Intersections of base rings associated to transversal polymatroids

by

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Abstract

In this paper, we study when the intersection of base rings associated to some transversal polymatroids from [5] is the base ring of a transversal polymatroid. This intersection is a Gorenstein ring and we compute its $a$-invariant.

Key Words: Base ring, transversal polymatroid, equations of a cone, $a$-invariant, canonical module, Hilbert series.

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1 Introduction

The discrete polymatroids and their base rings are studied recently in many papers (see [3], [4], [7] [8], [9]). It is important to give conditions when the base ring associated to a transversal polymatroid is Gorenstein (see [3]). In [5] we introduced a class of such base rings. In this paper we note that an intersection of such base rings (introduced in [5]) is Gorenstein and give necessary and sufficient conditions for the intersection of two base rings from [5] to be still a base ring of a transversal polymatroid. Also, we compute the $a$-invariant of those base rings. The results presented were discovered by extensive computer algebra experiments performed with Normaliz [2].

2 Preliminaries

Let $n \in \mathbb{N}$, $n \geq 3$, $\sigma \in S_n$, $\sigma = (1, 2, \ldots, n)$ the cycle of length $n$, $[n] := \{1, 2, \ldots, n\}$, $\sigma^i[i] := \{\sigma^i(1), \ldots, \sigma^i(i)\}$ for any $1 \leq i \leq n - 1$ and $\{\epsilon_i\}_{1 \leq i \leq n}$ be the canonical base of $\mathbb{R}^n$. For a vector $x \in \mathbb{R}^n$, $x = (x_1, \ldots, x_n)$, we will denote $|x| := x_1 + \ldots + x_n$. If $x^a$ is a monomial in $K[x_1, \ldots, x_n]$ we set $\log(x^a) = a$. Given a set $A$ of monomials, the log set of $A$, denoted $\log(A)$, consists of all
log($x^n$) with $x^n \in A$.

If $A_i$ are some nonempty subsets of $[n]$ for $1 \leq i \leq m$, $m \geq 2$, $A = \{A_1, \ldots, A_m\}$, then the set of the vectors $\sum_{k=1}^m e_{i_k}$ with $i_k \in A_k$ is the base of a polymatroid, called the transversal polymatroid presented by $A$. The base ring of a transversal polymatroid presented by $A$ is the ring

$$K[A] := K[x_{i_1} \cdots x_{i_m} \mid i_j \in A_j, 1 \leq j \leq m].$$

We know by [3] that the $K$–algebra $K[A]$ is normal and hence Cohen-Macaulay. From [5] we know that the transversal polymatroid presented by

$$A = \{A_k \mid A_{\sigma^t(k)} = [n], \text{ if } k \in [i] \cup \{n\} \text{ and } A_{\sigma^t(k)} = [n] \setminus \sigma^t[i], \text{ if } k \in [n-1] \setminus [i]\}$$

has the base ring associated $K[A]$ a Gorenstein ring for any $1 \leq i \leq n-1$ and $0 \leq t \leq n-1$.

We put

$$\nu_{\sigma^t[i]} = -(n - i - 1) \sum_{k=1}^i e_{\sigma^t(k)} + (i + 1) \sum_{k=i+1}^n e_{\sigma^t(k)}$$

for any $0 \leq t \leq n-1$ and $1 \leq i \leq n-1$.

The cone generated by the exponent vectors of the monomials defining the base ring $K[A]$,

$$A := \{\log(x_{j_1} \cdots x_{j_n}) \mid j_k \in A_k, k \in [n]\} \subset \mathbb{N}^n,$$

has the irreducible representation:

$$\mathbb{R}^+ A = \bigcap_{a \in N} H^+_a,$$

where $N = \{\nu_{\sigma^t[i]}, \nu_{\sigma^t[n-1]} \mid 0 \leq k \leq n-1\}$.

It is easy to see that for any $1 \leq i \leq n-1$ and $0 \leq t \leq n-1$

$$A = \{\alpha \in \mathbb{N}^n \mid |\alpha| = n, 0 \leq \alpha_{t+1} + \ldots + \alpha_{t+i} \leq i + 1\}, \text{ if } i + t \leq n$$

and

$$A = \{\alpha \in \mathbb{N}^n \mid |\alpha| = n, 0 \leq \sum_{s=1}^{i+t-n} \alpha_s + \sum_{s=t+1}^n \alpha_s \leq i + 1\}, \text{ if } i + t > n.$$ 

We denote by $\{x_{i_1}, \ldots, x_{i_r}\}^r$ all monomials of degree $r$ with the indeterminates $x_{i_1}, \ldots, x_{i_r}$.

3 Intersection of cones of dimension $n$ with $n+1$ facets

Let $r \geq 2, 1 \leq i_1, \ldots, i_r \leq n - 2$, $0 = t_1 \leq t_2, \ldots, t_r \leq n - 1$ and we consider $r$ presentations of transversal polymatroids:

$$A_s = \{A_{s,k} \mid A_{s,\sigma^2(k)} = [n], \text{ if } k \in [i_2] \cup \{n\},$$
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$A_s, \sigma^s \subset (k) = [n] \setminus \sigma^s[i_2], \text{ if } k \in [n-1] \setminus \{i_2\}$

for any $1 \leq s \leq r$. From [5] we know that the base rings $K[A_s]$ are Gorenstein rings and the cones generated by the exponent vectors of the monomials defining the base ring associated to a transversal polymatroid presented by $A_s$ are:

$$R_+ A_s = \bigcap_{a \in N_s} H_a^+,$$

where $N_s = \{\nu_{\sigma^s[i_s]}, \nu_{\sigma^k[n-1]} | 0 \leq k \leq n - 1\}$,

$A_s = \{\log(x_{j_1} \cdots x_{j_n}) | j_k \in A_s, 1 \leq k \leq n\} \subset \mathbb{N}^n$

for any $1 \leq s \leq r$.

We denote by $K[A_1 \cap \ldots \cap A_r]$, $K$–algebra generated by $x^a$ with $a \in A_1 \cap \ldots \cap A_r$.

