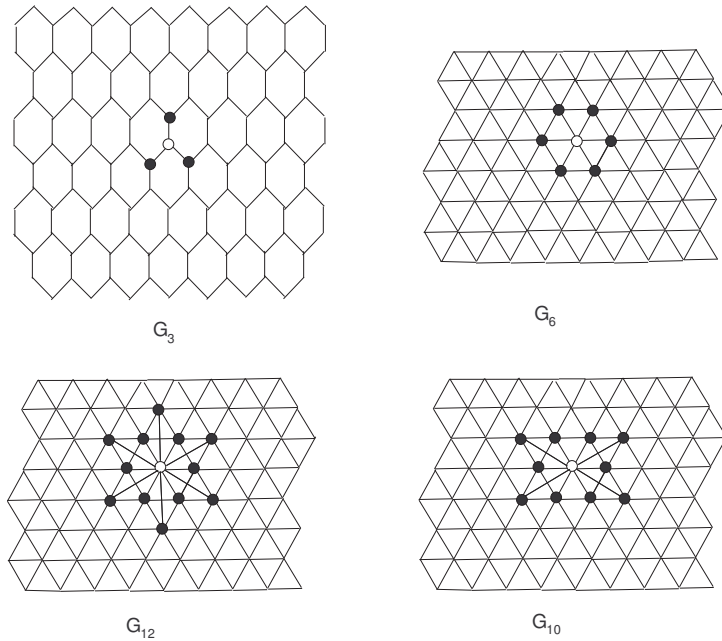


Figure 1: Graphs G_3, G_6, G_{10} and G_{12}

hexagons of unit side. $V(G_3)$ consists of the vertices of these hexagons, two vertices being adjacent if they are the extremities of a unit side of a hexagon in the tiling. Similarly, G_6 is a planar 6-regular infinite graph whose regions are equilateral triangles of unit side. It is also known as the triangular lattice graph. Three kinds of rhombuses of unit side appear in G_6 . By drawing all diagonals of these rhombuses we get a 12-regular infinite graph denoted by G_{12} ; if all vertical diagonals are deleted, the resulting 10-regular infinite graph is denoted by G_{10} (see Fig.1). The indices 3, 6, 10 and 12 represent the number of vertices at distance 1 from a given vertex. For sake of simplicity for G_{10} and G_{12} or their subgraphs we shall not draw the corresponding diagonals of the rhombuses; this will be clear from the context.

2 Main results

Lemma 2.1. *The graphs G_3, G_6, G_{10} and G_{12} have no finite metric basis, i.e., $\dim(G_3) = \dim(G_6) = \dim(G_{10}) = \dim(G_{12}) = \infty$.*

Proof: Figs. 2a) – d) represent two vertices x, y in G_3, G_6, G_{10}, G_{12} having their Euclidean distances equal to 1 and to $\sqrt{3}$, respectively and subgraphs $G_i(x, y)$ ($i = 3, 6, 10, 12$) consisting of vertices z such that $d(x, z) = d(z, y)$. Suppose that G_3 has a finite metric basis S . We can find two vertices x, y and a subset $T \subset G_3(x, y)$ consisting of all vertices $z \in G_3(x, y)$ such that $d(z, x) = d(z, y) \leq k$ for k large enough, such that $S \subset T$. This implies that $d(x, z) = d(y, z)$ for all $z \in S$, a contradiction. The proof is similar for other three graphs. $\square$

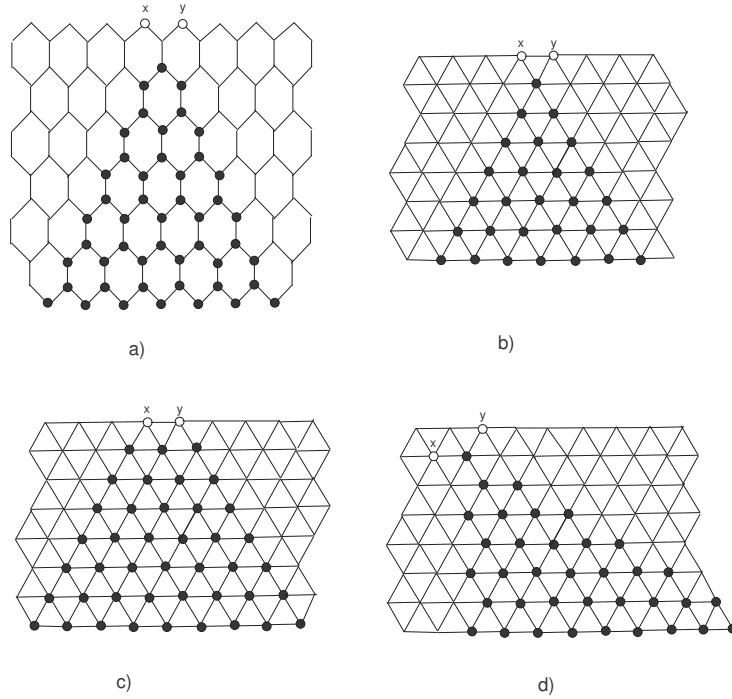
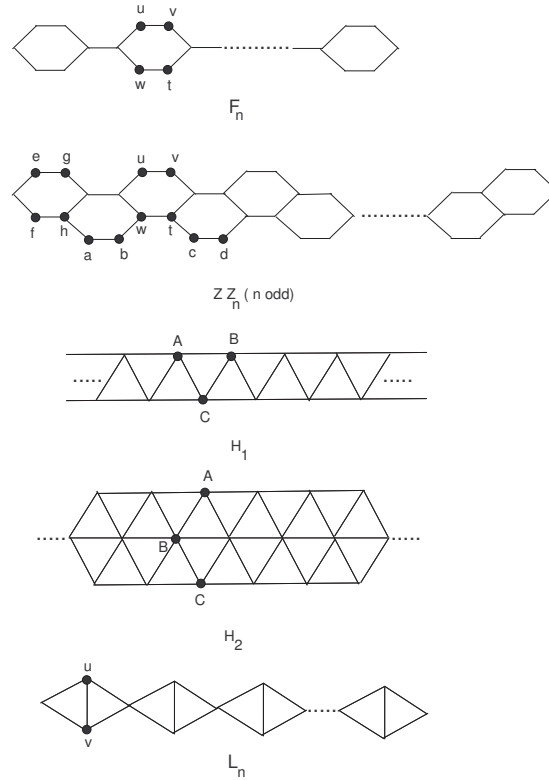


Figure 2: Subgraphs of vertices having equal distances to x and y .

Fig. 3 represents some induced subgraphs of G_3 and G_6 : F_n has n hexagons, ZZ_n is a zig-zag sequence of n hexagons, H_1 and H_2 are infinite subgraphs of G_6 and L_n is an induced subgraph of G_6 consisting of n rhombuses with one

Figure 3: Induced subgraphs of G_3 and G_6 .

diagonal.

We have $\dim(H_1) = \dim(H_2) = 3$ since vertices having the same distance to B are distinguished by A and C , respectively.

Theorem 2.2. a) For every $n \geq 2$ we have: $\dim(F_n) = n$ and F_n has $n!4^n$ metric bases; $\dim(L_n) = n$ and L_n has $n!2^n$ metric bases.
b) For every $n \geq 7$ $\dim(ZZ_n) = \lceil \frac{n}{3} \rceil$.

Proof: a) u and w have equal distances to all vertices of F_n different from v and t and v and t may be distinguished only by u or w if v and t do not belong to any basis of F_n . It follows that at least one of u, v, w, t must belong to any basis of F_n .

On the other hand, by choosing exactly one vertex of degree two in each hexagon of F_n (in 4^n ways), the set of these vertices is a metric basis of F_n . These vertices can be ordered in $n!$ ways and the result follows. The situation of L_n is similar

to F_n .

b) Suppose that the hexagons of ZZ_n are denoted by $H_1, \dots, H_n$ from left to right. The vertices e and f of H_1 (see Fig. 3) can be distinguished only by g, h, a or b and vertices g and h only by e, f, a or b . We deduce that any resolving set W of ZZ_n must contain at least one vertex from unavoidable set $\{e, f, g, h, a, b\}$ of the pair of hexagons (H_1, H_2) . This set will be denoted by $U(H_1, H_2)$. A similar situation occurs for the pair (H_{n-1}, H_n) . We shall say that W satisfies the 2 consecutive hexagons (shortly 2CH) property.