It is clear that the cone

$$R_+(A_1 \cap \ldots \cap A_r) \subseteq R_+ A_1 \cap \ldots \cap R_+ A_r = \bigcap_{a \in N_1 \cup \ldots \cup N_r} H_a^+.$$ 

Conversely, since

$A_1 \cap \ldots \cap A_r = \{\alpha \in \mathbb{N}^n | |\alpha| = n, H_{\nu_{\sigma^s[i_s]}}(\alpha) \geq 0, \text{ for any } 1 \leq s \leq r\}$,

we have that

$$R_+(A_1 \cap \ldots \cap A_r) \supseteq \bigcap_{a \in N_1 \cup \ldots \cup N_r} H_a^+$$

and so,

$$R_+(A_1 \cap \ldots \cap A_r) = \bigcap_{a \in N_1 \cup \ldots \cup N_r} H_a^+.$$ 

We claim that the intersection

$$\bigcap_{a \in N_1 \cup \ldots \cup N_r} H_a^+$$

is the irreducible representation of the cone $R_+(A_1 \cap \ldots \cap A_r)$. We prove by induction on $r \geq 1$. If $r = 1$, then the intersection

$$\bigcap_{a \in N_1} H_a^+$$

is the irreducible representation of the cone $R_+(A_1)$. (see [5], Lemma 4.1)

If $r > 1$ then we have two cases to study:
1) if we delete, for some $0 \leq k \leq n - 1$, the hyperplane with the normal $\nu_{\sigma^k[n-1]}$; then a coordinate of a $\log(x_{j_1} \cdots x_{j_k+1} \cdots x_{j_{n-1}}x_{j_n})$ would be negative, which
is impossible;
2) if we delete, for some 1 ≤ s ≤ r, the hyperplane with the normal \( \nu_{\sigma^s[i_s]} \), then

\[
\bigcap_{\alpha \in N_1 \cup \ldots \cup N_s \cup N_{s+1} \ldots N_r} H_a^+
\]

is by induction the irreducible representation of the cone \( \mathbb{R}_+(A_1 \cap \ldots \cap A_{s-1} \cap A_{s+1} \ldots \cap A_r) \), which is different from \( \mathbb{R}_+(A_1 \cap \ldots \cap A_r) \). Hence, the intersection

\[
\bigcap_{\alpha \in N_1 \cup \ldots \cup N_r} H_a^+
\]

is the irreducible representation of the cone \( \mathbb{R}_+(A_1 \cap \ldots \cap A_r) \).

**Lemma 3.1.** The \( K \)-algebra \( K[A_1 \cap \ldots \cap A_r] \) is a Gorenstein ring.

**Proof:** We will show that the canonical module \( \omega_{K[A_1 \cap \ldots \cap A_r]} \) is generated by \( (x_1 \cdots x_n)K[A_1 \cap \ldots \cap A_r] \). Since the semigroups \( \mathbb{N}(A_i) \) are normal for any 1 ≤ t ≤ r, it follows that \( \mathbb{N}(A_1 \cap \ldots \cap A_r) \) is normal. Then the \( K \)-algebra \( K[A_1 \cap \ldots \cap A_r] \) is normal (see [1] Theorem 6.1.4. p. 260 ) and using the Danilov – Stanley theorem we get that the canonical module \( \omega_{K[A_1 \cap \ldots \cap A_r]} \) is

\[
\omega_{K[A_1 \cap \ldots \cap A_r]} = (\{x^\alpha \mid \alpha \in \mathbb{N}(A_1 \cap \ldots \cap A_r) \cap ri(\mathbb{R}_+(A_1 \cap \ldots \cap A_r))\}).
\]

Let \( d_t \) be the greatest common divisor of \( n \) and \( i_t + 1 \), \( gcd(n, i_t + 1) = d_t \), for any 1 ≤ t ≤ r.

For any 1 ≤ s ≤ r, there exist two possibilities for the equation of the facet \( H_{\nu_{\sigma^s[i_s]}} \):

1) If \( i_s + t_s \leq n \), then the equation of the facet \( H_{\nu_{\sigma^s[i_s]}} \) is:

\[
H_{\nu_{\sigma^s[i_s]}}(y) : \frac{(i_s + 1)}{d_s} \sum_{k=1}^{t_s} y_k - \frac{(n - i_s - 1)}{d_s} \sum_{k=t_s + 1}^{i_s + t_s} y_k + \frac{(i_s + 1)}{d_s} \sum_{k=t_s + i_s + 1}^{n} y_k = 0.
\]

2) If \( i_s + t_s > n \), then the equation of the facet \( H_{\nu_{\sigma^s[i_s]}} \) is:

\[
H_{\nu_{\sigma^s[i_s]}}(y) :
\]

\[
- \frac{(n - i_s - 1)}{d_s} \sum_{k=1}^{i_s + t_s - n} y_k + \frac{(i_s + 1)}{d_s} \sum_{k=i_s + t_s - n + 1}^{t_s} y_k - \frac{(n - i_s - 1)}{d_s} \sum_{k=t_s + 1}^{n} y_k = 0.
\]

The relative interior of the cone \( \mathbb{R}_+(A_1 \cap \ldots \cap A_r) \) is:

\[
ri(\mathbb{R}_+(A_1 \cap \ldots \cap A_r)) = \{y \in \mathbb{R}^n \mid y_k > 0, H_{\nu_{\sigma^s[i_s]}}(y) > 0 \text{ for any } 1 \leq k \leq n \text{ and } 1 \leq s \leq r\}.
\]
We will show that
\[ N(A_1 \cap \ldots \cap A_r) \cap ri(\mathbb{R}_+(A_1 \cap \ldots \cap A_r)) =
\]
\[ (1, \ldots, 1) + (N(A_1 \cap \ldots \cap A_r) \cap \mathbb{R}_+(A_1 \cap \ldots \cap A_r)). \]

It is clear that \( ri(\mathbb{R}_+(A_1 \cap \ldots \cap A_r)) \supset (1, \ldots, 1) + \mathbb{R}_+(A_1 \cap \ldots \cap A_r) \).

If \((\alpha_1, \alpha_2, \ldots, \alpha_n) \in N(A_1 \cap \ldots \cap A_r) \cap ri(\mathbb{R}_+(A_1 \cap \ldots \cap A_r))\), then \(\alpha_k \geq 1\) for any \(1 \leq k \leq n\) and for any \(1 \leq s \leq r\) we have
\[
\frac{(i_s + 1)}{d_s} \sum_{k=1}^{t_s} \alpha_k - \frac{(n - i_s - 1)}{d_s} \sum_{k=t_s+1}^{t_s+i_s} \alpha_k + \frac{(i_s + 1)}{d_s} \sum_{k=t_s+i_s+1}^{n} \alpha_k \geq 1, \text{ if } i_s + t_s \leq n
\]
or
\[
- \frac{(n - i_s - 1)}{d_s} \sum_{k=1}^{t_s} \alpha_k + \frac{(i_s + 1)}{d_s} \sum_{k=t_s}^{t_s+i_s-n+1} \alpha_k - \frac{(n - i_s - 1)}{d_s} \sum_{k=t_s+i_s+1}^{n} \alpha_k \geq 1, \text{ if } i_s + t_s > n
\]
and
\[
\sum_{k=1}^{n} \alpha_k = t n \text{ for some } t \geq 1.
\]

We claim that there exist \((\beta_1, \beta_2, \ldots, \beta_n) \in N(A_1 \cap \ldots \cap A_r) \cap \mathbb{R}_+(A_1 \cap \ldots \cap A_r)\) such that \((\alpha_1, \alpha_2, \ldots, \alpha_n) = (\beta_1 + 1, \beta_2 + 1, \ldots, \beta_n + 1)\). Let \(\beta_k = \alpha_k - 1\) for all \(1 \leq k \leq n\).