Consider now a triplet of consecutive hexagons (H_2, H_3, H_4) . The vertices u and w of H_3 can be distinguished only by v, t, a, b, c or d and a similar situation also occurs for v and t . It follows that any resolving set W of ZZ_n must contain at least one vertex from unavoidable set $\{u, v, w, t, a, b, c, d\}$ of (H_2, H_3, H_4) and a similar property holds for any triplet (H_i, H_{i+1}, H_{i+2}) , where $2 \leq i \leq n-3$. This set will be denoted by $U(H_i, H_{i+1}, H_{i+2})$ and we shall say that W satisfies the 3 consecutive hexagons (3CH) property.

Let B be a metric basis of ZZ_n . We shall transform B into a set B' such that $|B| \geq |B'|$, B' contains only vertices of degree 2 and B' also satisfies 2CH and 3CH properties by the following algorithm:

Initially $B' = \emptyset$. We have $U(H_1, H_2) \cap U(H_2, H_3, H_4) = \{a, b\}$. If there exists $x \in \{a, b\}$ such that $x \in B$ then label x , assign $B' \leftarrow B' \cup \{x\}$ and consider the next triplet (H_2, H_3, H_4) . Otherwise, if there exists $y \in \{e, f, g\}$ such that $y \in B$ then label y , define $B' \leftarrow B' \cup \{y\}$ and consider the next triplet. Otherwise, $\{h\} = U(H_1, H_2) \cap B$. In this case label g and do $B' \leftarrow B' \cup \{g\}$. Consequently, only one of the vertices e, f, g, a, b were labeled and included in B' .

Consider now the triplet (H_2, H_3, H_4) . If $U(H_2, H_3, H_4)$ contains a labeled vertex (or equivalently, a or b has been selected in B'), then consider the next triplet (H_3, H_4, H_5) . Otherwise a and b are not labeled. We deduce $U(H_2, H_3, H_4) \cap U(H_3, H_4, H_5) = \{u, v, c, d\}$, $U(H_2, H_3, H_4) \cap U(H_4, H_5, H_6) = \{c, d\}$ and $U(H_2, H_3, H_4) \cap U(H_i, H_{i+1}, H_{i+2}) = \emptyset$ for $i \geq 5$. If there exists $x \in \{c, d\}$ such that $x \in B$ then label it, do $B' \leftarrow B' \cup \{x\}$ and consider the next triplet. Otherwise, if there exists $y \in \{u, v\}$ such that $y \in B$, then label it, assign $B' \leftarrow B' \cup \{y\}$ and consider the next triplet. If $u, v \notin B$ but there exists $z \in \{w, t\}$ such that $z \in B$ then label u (or v), $B' \leftarrow B' \cup \{u\}$ and consider the next triplet. Otherwise, there exists $s \in \{a, b\}$ such that $s \in B$. In this case label s , define $B' \leftarrow B' \cup \{s\}$ and consider the next triplet (H_3, H_4, H_5) .

Each time when the unavoidable set of a triplet has a vertex which has been labeled in the previous steps we shall consider the next triplet (or the last pair). We shall proceed in a similar way for all triplets (H_i, H_{i+1}, H_{i+2}) in the order $i = 3, \dots, n-4$ and also for the triplet $(H_{n-3}, H_{n-2}, H_{n-1})$ and the pair (H_{n-1}, H_n) (like as for (H_1, H_2) and (H_2, H_3, H_4)).

Note that h does not belong to $U(H_2, H_3, H_4)$, so its possible replacement by g does not affect the property of B' to satisfy 2CH and 3CH properties. Similarly, $w, t \notin U(H_3, H_4, H_5) \cup U(H_4, H_5, H_6)$, so their possible replacement by vertices of degree 2 u or v , respectively, cannot affect the property of B' to satisfy the 3CH

property and this holds for all triplets of hexagons of ZZ_n . Consequently, B' satisfies 2CH and 3CH properties; moreover, B' has the property that it contains only vertices of degree 2, i. e., it does not contain vertices common to neighboring hexagons H_i, H_{i+1} for $1 \leq i \leq n-1$ and each hexagon has at most one vertex in B' .

For $i = 1, \dots, n$ let z_i denote a binary variable representing the number of vertices of H_i included in B' . Since B' satisfies 2CH and 3CH properties we can write:

$$\begin{aligned} z_1 + z_2 &\geq 1 \\ z_2 + z_3 + z_4 &\geq 1 \\ z_3 + z_4 + z_5 &\geq 1 \\ &\dots \dots \dots \\ z_{n-3} + z_{n-2} + z_{n-1} &\geq 1 \\ z_{n-1} + z_n &\geq 1 \end{aligned}$$

By summing up these inequalities we get:

$$S = z_1 + 2z_2 + 2z_3 + 3(z_4 + \dots + z_{n-3}) + 2z_{n-2} + 2z_{n-1} + z_n \geq n - 2.$$

We have

$$3|B| \geq 3|B'| = 3 \sum_{i=1}^n z_i = S + 2z_1 + z_2 + z_3 + z_{n-2} + z_{n-1} + 2z_n \geq n$$

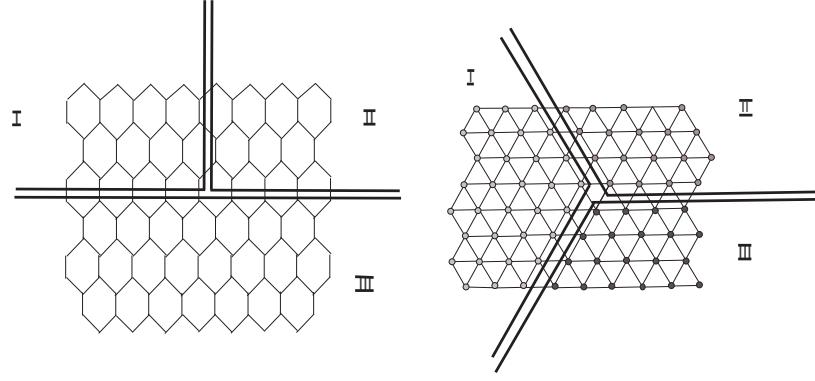
since $2z_1 + z_2 \geq z_1 + z_2 \geq 1$; $2z_n + z_{n-1} \geq z_n + z_{n-1} \geq 1$ and $z_3 + z_{n-2} \geq 0$. It follows that $\dim(ZZ_n) = |B| \geq |B'| \geq \lceil \frac{n}{3} \rceil$.

By direct evaluation we find that $\dim(ZZ_n)$ is equal to: 2 for $n = 1$ or $n = 2$; 3 for $3 \leq n \leq 9$; 4 for $10 \leq n \leq 12$ and so on. For every $n \geq 7$, we can construct a metric basis B of ZZ_n of cardinality $\lceil \frac{n}{3} \rceil$ as follows: we choose any vertex of degree two in the hexagons numbered by: $2, 5, 8, \dots, n-4, n-1$ for $n \equiv 0(mod 3)$; $2, 5, 8, \dots, n-5, n-2, n$ for $n \equiv 1(mod 3)$ and $2, 5, 8, \dots, n-6, n-3, n$ for $n \equiv 2(mod 3)$. It can be easily verified that any two vertices of ZZ_n having equal distances to any vertex of B can be distinguished by other vertices of B , which concludes the proof. $\square$

Lemma 2.3. *We have $pd(G_3) = pd(G_6) = 3$.*

Proof: In [4] it was shown that $pd(G) = 2$ if and only if G is a path and this property also holds for infinite graphs. It follows that $pd(G_3) \geq 3$ and $pd(G_6) \geq 3$.

Fig. 4 provides resolving 3-partitions of G_3 and G_6 , respectively. It follows that $pd(G_3) = pd(G_6) = 3$. $\square$

Figure 4: Resolving 3-partitions for $V(G_3)$ and $V(G_6)$.

The problem of determining partition dimension of G_{10} and G_{12} is much more difficult. We are only able to find some bounds in the next theorems. Consider now the Euclidean plane containing plane realizations of G_{10} and G_{12} endowed with a rectangular system of axes xOy .

Theorem 2.4. *We have $4 \leq pd(G_{10}) \leq 5$.*

Proof: The same argument used in Lemma 2.2 implies that $pd(G_{10}) \geq 3$. Suppose that there exists a resolving partition $\Pi = (S_1, S_2, S_3)$ with three classes of $V(G_{10})$.

We will show that this leads to a contradiction, which implies $pd(G_{10}) \geq 4$.

Claim 1. *In any unit rhombus $PQRS$ which is an induced subgraph of G_{10} , having two sides parallel to Ox there are not three vertices belonging to different classes of Π .*

Suppose that $P \in S_1, Q \in S_2$ and $R \in S_3$. S can belong to S_1, S_2 or S_3 . In any case there exist two vertices of $PQRS$ in S_i having unit distances to S_j and S_k respectively ($1 \leq i, j, k \leq 3, i, j$ and k pairwise distinct indices), a contradiction.