It is clear that \(\beta_k \geq 0\) and for any \(1 \leq s \leq r\),
\[
H_{\nu,s|s}[\beta] = H_{\nu,s|s}[\alpha] - H_{\nu,s|s}[1, \ldots, 1] = H_{\nu,s|s}[\alpha] - \frac{n}{d_s}.
\]

If \(H_{\nu,s|s}[\beta] = j_s\), for some \(1 \leq s \leq r\) and \(1 \leq j_s \leq \frac{n}{d_s} - 1\), then we will get a contradiction. Indeed, since \(n\) divides \(\sum_{k=1}^{n} \alpha_k\), it follows \(\frac{n}{d_s}\) divides \(j_s\), which is false.

Hence, we have \((\beta_1, \beta_2, \ldots, \beta_n) \in N(A_1 \cap \ldots \cap A_r) \cap \mathbb{R}_+(A_1 \cap \ldots \cap A_r)\) and \((\alpha_1, \alpha_2, \ldots, \alpha_n) \in N(A_1 \cap \ldots \cap A_r) \cap ri(\mathbb{R}_+(A_1 \cap \ldots \cap A_r))\).

Since \(N(A_1 \cap \ldots \cap A_r) \cap ri(\mathbb{R}_+(A_1 \cap \ldots \cap A_r)) = (1, \ldots, 1) + (N(A_1 \cap \ldots \cap A_r) \cap \mathbb{R}_+(A_1 \cap \ldots \cap A_r))\), we get that \(\omega_K[A_1 \cap \ldots \cap A_r] = (x_1 \cdots x_n)K[A_1 \cap \ldots \cap A_r] \).

Let \(S\) be a standard graded \(K\)-algebra over a field \(K\). Recall that the \(a\)–invariant of \(S\), denoted \(a(S)\), is the degree as a rational function of the Hilbert series of \(S\), see for instance ([6], p. 99). If \(S\) is Cohen-Macaulay and \(\omega_S\) is the canonical module of \(S\), then
\[
a(S) = - \min \{i \mid (\omega_S)_i \neq 0\},
\]
see ([1], 41) and ([6], Proposition 4.2.3). In our situation \( S = K[A_1 \cap \ldots \cap A_r] \) is normal and consequently Cohen-Macaulay, thus this formula applies. As consequence of Lemma 3.1, the we have the following:

**Corollary 3.2.** The \( a \)-invariant of \( K[A_1 \cap \ldots \cap A_r] \) is \( a(K[A_1 \cap \ldots \cap A_r]) = -1 \).

**Proof:** Let \( \{x^{a_1}, \ldots, x^{a_n}\} \) be the generators of \( K \)-algebra \( K[A_1 \cap \ldots \cap A_r] \).

**Example** 1. Let \( n = 4 \), \( \mathcal{A} = \{A_1, A_2, A_3, A_4\} \), \( \mathcal{B} = \{B_1, B_2, B_3, B_4\} \), where \( A_1 = A_4 = B_2 = B_3 = \{1, 2, 3, 4\}, A_2 = A_3 = \{2, 3, 4\}, B_1 = B_4 = \{1, 3, 4\} \) and \( K[\mathcal{A}], K[\mathcal{B}] \) the base rings associated to transversal polymatroids presented by \( \mathcal{A} \), respectively \( \mathcal{B} \). It is easy to see that the exponent vectors of monomials defining the base rings \( K[\mathcal{A}] \), respectively \( K[\mathcal{B}] \) and \( K[\mathcal{A} \cap \mathcal{B}] \) the \( K \)-algebra generated by \( x^a \) with \( a \in \mathcal{A} \cap \mathcal{B} \).

**Question:** There exists a transversal polymatroid such that its base ring is the \( K \)-algebra \( K[\mathcal{A} \cap \mathcal{B}] \)?

In the following we give two suggestive examples.

**Example 2.** Let \( n = 4 \), \( \mathcal{A} = \{A_1, A_2, A_3, A_4\} \), \( \mathcal{B} = \{B_1, B_2, B_3, B_4\} \) where \( A_1 = A_2 = A_4 = B_1 = B_2 = B_3 = \{1, 2, 3, 4\}, A_3 = \{3, 4\} \) and \( K[\mathcal{A}], K[\mathcal{B}] \) the base rings associated to transversal polymatroids presented by \( \mathcal{A} \), respectively \( \mathcal{B} \). It is easy to see that the generators set of \( K[\mathcal{A}] \), respectively \( K[\mathcal{B}] \) are given by \( A = \{y \in \mathbb{N}^4 \mid \gamma y = 4, 0 \leq y_1 \leq 2, y_k \geq 0, 1 \leq k \leq 4\} \), respectively \( B = \{y \in \mathbb{N}^4 \mid \gamma y = 4, 0 \leq y_2 \leq 2, y_k \geq 0, 1 \leq k \leq 4\} \). We show that the \( K \)-algebra \( K[\mathcal{A} \cap \mathcal{B}] \) is the base ring of the transversal polymatroid presented by \( \mathcal{C} = \{C_1, C_2, C_3, C_4\} \), where \( C_1 = C_4 = \{1, 3, 4\}, C_2 = C_3 = \{2, 3, 4\} \). Since the base ring associated to the transversal polymatroid presented by \( \mathcal{C} \) has the exponent set \( C = \{y \in \mathbb{N}^4 \mid \gamma y = 4, 0 \leq y_1 \leq 2, 0 \leq y_2 \leq 2, y_k \geq 0, 1 \leq k \leq 4\} \), it follows that \( K[\mathcal{A} \cap \mathcal{B}] = K[\mathcal{C}] \). Thus, in this example \( K[\mathcal{A} \cap \mathcal{B}] \) is the base ring of a transversal polymatroid.

We claim that there exists no transversal polymatroid \( \mathcal{P} \) such that \( K \)-algebra \( K[\mathcal{A} \cap \mathcal{B}] \) is its
base ring. Suppose, on the contrary, let $\mathcal{P}$ be presented by $\mathcal{C} = \{C_1, C_2, C_3, C_4\}$ with each $C_k \subset [4]$. Since $(3, 0, 1, 0), (3, 0, 0, 1) \in \mathcal{P}$ and $(3, 1, 0, 0) \notin \mathcal{P}$, we may assume that changing the numerotation of $\{C_i\}_{i=1}^4$ that $1 \in C_1, 1 \in C_2, 1 \in C_4$ and $C_3 = \{3, 4\}$. Since $(0, 3, 0, 1) \in \mathcal{P}$, we assume that $2 \in C_1, 2 \in C_2, 2 \in C_4$. Hence $(0, 3, 1, 0) \in \mathcal{P}$, a contradiction.