Claim 2. *In any unit rhombus $PQRS$, induced subgraph of G_{10} , no three vertices belong to a class and the fourth to another class of Π .*

Suppose that $P, Q, R \in S_1$ and $S \in S_2$. Since P, Q, R have equal unit distances to the class S_2 , we deduce that $d(P, S_3) \neq d(Q, S_3)$, $d(P, S_3) \neq d(R, S_3)$ and $d(Q, S_3) \neq d(R, S_3)$. Since any distance between vertices P, Q and R is 1, one obtains that $d(P, S_3)$ and $d(Q, S_3)$ and $d(R, S_3)$ may differ by at most one, which contradicts these inequalities. A consequence of Claims 1 and 2 follows.

Claim 3. *Suppose that $P, Q \in V(G_{10})$ and $d(P, Q) = 1$ such that P and Q belong to the same line having a slope equal to $\sqrt{3}$ or $-\sqrt{3}$ and $P \in S_1, Q \in S_2$. Then all the vertices of G_{10} lying on the horizontal lines l_P and l_Q passing through P and Q belong to the classes S_1 and S_2 and each unit rhombus, induced subgraph*

of G_{10} with vertices on l_P and l_Q has two vertices in S_1 and two in S_2 . A similar conclusion holds if P and Q belong to the same horizontal line by considering the lines having slope $\sqrt{3}$ or $-\sqrt{3}$ passing through P and Q .

Without loss of generality, we can assume that there exist two vertices of G_{10} , $P(x, y), Q(x - \frac{1}{2}, y - \frac{\sqrt{3}}{2})$ on the same line having slope $\sqrt{3}$ such that $P \in S_1$, $Q \in S_2$ and $d(P, Q) = 1$. Let P, Q, R, S, P be a 4-cycle containing PQ such that $S(x+1, y)$. By Claims 1 and 2 vertices R and S belong to different classes S_1, S_2 of Π .

Without loss of generality we can suppose that $R \in S_2$ and $S \in S_1$. Since $d(P, S_2) = d(S, S_2) = 1$ it follows that $d(P, S_3) \neq d(S, S_3)$. Because $d(P, S) = 1$ we deduce that $d(P, S_3)$ and $d(S, S_3)$ differ by 1, e.g., $d(P, S_3) = d(S, S_3) + 1$. We obtain that there exists a vertex $T \in S_3$ such that $d(P, S_3) = d(P, T)$, $d(S, S_3) = d(S, T)$ and $d(P, T) = d(S, T) + 1$. By Claim 3 all vertices of G_{10} lying on lines QR and SP belong to $S_1 \cup S_2$.

If $d(S, T) = 1$, since all vertices of G_{10} on PS belong to $S_1 \cup S_2$, we deduce that the slope of ST is $\pm 1/\sqrt{3}$. Without loss of generality we can suppose that this slope is equal to $1/\sqrt{3}$. Let J be on the line PS such that $d(S, J) = d(T, J) = 1$. Because $S \in S_1$ and $T \in S_3$, by Claims 1 and 3 it follows that $J \in S_1$. In this case $S, J \in S_1$, but $d(S, S_2) = d(J, S_2) = 1$ and $d(S, S_3) = d(J, S_3) = 1$, a contradiction.

Suppose that $d(S, T) \geq 2$; then there exists a shortest path between S and T such that the last edge UT of this path incident to T has the slope equal to $1/\sqrt{3}$. Consider the unit rhombus $TVUW$ having diagonal TU and such that UV is parallel to PS . Since $U \in S_1 \cup S_2$, we can suppose that $U \in S_1$. By Claim 1 we have $V \in S_1 \cup S_3$.

If $V \in S_1$ then $W \in S_3$, but in this case either $d(S, W) = d(S, T) - 1$ or $d(P, W) = d(S, W) = d(S, T) = d(S, S_3)$, a contradiction.

It follows that $V \in S_3$. Consider the unit rhombus determined by the intersection of the lines PS, QR and WU, TV . By Claim 3 two vertices of this rhombus are in S_1 and two in S_3 and in the same time, two belong to S_1 and two belong to S_2 , a contradiction.

A similar conclusion holds if $U \in S_2$. It follows that $pd(G_{10}) \geq 4$.