Let $1 \leq i_1, i_2 \leq n - 2$, $0 \leq t_2 \leq n - 1$ and $\tau \in S_{n-2}$, $\tau = (1, 2, \ldots, n - 2)$ the cycle of length $n - 2$. We consider two transversal polymatroids presented by:

$$A = \{A_k \mid A_k = [n], \text{ if } k \in [i_1] \cup \{n\}, \text{ } A_k = [n] \setminus [i_1], \text{ if } k \in [n-1] \setminus [i_1]\}$$

and

$$B = \{B_k \mid B_{\sigma^{[\tau][k]} = [n], \text{ if } k \in [i_2] \cup \{n\}, \text{ } B_{\sigma^{[\tau][k]} = [n] \setminus \sigma^{[\tau][i_2]}, \text{ if } k \in [n-1] \setminus [i_2]\}$$

such that $A$, respectively $B$ is the exponent vectors of the monomials defining the base rings associated to transversal polymatroid presented by $A$, respectively $B$. From [5] we know that the base rings $K[A]$, respectively $K[B]$ are Gorenstein rings and the cones generated by the exponent vectors of the monomials defining the base ring associated to the transversal polymatroids presented by $A$, respectively $B$ are:

$$R^+_A = \bigcap_{\alpha \in N_1} H^+_\alpha, \quad R^+_B = \bigcap_{\alpha \in N_2} H^+_\alpha,$$

where $N_1 = \{\nu_{\sigma^{[\tau][i_1]}}, \nu_{\sigma^{[\tau][n-1]}}, 0 \leq k \leq n - 1\}, \quad N_2 = \{\nu_{\sigma^{[\tau][i_2]}}, \nu_{\sigma^{[\tau][n-1]}}, 0 \leq k \leq n - 1\}$.

$$A = \{\log(x_j, \ldots x_j) \mid j_k \in A_k, 1 \leq k \leq n\} \subset \mathbb{N}^n \text{ and }$$

$$B = \{\log(x_j, \ldots x_j) \mid j_k \in B_k, 1 \leq k \leq n\} \subset \mathbb{N}^n.$$ 

It is easy to see that $A = \{\alpha \in \mathbb{N}^n \mid 0 \leq \alpha_1 + \ldots + \alpha_{i_1} \leq i_1 + 1 \text{ and } |\alpha| = n\}$ and

$$B = \{\alpha \in \mathbb{N}^n \mid 0 \leq \alpha_{t_2+1} + \ldots + \alpha_{t_2+i_2} \leq i_2 + 1 \text{ and } |\alpha| = n\}, \text{ if } i_2 + t_2 \leq n$$
or

$$B = \{\alpha \in \mathbb{N}^n \mid 0 \leq \sum_{s=1}^{i_2+t_2-n} \alpha_s + \sum_{s=t_2+1}^{n} \alpha_s \leq i_2 + 1 \text{ and } |\alpha| = n\}, \text{ if } i_2 + t_2 \geq n.$$ 

For any base ring $K[A]$ of a transversal polymatroid presented by $A = \{A_1, \ldots, A_n\}$ we associate a $(n \times n)$ square tiled by closed unit subsquares, called boxes, colored with colors, "white" and "black", as follows: the box of coordinate $(i, j)$ is "white" if $j \in A_i$, otherwise the box is "black". We will call this square the polymatroidal diagram associated to the presentation $A = \{A_1, \ldots, A_n\}$.

Next we give necessary and sufficient conditions such that the $K$–algebra $K[A \cap B]$ is the base ring associated to some transversal polymatroid.
Theorem 4.1. Let $1 \leq i_1, i_2 \leq n - 2$, $0 \leq t_2 \leq n - 1$. We consider two presentation of transversal polymatroids presented by: $A = \{A_k \mid A_k = [n], \text{ if } k \in [i_1] \cup \{n\}, A_k = [n] \setminus [i_1], \text{ if } k \in [n-1] \setminus [i_1]\}$ and $B = \{B_k \mid B_{\sigma^{t_2}(k)} = [n], \text{ if } k \in [i_2] \cup \{n\}, B_{\sigma^{t_2}(k)} = [n] \setminus \sigma^{t_2}[i_2], \text{ if } k \in [n-1] \setminus [i_2]\}$ such that $A$, respectively $B$ are the exponent vectors of the monomials defining the base ring associated to the transversal polymatroids presented by $A$, respectively $B$.

Then, the $K$–algebra $K[A \cap B]$ is the base ring associated to a transversal polymatroid if and only if one of the following conditions hold:

a) $i_1 = 1$.

b) $i_1 \geq 2$ and $t_2 = 0$.

c) $i_1 \geq 2$ and $t_2 = i_1$.

d) $i_1 \geq 2$, $1 \leq t_2 \leq i_1 - 1$ and $i_2 \in \{1, \ldots, i_1 - t_2\} \cup \{n - t_2, \ldots, n - 2\}$;

e) $i_1 \geq 2$, $i_1 + 1 \leq t_2 \leq n - 1$ and $i_2 \in \{1, \ldots, n - t_2\} \cup \{n - t_2 + i_1, \ldots, n - 2\}$.

The proof follows from the following three lemmas.

Lemma 4.2. Let $A$ and $B$ be like above. If $i_1 \geq 2$, $1 \leq t_2 \leq i_1 - 1$, then the $K$–algebra $K[A \cap B]$ is the base ring associated to some transversal polymatroid if and only if $i_2 \in \{1, \ldots, i_1 - t_2\} \cup \{n - t_2, \ldots, n - 2\}$.

Proof: $\iff$ Let $i_2 \in \{1, \ldots, i_1 - t_2\} \cup \{n - t_2, \ldots, n - 2\}$. We will prove that there exists a transversal polymatroid $\mathcal{P}$ presented by $\mathcal{C} = \{C_1, \ldots, C_n\}$ such that the base ring associated to $\mathcal{P}$ is $K[A \cap B]$.

We have two cases to study:

Case 1. If $i_2 + t_2 \leq i_1$, then let $\mathcal{P}$ be the transversal polymatroid presented by $\mathcal{C} = \{C_1, \ldots, C_n\}$, where

$$C_1 = \ldots = C_{i_2} = C_n = [n],$$

$$C_{i_2+1} = \ldots = C_{i_1} = [n] \setminus \sigma^{t_2}[i_2],$$

$$C_{i_1+1} = \ldots = C_{n-1} = [n] \setminus [i_1].$$

The polymatroidal diagram associated is the following:
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It is easy to see that the base ring associated to the transversal polymatroid $\mathcal{P}$ presented by $\mathcal{C}$ is generated by the following set of monomials:

$$\{x_{t_2+1}, \ldots, x_{t_2+i_2}\}^{i_2+1-k}\{x_1, \ldots, x_{t_2}, x_{t_2+i_2+1}, \ldots, x_{i_1}\}^{i_1-i_2+k-s}\{x_{i_1+1}, \ldots, x_n\}^{n-1-i_1+s}$$

for any $0 \leq k \leq i_2 + 1$ and $0 \leq s \leq i_1 - i_2 + k$. If $x^\alpha \in K[\mathcal{C}]$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, then there exists $0 \leq k \leq i_2 + 1$ and $0 \leq s \leq i_1 - i_2 + k$ such that

$$\alpha_{t_2+1} + \ldots + \alpha_{t_2+i_2} = i_2 + 1 - k \text{ and } \alpha_1 + \ldots + \alpha_{i_1} = i_1 + 1 - s$$

and thus, $K[\mathcal{C}] \subset K[A \cap B]$.

Conversely, if $\alpha \in A \cap B$ then $\alpha_{t_2+1} + \ldots + \alpha_{t_2+i_2} \leq i_2 + 1$ and $\alpha_1 + \ldots + \alpha_{i_1} \leq i_1 + 1$ and thus there exists $0 \leq k \leq i_2 + 1$ and $0 \leq s \leq i_1 - i_2 + k$ such that

$$x^\alpha \in \{x_{t_2+1}, \ldots, x_{t_2+i_2}\}^{i_2+1-k}\{x_1, \ldots, x_{t_2}, x_{t_2+i_2+1}, \ldots, x_{i_1}\}^{i_1-i_2+k-s}\{x_{i_1+1}, \ldots, x_n\}^{n-1-i_1+s}.$$ 

Thus, $K[\mathcal{C}] \supset K[A \cap B]$ and so $K[\mathcal{C}] = K[A \cap B]$.

**Case 2.** If $i_2 + t_2 > i_1$, then it follows that $i_2 \geq n - t_2$ and $n - i_2 \leq t_2 < i_1$. Let $\mathcal{P}$ be the transversal polymatroid presented by $\mathcal{C} = \{C_1, \ldots, C_n\}$, where

$$C_1 = \ldots = C_{n-i_2-1} = [n] \setminus \sigma^{t_2}[i_2],$$

$$C_{n-i_2} = \ldots = C_{i_1} = C_n = [n],$$

$$C_{i_1+1} = \ldots = C_{n-1} = [n] \setminus [i_1].$$

The polymatroidal diagram associated is the following:
It is easy to see that the base ring associated to the transversal polymatroid \( \mathcal{P} \) presented by \( \mathcal{C} \) is generated by the following set of monomials:

\[
\{x_{i_2+t_2-n+1}, \ldots, x_{t_2}\}^{i_1+1-k}\{x_1, \ldots, x_{i_2+t_2-n}, x_{t_2+1}, \ldots, x_{i_1}\}^{k-s}\{x_{i_1+1}, \ldots, x_n\}^{n-1-i_1+s}
\]

for any \( 0 \leq k \leq i_1 + i_2 - n + 2 \) and \( 0 \leq s \leq k \). Since \( i_2 + t_2 \geq n \) and \( 0 \leq s \leq k \leq i_1 + i_2 - n + 2 \) it follows that for any \( x^\alpha \in K[\mathcal{C}] \) we have \( \alpha_1 + \ldots + \alpha_{i_1} = i_1 + 1 - s \leq i_1 + 1, \alpha_1 + \ldots + \alpha_{i_2+t_2-n} + \alpha_{t_2+1} + \ldots + \alpha_n = n - 1 - i_1 + k \leq i_2 + 1 \) and thus, \( K[\mathcal{C}] \subset K[A \cap B] \).

Conversely, if \( \alpha \in A \cap B \) then \( \alpha_1 + \ldots + \alpha_{i_1} \leq i_1 + 1, \alpha_1 + \ldots + \alpha_{i_2+t_2-n} + \alpha_{t_2+1} + \ldots + \alpha_n \leq i_2 + 1 \) and thus there exists \( 0 \leq k \leq i_1 + i_2 - n + 2 \) and \( 0 \leq s \leq k \) such that

\[
x^\alpha \in \{x_{i_2+t_2-n+1}, \ldots, x_{t_2}\}^{i_1+1-k}\{x_1, \ldots, x_{i_2+t_2-n}, x_{t_2+1}, \ldots, x_{i_1}\}^{k-s}\{x_{i_1+1}, \ldots, x_n\}^{n-1-i_1+s}.
\]

Thus, \( K[\mathcal{C}] \supset K[A \cap B] \) and so \( K[\mathcal{C}] = K[A \cap B] \).

Now suppose that there exists a transversal polymatroid \( \mathcal{P} \) given by \( \mathcal{C} = \{C_1, \ldots, C_n\} \) such that its associated base ring is \( K[A \cap B] \) and we will prove that \( i_2 \in \{1, \ldots, i_1 - t_2\} \cup \{n - t_2, \ldots, n - 2\} \).

Suppose, on the contrary, let \( i_1 + 1 - t_2 \leq i_2 \leq n - t_2 - 1 \). We have two cases to study:

**Case 1'**. If \( n - i_1 - 1 \leq i_2 + 1 \), then since \( (i_1 + 1)e_1 + (n - i_1 - 1)e_k \in \mathcal{P} \) for any \( i_1 + 1 \leq k \leq n \) and \( (i_1 + 1)e_1 + e_s + (n - i_1 - 2)e_k \notin \mathcal{P} \) for any \( 2 \leq s \leq i_1 \) and \( i_1 + 1 \leq k \leq n \), we may assume \( 1 \in C_1, \ldots, 1 \in C_{i_1}, 1 \in C_n \) and \( C_{i_1+1} = \ldots = C_{n-1} = [n] \setminus [i_1] \).

If \( i_1 \leq i_2 \), then since \( (i_1 + 1)e_{i_2+1} + (n - i_1 - 1)e_{i_2+i_2+1} \in \mathcal{P} \), we may assume \( t_2+1 \in C_1, \ldots, t_2+1 \in C_{i_1}, t_2+1 \in C_n \). Then \( (i_1 + 1)e_{i_2+1} + (n - i_1 - 1)e_{i_1+1} \in \mathcal{P} \), which is false. If \( i_1 > i_2 \), then since \( (i_2 + 1)e_{i_2+1} + (n - i_1 - 1)e_{i_2+i_2+1} \in \mathcal{P} \), we may assume \( t_2+1 \in C_1, \ldots, t_2+1 \in C_{i_2}, t_2+1 \in C_n \). Then \( (i_2 + 1)e_{i_2+1} + (i_1 - i_2)e_1 + (n - i_1 - 1)e_{i_1+1} \in \mathcal{P} \), which is false.
Case 2’. If \( n-i_1-1 > i_2+1 \), then since \((i_1+1)e_1+(i_2+1)e_{i_1+1}+(n-i_1-i_2-2)ek \in \mathcal{P}\) for any \( t_2+i_2+1 \leq k \leq n \) and \((i_1+1)e_1+e_s+(i_2+1)e_{i_1+1}+(n-i_1-i_2-3)ek \notin \mathcal{P}\) for any \( 1 \leq s \leq i_1 \) and \( t_2+i_2+1 \leq k \leq n \), we may assume \( 1 \in C_1, \ldots, 1 \in C_n, C_{i_1+1} = \ldots = C_{i_1+i_2+1} = [n] \setminus [i_1] \) and \( C_{i_1+i_2+2} = \ldots = C_{n-1} = [n] \setminus [t_2+i_2] \).