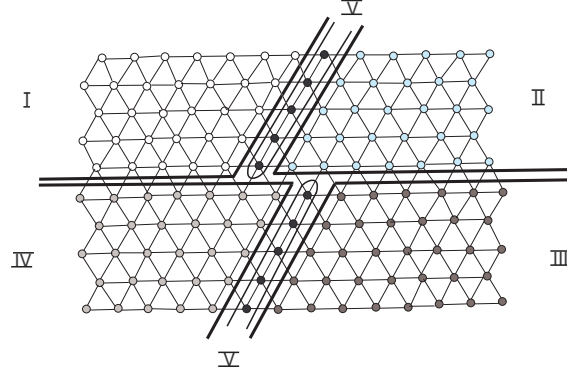
Figure 5 shows that $pd(G_{10}) \leq 5$, which concludes the proof. $\square$

Theorem 2.5. *The following inequalities hold:*

$$4 \leq pd(G_{12}) \leq 6.$$

Proof: As for Theorem 2.2 we deduce that $pd(G_{12}) \geq 3$. Suppose that there exists a resolving partition $\Pi = (S_1, S_2, S_3)$ of $V(G_{12})$ and we will deduce a contradiction.

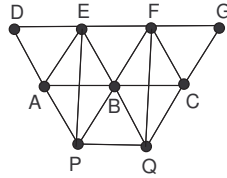
Since $V(G_{12}) = V(G_{10})$ and all distances equal to 1 in G_{10} are also equal to 1 in G_{12} , we deduce that Claims 1, 2 and 3 hold for G_{12} . In G_{12} all vertical diagonals of unit rhombus are edges; this leads to a supplementary property of

Figure 5: A resolving 5-partition of $V(G_{10})$.

G_{12} expressed as follows:

Claim 4. *Let $P, Q \in V(G_{12})$ such that $d(P, Q) = 1$, line PQ is horizontal and P, Q belong to the same class of Π , say S_1 . Then all vertices of G_{12} lying on vertical lines passing through P and Q also belong to S_1 .*

Consider the subgraph of G_{12} having vertices $P, Q, A, B, C, D, E, F, G$, drawn in Fig. 6. Suppose that $F \notin S_1$. Without loss of generality we can suppose that

Figure 6: A subgraph of G_{12}

$F \in S_2$. By Claim 2 we have $C \notin S_1$. If $C \in S_2$ then by Claims 1 and 2 we deduce $B \in S_2$; thus in $BCGF$ three vertices belong to a class, which contradicts Claim 2. If $C \in S_3$ it follows that $B \in S_3$ and also $G \in S_2$. In this case $B, C \in S_3$ and $d(B, S_1) = d(C, S_1) = 1$ and $d(B, S_2) = d(C, S_2) = 1$, which contradicts the fact that Π is a resolving partition of G_{12} .

In a similar way, by considering the parallelogram $PQED$ we deduce that $E \in S_1$. By repeating this argument it follows that all vertices of G_{12} lying on lines EP and FQ belong to S_1 . This concludes the proof of this claim.

Now the proof of the theorem follows the same lines as the proof of Theorem 2.2.

In the case when $T \in S_3$ is such that $d(P, T) = d(S, T) + 1$, by Claim 4 vertex T cannot belong to the vertical line passing through S and we have the same subcases as for Theorem 2.2.

Fig. 7 represents a resolving 6- partition of $V(G_{12})$, hence $pd(G_{12}) \leq 6$. $\square$

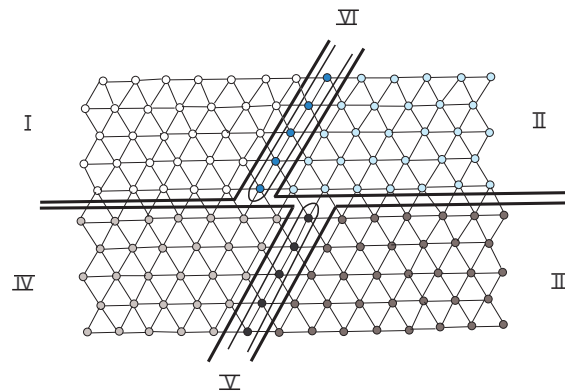


Figure 7: A resolving 6-partition of $V(G_{12})$.

We propose the following conjecture:

Conjecture. *The following equalities hold:*

$$pd(G_{10}) = 5 \text{ and } pd(G_{12}) = 6.$$

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