If \( i_1 \leq i_2 \), then since \((i_1+1)e_{t_2+1}+(n-i_1-1)e_{t_2+i_2+1} \in \mathcal{P}\), we may assume \( t_2+1 \in C_1, \ldots, t_2+1 \in C_n \). Then \((i_1+1)e_{t_2+1}+(i_2+1)e_{i_1+1}+(n-i_1-i_2-2)e_{t_2+i_2+1} \in \mathcal{P}\), which is false. If \( i_1 > i_2 \), then since \((i_2+1)e_{t_2+1}+(i_1-i_2)e_1+(n-i_1-1)e_{t_2+i_2+1} \in \mathcal{P}\), we may assume \( t_2+1 \in C_1, \ldots, t_2+1 \in C_{i_2}, t_2+1 \in C_n \). Then \((i_1-i_2)e_1+(i_2+1)e_{t_2+1}+(i_2+1)e_{t_2+i_2}+(n-i_1-i_2-2)e_{t_2+i_2+1} \in \mathcal{P}\), which is false.

Lemma 4.3. Let \( A \) and \( B \) be like above. If \( i_1 \geq 2, i_1+1 \leq t_2 \leq n-1 \), then the \( K \)-algebra \( K[A \cap B] \) is the base ring associated to some transversal polymatroid if and only if \( i_2 \in \{1, \ldots, n-t_2\} \cup \{n-t_2+i_1, \ldots, n-2\} \).

Proof: ” \( \Leftarrow \) ” Let \( i_2 \in \{1, \ldots, n-t_2\} \cup \{n-t_2+i_1, \ldots, n-2\} \). We will prove that there exists a transversal polymatroid \( \mathcal{P} \) presented by \( \mathcal{C} = \{C_1, \ldots, C_n\} \) such that its associated base ring is \( K[A \cap B] \). We distinct three cases to study:

Case 1. If \( i_2+t_2 \leq n \) and \( i_1+1+i_2 \neq n \), then let \( \mathcal{P} \) be the transversal polymatroid presented by \( \mathcal{C} = \{C_1, \ldots, C_n\} \), where

\[
\begin{align*}
C_1 &= \ldots = C_{i_1} = C_n = [n] \setminus \sigma^{t_2}[i_2], \\
C_{i_1+1} &= \ldots = C_{i_1+i_2+1} = [n] \setminus [i_1], \\
C_{i_1+i_2+2} &= \ldots = C_{n-1} = [n] \setminus ([i_1] \cup \sigma^{t_2}[i_2]).
\end{align*}
\]

The polymatroidal diagram associated is the following:

\[
\begin{array}{c}
t_2 - \text{columns} \\
\downarrow \\
\begin{array}{c}
\begin{array}{c}
i_1 - \text{rows}
\end{array}
\end{array}
\end{array}
\]
It is easy to see that the base ring $K[C]$ associated to the transversal polymatroid $P$ presented by $C$ is generated by the following set of monomials:

\[
\begin{align*}
\{x_1, \ldots, x_{i_1}\}^{i_1+1-k} & \{x_{t_2+1}, \ldots, x_{t_2+i_2}\}^{i_2+1-s} \\
\{x_{i_1+1}, \ldots, x_{t_2}, x_{t_2+i_2+1}, \ldots, x_n\}^{n-i_1-i_2-2+k+s}
\end{align*}
\]

for any $0 \leq k \leq i_1 + 1$ and $0 \leq s \leq i_2 + 1$. If $x^\alpha \in K[C]$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, then there exists $0 \leq k \leq i_1 + 1$ and $0 \leq s \leq i_2 + 1$ such that

\[
\alpha_{t_2+1} + \ldots + \alpha_{t_2+i_2} = i_2 + 1 - s, \quad \alpha_1 + \ldots + \alpha_{i_1} = i_1 + 1 - k
\]

and thus, $K[C] \subseteq K[A \cap B]$. Conversely, if $\alpha \in A \cap B$ then $\alpha_{t_2+1} + \ldots + \alpha_{t_2+i_2} \leq i_2 + 1$, $\alpha_1 + \ldots + \alpha_{i_1} \leq i_1 + 1$; thus there exists $0 \leq k \leq i_1 + 1$ and $0 \leq s \leq i_2 + 1$ such that

\[
\alpha_{t_2+1} + \ldots + \alpha_{t_2+i_2} = i_2 + 1 - s, \quad \alpha_1 + \ldots + \alpha_{i_1} = i_1 + 1 - k
\]

and since $|\alpha| = n$ it follows that

\[
\begin{align*}
x^\alpha & \in \{x_1, \ldots, x_{i_1}\}^{i_1+1-k} \{x_{t_2+1}, \ldots, x_{t_2+i_2}\}^{i_2+1-s} \\
& \{x_{i_1+1}, \ldots, x_{t_2}, x_{t_2+i_2+1}, \ldots, x_n\}^{n-i_1-i_2-2+k+s}.
\end{align*}
\]

Thus, $K[C] \supseteq K[A \cap B]$ and so $K[C] = K[A \cap B]$.

**Case 2.** If $i_2 + t_2 \leq n$ and $i_1 + 1 + i_2 = n$, then $t_2 = i_1 + 1$ and let $P$ be the transversal polymatroid presented by $C = \{C_1, \ldots, C_n\}$, where

\[
\begin{align*}
C_1 = \ldots = C_{i_1} & = [n] \setminus \sigma^{t_2}[i_2], \\
C_{i_1+1} = \ldots = C_{n-1} & = [n] \setminus [i_1], \\
C_n & = [n].
\end{align*}
\]

The polymatroidal diagram associated is the following:
It is easy to see that the base ring \( K[C] \) associated to the transversal polymatroid \( \mathcal{P} \) presented by \( C \) is generated by the following set of monomials:

\[
\{x_1, \ldots, x_{i_1}\}^{i_1+1-k} \cdot x_{i_1+1}^{n-i_1-1+k-s} \cdot \{x_{i_1+2}, \ldots, x_n\}^s
\]

for any \( 0 \leq k \leq i_1 + 1 \) and \( 0 \leq s \leq n - i_1 \). If \( x^\alpha \in K[C] \), \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \), then there exists \( 0 \leq k \leq i_1 + 1 \) and \( 0 \leq s \leq i_2 + 1 (= n - i_1) \) such that

\[
\alpha t_2 + 1 + \ldots + \alpha_{t_2+i_2} = \alpha_{i_1+2} + \ldots + x_n = s \leq i_2 + 1 \quad \alpha_1 + \ldots + \alpha_{i_1} = i_1 + 1 - k
\]

and thus, \( K[C] \subset K[A \cap B] \). Conversely, if \( \alpha \in A \cap B \) then \( \alpha_{t_2+1} + \ldots + \alpha_{t_2+i_2} = \alpha_{i_1+2} + \ldots + x_n \leq i_2 + 1 \) and \( \alpha_1 + \ldots + \alpha_{i_1} \leq i_1 + 1 \); thus there exists \( 0 \leq k \leq i_1 + 1 \) and \( 0 \leq s \leq i_2 + 1 \) such that

\[
x^\alpha \in \{x_1, \ldots, x_{i_1}\}^{i_1+1-k} \cdot x_{i_1+1}^{n-i_1-1+k-s} \cdot \{x_{i_1+2}, \ldots, x_n\}^s.
\]

Thus, \( K[C] \supset K[A \cap B] \) and so \( K[C] = K[A \cap B] \).

**Case 3.** If \( i_2 + t_2 > n \), then let \( \mathcal{P} \) be the transversal polymatroid presented by \( C = \{C_1, \ldots, C_n\} \), where

\[
C_1 = \ldots = C_{i_1} = C_n = [n],
\]

\[
C_{i_1+1} = \ldots = C_{i_1+i_2-1} = [n] \setminus \sigma^{t_2}[i_2],
\]

\[
C_{i_1+i_2} = \ldots = C_{n-1} = [n] \setminus [i_1].
\]

The polymatroidal diagram associated is the following:

Since \( i_2 + t_2 > n \) it follows that \( i_2 + t_2 \geq n + i_1 \) and \( i_2 - i_1 \geq n - t_2 \geq 1 \). It is easy to see that the base ring \( K[C] \) associated to the transversal polymatroid \( \mathcal{P} \) presented by \( C \) is generated by the following set of monomials:

\[
\{x_1, \ldots, x_{i_1}\}^{i_1+1-k} \cdot x_{i_1+1}^{i_2+t_2-n-x_{i_2+t_2-n}} \cdot x_{i_2+t_2+1}^{i_2-t_2+1} \cdot x_n^{i_2-k-s}
\]
\[ \{x_{i_2+t_2-n+1}, \ldots, x_{t_2}\}^{n-i_2-1+s} \]
for any \(0 \leq k \leq i_1+1\) and \(0 \leq s \leq i_2-i_1+k\). If \(x^\alpha \in K[C]\), \(\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\), then there exists \(0 \leq k \leq i_1+1\) and \(0 \leq s \leq i_2-i_1+k\) such that

\[ \alpha_1 + \ldots + \alpha_{i_1} = i_1+1-k, \quad \alpha_1 + \ldots + \alpha_{i_2+t_2-n} + \alpha_{t_2+1} + \ldots + \alpha_n = i_2+1-s \]
and thus, \(K[C] \subset K[A \cap B]\). Conversely, if \(\alpha \in A \cap B\) then \(\alpha_1 + \ldots + \alpha_{i_1} \leq i_1+1\) and \(\alpha_1 + \ldots + \alpha_{i_2+t_2-n} + \alpha_{t_2+1} + \ldots + \alpha_n \leq i_2+1\). Then there exists \(0 \leq k \leq i_1+1\) and \(0 \leq s \leq i_2-i_1+k\) such that

\[ \alpha_1 + \ldots + \alpha_{i_1} = i_1+1-k, \quad \alpha_1 + \ldots + \alpha_{i_2+t_2-n} + \alpha_{t_2+1} + \ldots + \alpha_n = i_2+1-s \]
and since \(| \alpha | = n\) it follows that

\[ x^\alpha \in \{x_1, \ldots, x_i\}^{i_1+1-k} \{x_{i_1+1}, \ldots, x_{i_2+t_2-n}, x_{t_2+1}, \ldots, x_n\}^{i_2-i_1-k-s} \{x_{i_2+t_2-n+1}, \ldots, x_{t_2}\}^{n-i_2-1+s}. \]

Thus, \(K[C] \supset K[A \cap B]\) and so \(K[C] = K[A \cap B]\).

"\(\Rightarrow\)" Now suppose that there exists a transversal polymatroid \(\mathcal{P}\) presented by \(C = \{C_1, \ldots, C_n\}\) such that its associated base ring is \(K[A \cap B]\) and we will prove that \(i_2 \in \{1, \ldots, n-t_2\} \cup \{n-t_2+i_1, \ldots, n-2\}\). Suppose, on the contrary, let \(i_2 \in \{n-t_2+1, \ldots, n-t_2+i_1-1\}\). We may assume that \(1 \not\in C_{i_2+t_2-n+1}, \ldots, 1 \not\in C_{i_1}, 1 \not\in C_{i_1+1}, \ldots, 1 \not\in C_{n-1}\). If \(i_1 < i_2\), then \(x_1^{i_1+1}x_{n-i_1-1}^{i_2+1} \in K[A \cap B]\). But \(x_1^{i_1+1}x_{n-i_1-1}^{i_2+1} \not\in K[C]\) because the maximal power of \(x_1\) in a minimal generator of \(K[C]\) is \(\leq i_2 + t_2 - n + 1 \leq i_1\), which is false. If \(i_1 \geq i_2\), then \(x_1^{i_2+1}x_{n-i_2-1}^{i_1+1} \in K[A \cap B]\). But \(x_1^{i_2+1}x_{n-i_2-1}^{i_1+1} \not\in K[C]\) because the maximal power of \(x_1\) in a minimal generator of \(K[C]\) is \(\leq i_2 + t_2 - n + 1 \leq i_2\), which is false. Thus, \(i_2 \in \{1, \ldots, n-t_2\} \cup \{n-t_2+i_1, \ldots, n-2\}\).

**Lemma 4.4.** Let \(A\) and \(B\) like above. If either \(i_1 = 1\) and \(0 \leq t_2 \leq n - 1\), or \(i_1 \geq 2\) and \(t_2 = i_1\), or \(i_1 \geq 2\) and \(t_2 = 0\), then the \(K\)-algebra \(K[A \cap B]\) is the base ring associated to some transversal polymatroid.

**Proof.** We have three cases to study:

**Case 1.** \(i_1 = 1\) and \(0 \leq t_2 \leq n - 1\). Then we distinct five subcases:

**Subcase 1.a.** If \(t_2 = 0\), then we find a transversal polymatroid \(\mathcal{P}\) like in the Subcase 3.b. when \(i_1 = 1\).

**Subcase 1.b.** If \(t_2 > 0\) and \(t_2 + i_2 \leq n\) with \(i_2 \neq n - 2, n - 3\), then let \(\mathcal{P}\) be the transversal polymatroid presented by \(C = \{C_1, \ldots, C_n\}\), where

\[ C_1 = C_n = [n] \setminus \sigma^{t_2}[i_2], \]
\[ C_2 = \ldots = C_{i_2+2} = [n] \setminus [1], \]
\[ C_{i_2+3} = \ldots = C_{n-1} = [n] \setminus \{\{1\} \cup \sigma^{t_2}[i_2]\}. \]
It is easy to see that the polymatroid $P$ is the same like in Lemma 4.3. when $i_2 + t_2 < n - 1$. Thus $K[A \cap B] = K[C]$.  

**Subcase 1.c.** If $t_2 > 0$ and $t_2 + i_2 \leq n$ with $i_2 = n - 2$, then let $P$ be the transversal polymatroid presented by $C = \{C_1, \ldots, C_n\}$, where

$$C_1 = [n] \setminus \sigma^{t_2}[n - 2], \quad C_n = [n],$$

$$C_2 = \ldots = C_{n-1} = [n] \setminus \{1\}.$$ 

It is easy to see that the polymatroid $P$ is the same like in Lemma 4.3. when $i_2 + t_2 \leq n$ and $i_1 + 1 + i_2 \neq n$. Thus $K[A \cap B] = K[C]$.  

**Subcase 1.d.** If $t_2 > 0$ and $t_2 + i_2 \leq n$ with $i_2 = n - 3$, then let $P$ be the transversal polymatroid presented by $C = \{C_1, \ldots, C_n\}$, where

$$C_1 = C_n = [n] \setminus \sigma^{t_2}[n - 3],$$

$$C_2 = \ldots = C_{n-1} = [n] \setminus \{1\}.$$ 

It is easy to see that the polymatroid $P$ is the same like in Lemma 4.3. when $i_2 + t_2 \leq n$ and $i_1 + 1 + i_2 \neq n$. Thus $K[A \cap B] = K[C]$.  

**Subcase 1.e.** If $t_2 > 0$ and $t_2 + i_2 > n$, then let $P$ be the transversal polymatroid presented by $C = \{C_1, \ldots, C_n\}$, where

$$C_1 = C_n = [n],$$

$$C_2 = \ldots = C_{n-i_2} = [n] \setminus \sigma^{t_2}[i_2],$$

$$C_{n-i_2+1} = \ldots = C_{n-1} = [n] \setminus \{1\}.$$ 

It is easy to see that the polymatroid $P$ is the same like in Lemma 4.3. when $i_2 + t_2 > n$ and $i_1 = 1$. Thus $K[A \cap B] = K[C]$.  

**Case 2.** $i_1 \geq 2$ and $t_2 = i_1$. Then we distinct three subcases:  

**Subcase 2.a.** If $i_2 + t_2 < n - 1$, then let $P$ be the transversal polymatroid presented by $C = \{C_1, \ldots, C_n\}$, where

$$C_1 = \ldots = C_{i_1} = C_n = [n] \setminus \sigma^{t_2}[i_2],$$

$$C_{i_1+1} = \ldots = C_{i_1+i_2+1} = [n] \setminus [i_1],$$

$$C_{i_1+i_2+2} = \ldots = C_{n-1} = [n] \setminus [i_1 + i_2].$$ 

It is easy to see that the polymatroid $P$ is the same like in Lemma 4.3. when $i_2 + t_2 \leq n$ and $i_1 + 1 + i_2 \neq n$. Thus $K[A \cap B] = K[C]$.  

**Subcase 2.b.** If $i_2 + t_2 = n - 1$, then let $P$ be the transversal polymatroid presented by $C = \{C_1, \ldots, C_n\}$, where

$$C_1 = \ldots = C_{i_1} = [n] \setminus \sigma^{t_2}[i_2],$$
\[ C_{i_1+1} = \ldots = C_{n-1} = [n] \setminus [i_1], \]

\[ C_n = [n]. \]

It is easy to see that the polymatroid \( P \) is the same like in Lemma 4.3. when \( i_2 + t_2 \leq n \) and \( i_1 + 1 + i_2 = n \). Thus \( K[A \cap B] = K[\mathcal{C}] \).

**Subcase 2.c.** If \( i_2 + t_2 \geq n \), then let \( P \) be the transversal polymatroid presented by \( \mathcal{C} = \{C_1, \ldots, C_n\} \), where

\[ C_1 = \ldots = C_{n-i_2-1} = [n] \setminus \sigma^{i_2}[i_2], \]

\[ C_{n-i_2} = \ldots = C_{i_1} = C_n = [n], \]

\[ C_{i_1+1} = \ldots = C_{n-1} = [n] \setminus [i_1]. \]

It is easy to see that the polymatroid \( P \) is the same like in Lemma 4.2. when \( i_2 + t_2 > i_1 \). Thus \( K[A \cap B] = K[\mathcal{C}] \).

**Case 3.** \( i_1 \geq 2 \) and \( t_2 = 0 \). Then we distinct two subcases:

**Subcase 3.a.** If \( i_2 \leq i_1 \), then let \( P \) be the transversal polymatroid presented by \( \mathcal{C} = \{C_1, \ldots, C_n\} \), where

\[ C_1 = \ldots = C_{i_2} = C_n = [n], \]

\[ C_{i_2+1} = \ldots = C_{i_1} = [n] \setminus [i_2], \]

\[ C_{i_1+1} = \ldots = C_{n-1} = [n] \setminus [i_1]. \]

It is easy to see that the polymatroid \( P \) is the same like in Lemma 4.2. when \( i_2 + t_2 \leq i_1 \). Thus \( K[A \cap B] = K[\mathcal{C}] \).

**Subcase 3.b.** If \( i_2 > i_1 \), then let \( P \) be the transversal polymatroid presented by \( \mathcal{C} = \{C_1, \ldots, C_n\} \), where

\[ C_1 = \ldots = C_{i_1} = C_n = [n], \]

\[ C_{i_1+1} = \ldots = C_{i_2} = [n] \setminus [i_1], \]

\[ C_{i_2+1} = \ldots = C_{n-1} = [n] \setminus [i_2]. \]

The polymatroidal diagram associated is the following:
It is easy to see that the base ring $K[C]$ associated to the transversal polymatroid $\mathcal{P}$ presented by $C$ is generated by the following set of monomials:

$$\{x_1, \ldots, x_{i_1}\}^{i_1+1-k}\{x_{i_1+1}, \ldots, x_{i_2}\}^{i_2-i_1+k-s}\{x_{i_2+1}, \ldots, x_n\}^{n-i_2+s-1}$$

for any $0 \leq k \leq i_1+1$ and $0 \leq s \leq i_2-i_1+k$. If $x^\alpha \in K[C]$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, then there exists $0 \leq k \leq i_1 + 1$ and $0 \leq s \leq i_2 - i_1 + k$ such that

$$\alpha_1 + \ldots + \alpha_{i_2} = i_2 + 1 - s \quad \text{and} \quad \alpha_1 + \ldots + \alpha_{i_1} = i_1 + 1 - k$$

and thus, $K[C] \subset K[A \cap B]$. Conversely, if $\alpha \in A \cap B$ then $\alpha_1 + \ldots + \alpha_{i_2} \leq i_2 + 1$ and $\alpha_1 + \ldots + \alpha_{i_1} \leq i_1 + 1$ and so there exists $0 \leq k \leq i_1 + 1$ and $0 \leq s \leq i_2 - i_1 + k$ such that

$$\alpha_1 + \ldots + \alpha_{i_2} = i_2 + 1 - s \quad \text{and} \quad \alpha_1 + \ldots + \alpha_{i_1} = i_1 + 1 - k$$

and since $|\alpha| = n$ it follows that

$$\{x_1, \ldots, x_{i_1}\}^{i_1+1-k}\{x_{i_1+1}, \ldots, x_{i_2}\}^{i_2-i_1+k-s}\{x_{i_2+1}, \ldots, x_n\}^{n-i_2+s-1}$$

Thus, $K[C] \supset K[A \cap B]$ and so $K[C] = K[A \cap B]$.

\[\square\]

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**References**


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